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Describing families of algebraic points of given degree on a hyperelliptic curve

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Abstract. In this paper, we give a parametrization of the set of algebraic points of degree at most ℓ over $\mathbb{Q}$ on the hyperelliptic curve

$$y^2 = x(x^2 + x - 4)(x^2 - x + 45).$$

This curve was studied by Bruin and Flynn in [4] where the heights explicitly described the set of rational points, i.e. $\mathcal{C}^{(1)}(\mathbb{Q})$. Drawing on the work of Arnth-Jensen and Flynn based on [2] and one of Abel-Jacobi's fundamental theorems in [1, 8], we extend the results of [4] to algebraic points of given degree which we denote $\mathcal{C}^{(\ell)}(\mathbb{Q})$.

Keywords: Mordell-Weill group, Rational Points, Divisors and Linear Systems, Jacobian, Special algebraic curves and curves of low genus.

1. **Introduction**

Let $\mathcal{C}$ be a smooth projective plane curve defined over $\mathbb{Q}$. For all algebraic extension field $\mathbb{K}$ of $\mathbb{Q}$, we denote by $\mathcal{C}(\mathbb{K})$ the set of $\mathbb{K}$-rational points of $\mathcal{C}$ on $\mathbb{K}$ and by $\mathcal{C}^{(\ell)}(\mathbb{Q})$ the set of algebraic points of degree ℓ over $\mathbb{Q}$ *i.e*

$$\mathcal{C}^{(\ell)}(\mathbb{Q}) = \bigcup_{[\mathbb{Q}(R):\mathbb{Q}]\leq \ell} \mathcal{C}(\mathbb{K}).$$

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The degree of an algebraic point R is the degree of its field of definition on $\mathbb{Q}$ *i.e* $\deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$. One of the most important problems in algebraic geometry is to determine the set of algebraic points of degree d over $\mathbb{Q}$ on a curve $\mathcal{C}$. At present, there is no general method to determine the set $\mathcal{C}^{(\ell)}(\mathbb{Q})$. In the case $g \geq 2$, Faltings proved Mordell's conjecture in 1982 that the set $\mathcal{C}(\mathbb{K})$ is finite [7] and this proof was completed a few years later by P. Vojta [11]. But these proofs are ineffective in the sense that they only give an upper bound on the number of rational points. In special case where the rank of the curve is null, we can determine for certain smooth plane curves the set of algebraic points of degree at most ℓ over $\mathbb{Q}$ by using Abel Jacobi's theorem and the Riemann Roch spaces [9]. It's proof in [4] that the curve $\mathcal{C}$ has rank null and genus 2. However, we also note that the authors proved in [4] the following result: the set of rational points $\mathcal{C}$ on $\mathbb{Q}$ denoted $\mathcal{C}^{(1)}(\mathbb{Q})$ is equal to $\{(0,0), \infty\}$. In this article, we generalise this result by giving a parametrisation of the set of algebraic points $\mathcal{C}^{(\ell)}(\mathbb{Q})$ on the hyperelliptic curve $\mathcal{C}$ of affine equation $y^2 = x(x^2 + x - 4)(x^2 - x + 45)$ described [4].

2. **Main Result**

Our main result is the following theorem:

Theorem 2.1. *The set of algebraic points of degree at most ℓ over $\mathbb{Q}$ on the curve $\mathcal{C}$ is given by* $\mathcal{C}^{(\ell)}(\mathbb{Q}) = \bigcup_{k=0}^{1} \left(\mathcal{M}_k \bigcup \left(\bigcup_{\tau=0}^{1} \mathcal{D}_{k,\tau} \right) \right)$, *with :*

$$\mathcal{M}_k = \left\{ \left(x,\ -\frac{\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j} \right) \middle| \begin{array}{c} a_i,\ b_j \in \mathbb{Q}^*,\ a_{\frac{\ell+k}{2}} \neq 0 \text{ if } \ell \text{ is even,} \\ b_{\frac{\ell+k-5}{2}} \neq 0 \text{ if } \ell \text{ is odd and } x \text{ is a} \\ \text{solution of the equation:} \end{array} \right. \\ \left. \left(\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^{i-\frac{k}{2}} \right)^2 = x^{1-k} \left(\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j \right)^2 \prod_{\nu=1}^{2} (x - \zeta_\nu) \prod_{\mu=3}^{4} (x - \zeta_\mu) \right\}$$

$$\mathcal{D}_{k,\tau} = \left\{ \left(x,\ -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_{k,\tau}^i \right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \middle| \begin{array}{c} \psi_{k,\tau}^i = -\frac{1}{2} \sum_{\tau=1}^{2} \left(\frac{1 + (-1)^\tau \imath^k \sqrt{17 + 162k}}{2} \right)^i \\ a_i,\ b_j \in \mathbb{Q}^*,\ a_{\frac{\ell+2}{2}} \neq 0 \text{ if } \ell \text{ is even,} \\ b_{\frac{\ell-3}{2}} \neq 0 \text{ if } \ell \text{ is odd, and } x \text{ is a root} \\ \text{of the equation:} \end{array} \right. \\ \left. \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(\frac{x^i + \psi_{k,\tau}^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 \prod_{\nu=1}^{2} (x - \zeta_\nu) \prod_{\mu=3}^{4} (x - \zeta_\mu) \right\}$$

3. **Auxiliary Results**

Definition 3.1. *For a divisor $D \in Div(\mathcal{C})$, we define the $\mathbb{Q}$-vector space denoted $\mathcal{L}(D)$ by:*

$$\mathcal{L}(D) \ := \ \{f \in \mathbb{K}(\mathcal{C})\setminus\{0\} \ \mid div(f) \geq -D\} \cup \{0\}.$$

Lemma 3.2. *According to [2], we have: $\mathcal{J}(\mathbb{Q}) \ \cong \ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. The proof of this lemma can be found in [2]. □

Consider the affine expression for curve $\mathcal{C}$ given by:

$$y^2 = x\prod_{\nu=1}^{2}(x-\zeta_\nu)\prod_{\mu=3}^{4}(x-\zeta_\mu), \tag{3.1}$$

with $\zeta_\mu = \left(\frac{-1+(-1)^\nu\sqrt{17}}{2}\right)$, $\zeta_\nu = \left(\frac{1+(-1)^\mu \imath\sqrt{179}}{2}\right)$ and $\imath^2 = -1$.
Let $x = \dfrac{X}{Z}$ and $y = \dfrac{Y}{Z}$ be rational functions defined on $\mathbb{Q}$. The projective equation of the curve deducted by (3.1) is given by:

$$Z^3Y^2 = X\prod_{\nu=1}^{2}(X-\zeta_\nu Z)\prod_{\mu=3}^{4}(X-\zeta_\mu Z). \tag{3.2}$$

From the equation (3.2), we define the projective points $P_0,\ P_\nu,\ P_\mu$ and ∞ of $\mathcal{C}$ by: $P_0 = [0:0:1]$, $P_\nu = [\zeta_\nu : 0 : 1]$, $P_\mu = [\zeta_\mu : 0 : 1]$ and $\infty = [0:1:0]$.

Lemma 3.3. *For the curve $\mathcal{C}\ :\ y^2 = x\prod_{\nu=1}^{2}(x-\zeta_\nu)\prod_{\mu=3}^{4}(x-\zeta_\mu)$, we have:*

- *i*: *$div(x-\zeta_k) = 2P_k - 2\infty$ with $k \in \{0,\ldots,4\}$,*
- *ii*: *$div(y) = \sum_{k=0}^{4} P_k - 5\infty$.*

Proof. see [6]. □

Definition 3.4. *Let P be an element of $\mathcal{C}$. We then define the application j associating P with the class $[P-\infty]$ by the Jacobian fold of $\mathcal{C}$ onto $\mathcal{J}(\mathbb{Q})$:*

$$\begin{array}{rccc} j: & \mathcal{C} & \longrightarrow & \mathcal{J}(\mathbb{Q}) \\ & P & \longmapsto & [P-\infty] \end{array} \tag{3.3}$$

Corollary 3.5. *The following results are the consequences of the* Lemma 3.3

- **a:** $\sum_{k=0}^{4} j(P_k) = 0$,
- **b:** $2j(P_k) = 0$ *where* $k \in \{0,\ldots,4\}$.

Proof. There are direct consequences of Lemma 3.3 associating the Jacobian plunge expression (3.3). □

Thus, The $j(P_k)$ with $k \in \{0, \ldots, 4\}$ generate the same subgroup $\mathcal{J}(\mathbb{Q})$.

Remark 3.6. *The generator of the torsion group of rational points on the Jacobian $\mathcal{J}(\mathbb{Q})_{tor}$ described in [2] is given by:*

$$\mathcal{J}(\mathbb{Q})_{tor} \simeq \left\langle [P_0 - \infty], \ \left[\sum_{\nu=1}^{2} P_\nu - 2\infty\right]\right\rangle.$$

From Lemma 3.2 and Remark 3.6, we deduce the following lemma.

Lemma 3.7. *The Mordell-Weil group $\mathcal{J}(\mathbb{Q})$ is given by:*

$$\mathcal{J}(\mathbb{Q}) = \left\{\alpha j(P_0) + \beta \sum_{\nu=1}^{2} j(P_\nu), \ \textit{with} \ \ \alpha, \ \beta \in \{0, \ 1\}\right\}.$$

Lemma 3.8. *A $\mathbb{Q}$-base of $\mathcal{L}(m\infty)$ is given by:*

$$\mathcal{B}_m = \left\{x^i \,\middle|\, i \in \mathbb{N} \textit{ and } i \leq \frac{m}{2}\right\} \bigcup \left\{yx^j \,\middle|\, j \in \mathbb{N} \textit{ and } j \leq \frac{m-5}{2}\right\}$$

Proof. These follow from Lemma 3.3 in combination with Clifford's theorem [5] for $1 \leq \ell \leq 2g-2$ and with the use of the Riemann-Roch theorem [3] for $\ell \geq 2g-2$. For more details, see [10]. $\square$

4. **Proof of Theorem**

The following proof corresponds to proving our main theorem; the Theorem 2.1

Proof. Let $R \in \mathcal{C}(\bar{\mathbb{Q}})$ such that $[\mathbb{Q}(R) : \mathbb{Q}] = \ell$ and $R \notin \{P_k, \ \infty\}$ where $k \in \{0, \ldots, 5\}$. Let's consider R_n such that $n \in \{1, \ldots, \ell\}$ the Galois conjugates of R and let $\lambda = \left[\sum_{n=0}^{\ell} R_n - \ell\infty\right] \in \mathcal{J}(\mathbb{Q})$. From Lemma 3.7, we have $\lambda = -\alpha j(P_0) - \beta \sum_{\nu=1}^{2} j(P_\nu)$ with $\alpha, \beta \in \{0, \ 1\}$. Note that with the Jacobian fold expression (3.3), it follows that:

$$\left[\sum_{n=0}^{\ell} R_n - \ell\infty\right] = \left[(\alpha + 2\beta)\infty - \alpha P_0 - \beta \sum_{\nu=1}^{2} P_\nu\right]. \tag{4.1}$$

By reducing the classes in the expression (4.1), we obtain the following:

$$\left[\sum_{n=0}^{\ell} R_n + \alpha P_0 + \beta \sum_{\nu=1}^{2} P_\nu - (\ell + \alpha + 2\beta)\infty\right] = 0. \tag{4.2}$$

From the nullity of the class (4.2), there exists, according to the Abel-Jacobi theorem [1, 8], there exists a rational function $f_{\alpha,\beta}(x,y)$ defined on $\mathbb{Q}$ such that:

$$div(f_{\alpha,\beta}) = \sum_{n=0}^{\ell} R_n + \alpha P_0 + \beta \sum_{\nu=1}^{2} P_\nu - (\ell+\alpha+2\beta)\infty, \tag{4.3}$$

From the expression (4.3), we deduce that $f_{\alpha,\beta} \in \mathcal{L}((\ell+\alpha+2\beta)\infty)$. Then from the **Lemma 3.8**, we deduce that:

$$f_{\alpha,\beta}(x,y) = \sum_{i=0}^{\frac{\ell+\alpha+2\beta}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell+\alpha+2\beta-5}{2}} b_j y x^j, \tag{4.4}$$

with $a_i,\ b_j \in \mathbb{Q}$ and $\alpha, \beta \in \{0,\ 1\}$. Thus, depending on the parameters α and β, the cases can be grouped into two families.

Case 1 : Let's first consider the case where the parameters α and β belong to the family $\{(0,0),(1,0)\}$;

- If $(\alpha,\beta) = (0,\ 0)$, then equation (4.4) becomes:

$$f_{\alpha,\beta}(x,y) = \sum_{i=0}^{\frac{\ell}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-5}{2}} b_j y x^j, \tag{4.5}$$

  with $a_i,\ b_j \in\ \mathbb{Q}^*$ (otherwise of the R_n's should be equal to P_0, which would be absurd), $a_{\frac{\ell}{2}} \neq 0$ and $b_{\frac{\ell-5}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be equal to ∞, which would be absurd).
- If $(\alpha,\beta) = (1,\ 0)$, then equation (4.4) becomes:

$$f_{\alpha,\beta}(x,y) = \sum_{i=0}^{\frac{\ell+1}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-4}{2}} b_i y x^j, \tag{4.6}$$

  and since $ord_{P_0} f_{\alpha,\beta} = 1$, which implies that $a_0 = 0$, so we obtain

$$f_{\alpha,\beta}(x,y) = \sum_{i=1}^{\frac{\ell+1}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-4}{2}} b_j y x^j, \tag{4.7}$$

  with $a_{i_{i\geq 1}},\ b_j \in\ \mathbb{Q}^*$ (otherwise of the R_n's should be at P_0, which would be absurd), $a_{\frac{\ell+1}{2}} \neq 0$ and $b_{\frac{\ell-4}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd).

Thus, for any pair of parameters (α,β), belonging to the set $\{(0,0),(1,0)\}$, of the combination of equations (4.5) and (4.7), we deduce the expression of f_k

as a function of the parameter k with $k \in \{0,\ 1\}$ as follows:

$$f_k(x,y) = \sum_{i=k}^{\frac{\ell+k}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell+k-5}{2}} b_j y x^j. \tag{4.8}$$

At the points R_n, which are zeros of f_k in the expression (4.7), we have $f_k(x,y)=0$, which allows us to express y as a function of x as follows:

$$y = -\frac{\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j}. \tag{4.9}$$

By replacing the expression for y of (4.9) in the equation (3.1), we obtain:

$$\left(\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^i\right)^2 = x\left(\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j\right)^2 \prod_{\nu=1}^{2}(x-\zeta_\nu)\prod_{\mu=3}^{4}(x-\zeta_\mu). \tag{4.10}$$

Since the equation (4.10) is not of degree ℓ, we deduce the following equation:

$$\left(\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^{i-\frac{k}{2}}\right)^2 = x^{1-k}\left(\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j\right)^2 \prod_{\nu=1}^{2}(x-\zeta_\nu)\prod_{\mu=3}^{4}(x-\zeta_\mu). \tag{4.11}$$

The degree of the equation (4.11) is ℓ. Indeed, the first member is of degree $2\left(\frac{\ell+k}{2}-\frac{k}{2}\right)=\ell$ and the second member is of degree $2\left(\frac{\ell+k-5}{2}\right)-5+k=\ell$. This gives a third family of points of degree ℓ:

$$\mathcal{M}_k = \left\{ \left(x,\ -\frac{\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j}\right) \middle| \begin{array}{c} a_i,\ b_j \in \mathbb{Q}^*,\ a_{\frac{\ell+k}{2}} \neq 0 \text{ if } \ell \text{ is even}, \\ b_{\frac{\ell+k-5}{2}} \neq 0 \text{ if } \ell \text{ is odd and } x \text{ is a} \\ \text{solution of the equation:} \end{array} \right.$$
$$\left. \left(\sum_{i=k}^{\frac{\ell+k}{2}} a_i x^{i-\frac{k}{2}}\right)^2 = x^{1-k}\left(\sum_{j=0}^{\frac{\ell+k-5}{2}} b_j x^j\right)^2 \prod_{\nu=1}^{2}(x-\zeta_\nu)\prod_{\mu=3}^{4}(x-\zeta_\mu) \right\}$$

Case 2 : Let's now consider the case where the parameters α and β belong to the family $\{(0,1),(1,1)\}$;

- If $(\alpha,\beta)=(0,\ 1)$, then equation (4.4) becomes:

$$f_{\alpha,\beta}(x,y) = \sum_{i=0}^{\frac{\ell+2}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{4.12}$$

and since $ord_{P_1} f_{\alpha,\beta} = ord_{P_2} f_{\alpha,\beta} = 1$, which implies that

$$a_0 = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \psi_0^i$$

with

$$\psi_0^i = -\frac{1}{2} \sum_{\nu=1}^{2} \left(\frac{-1 + (-1)^{\nu}\sqrt{17}}{2} \right)^i ,$$

so the equation (4.12) becomes:

$$f_{\alpha,\beta}(x, y) = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_0^i \right) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j , \tag{4.13}$$

with $a_{\frac{\ell+2}{2}} \neq 0$ and $b_{\frac{\ell-3}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd).

- If $(\alpha, \beta) = (1,\ 1)$, from Corollary 3.5 we have

$$div(f_{\alpha,\beta}) = \sum_{n=0}^{\ell} R_n + \sum_{\mu=3}^{4} P_\mu - (\ell + 2)\infty. \tag{4.14}$$

From this new expression for the rational divisor of (4.14), we deduce that $f_{\alpha,\beta} \in \mathcal{L}((\ell + 2)\infty)$, which, according to the Lemma 3.8, induces the transformation of the expression (4.4) becomes as follows:

$$f_{\alpha,\beta}(x, y) = \sum_{i=0}^{\frac{\ell+2}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j , \tag{4.15}$$

and since $ord_{P_3} f_{\alpha,\beta} = ord_{P_4} f_{\alpha,\beta} = 1$, which implies that

$$a_0 = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \psi_1^i$$

with

$$\psi_1^i = -\frac{1}{2} \sum_{\mu=3}^{4} \left(\frac{1 + (-1)^{\mu}\imath\sqrt{179}}{2} \right)^i ,$$

hence the equation (4.15) becomes:

$$f_{\alpha,\beta}(x, y) = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_1^i \right) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j , \tag{4.16}$$

with $a_{\frac{\ell+2}{2}} \neq 0$ and $b_{\frac{\ell-3}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd).

Thus, for any pair of parameters (α, β), belonging to the set $\{(0,1),(1,1)\}$, of the combination of equations (4.5) and (4.7), we deduce the expression of $f_{k,\tau}$ as a function of the parameters k and τ with $k,\ \tau \in \{0,\ 1\}$ as follows:

$$f_{k,\tau}(x,y) = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_{k,\tau}^i\right) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{4.17}$$

with

$$\psi_{k,\tau}^i = -\frac{1}{2} \sum_{\tau=1}^{2} \left(\frac{1 + (-1)^\tau \imath^k \sqrt{17 + 162k}}{2} \right)^i.$$

Similarly, at the points R_n, which are zeros of $f_{k,\tau}$ in the expression (4.17), we have $f_{k,\tau}(x,y) = 0$, which allows us to express y as a function of x as follows:

$$y = -\frac{\displaystyle\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_{k,\tau}^i\right)}{\displaystyle\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}. \tag{4.18}$$

By replacing the expression for y of (4.18) in the equation (3.1), we obtain:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_{k,\tau}^i\right) \right)^2 = x \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j \right)^2 \prod_{\nu=1}^{2} (x - \zeta_\nu) \prod_{\mu=3}^{4} (x - \zeta_\mu). \tag{4.19}$$

Since the equation (4.19) is not of degree ℓ, we deduce the following equation:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(\frac{x^i + \psi_{k,\tau}^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 \prod_{\nu=1}^{2} (x - \zeta_\nu) \prod_{\mu=3}^{4} (x - \zeta_\mu). \tag{4.20}$$

The degree of the equation (4.20) is ℓ. Indeed, the first member is of degree $2\left(\frac{\ell+2}{2} - 1\right) = \ell$ and the second member is of degree $2\left(\frac{\ell-3}{2} - \frac{1}{2}\right) + 4 = \ell$. This gives a third family of points of degree ℓ:

$$\mathcal{D}_{k,\tau} = \left\{ \left(x,\ -\frac{\displaystyle\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \psi_{k,\tau}^i\right)}{\displaystyle\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \middle| \begin{array}{c} \psi_{k,\tau}^i = -\frac{1}{2} \displaystyle\sum_{\tau=1}^{2} \left(\frac{1 + (-1)^\tau \imath^k \sqrt{17 + 162k}}{2} \right)^i \\ a_i,\ b_j \in \mathbb{Q}^*,\ a_{\frac{\ell+2}{2}} \neq 0 \text{ if } \ell \text{ is even}, \\ b_{\frac{\ell-3}{2}} \neq 0 \text{ if } \ell \text{ is odd, and } x \text{ is a root} \\ \text{of the equation:} \end{array} \right. \\ \left. \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(\frac{x^i + \psi_{k,\tau}^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 \prod_{\nu=1}^{2} (x - \zeta_\nu) \prod_{\mu=3}^{4} (x - \zeta_\mu) \right\}$$

$\square$

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