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On projectively PR-flat Douglas sprays

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Abstract. Deformation of every spray into a projective spray can be done using a volume form on a manifold. The Riemann curvature of a projective spray is called the projective Riemmann curvature. In this paper, we are going to present a global rigidity result for the projectively PR-flat sprays that have vanishing Douglas curvature. Then we characterize projectively PR-flat Randers metrics of Douglas curvature.

Keywords: Sprays, projective Riemann curvature, Douglas curvature.

1. Introduction

The S-curvature, an important non-Riemannian quantity, is derived as a free index form by the geodesic fields [3][6]. Let $\mathbf{G}$ be a spray on an n-dimensional manifold M. Z. Shen has introduced a projectively equivalent spray $\hat{\mathbf{G}}$ with respect to a fixed volume form dV on a manifold M^n, [9]:

$$\hat{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y},$$

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where $\mathbf{S} = \mathbf{S}_{(\mathbf{G},dV)}$ is the S-curvature of $(\mathbf{G}, dV)$ and $Y = y^i \frac{\partial}{\partial y^i}$ is the vertical vector field on TM^n. Thus, the curvatures of $\hat{\mathbf{G}}$ are the projective invariants of the spray $\mathbf{G}$ with respect to a fixed volume form dV.

The Riemann curvature of $\hat{\mathbf{G}}$ is called projective Riemann curvature of $(\mathbf{G}, dV)$:

$$PR^i{}_{k_{(\mathbf{G},dV)}} := R^i{}_{k_{\hat{G}}},$$

that can be expressed as follows:

$$\begin{aligned} PR^i{}_{k_{(\mathbf{G},dV)}} &= R^i{}_k + \frac{1}{n+1}\Big[-2\frac{\partial S}{\partial x^k} y^i + \frac{\partial^2 S}{\partial x^j \partial y^k} y^i y^j + \frac{\partial S}{\partial x^j} y^j \delta^i{}_k \\ &\quad -2G^j \frac{\partial^2 S}{\partial y^j \partial y^k} y^i - 2G^j \frac{\partial S}{\partial y^j} \delta^i{}_k + \frac{\partial S}{\partial y^j}\frac{\partial G^j}{\partial y^k} y^i \Big] \\ &\quad + \frac{1}{(n+1)^2}\Big[S^2 \delta^i{}_k - S\frac{\partial S}{\partial y^k} y^i \Big]. \end{aligned}$$

Hence, the projective Riemann curvature of (G, dV) is given by [8]

$$PR^i{}_{k_{(\mathbf{G},dV)}} = R^i{}_k + \Xi \delta^i{}_k - \frac{1}{2}\Xi_{.k} y^i + \frac{3\chi_k}{n+1} y^i, \tag{1.1}$$

where $R^i{}_k = R^i{}_{k_{\mathbf{G}}}$ is the Riemann curvature of the spray $\mathbf{G}$ and

$$\Xi = \frac{S_{|0}}{n+1} + \left[\frac{S}{n+1}\right]^2,$$

where $S_{|0}$ is the covariant derivative of S along the geodesic $\mathbf{G}$. $\mathbf{G}$ is of *projectively PR*-flat if there is a volume form dV on M such that

$$PR^i{}_{k_{(\mathbf{G},dV)}} = 0.$$

Similarly, the Ricci curvature of $\hat{\mathbf{G}}$ is called the projective Ricci-curvature of $(\mathbf{G}, dV)$:

$$PRic_{(\mathbf{G},dV)} := Ric_{\hat{\mathbf{G}}},$$

that can be expressed by (1.1) as follows:

$$PRic_{(\mathbf{G},dV)} = Ric + (n-1)\left\{ \frac{S_{|0}}{n+1} + \Big[\frac{S}{n+1}\Big]^2 \right\}. \tag{1.2}$$

It can be easily checked that if $\mathbf{G}$ has the condition Ricci-flat and $S = dh$ for some scalar function $h = h(x)$, then $\mathbf{G}$ is projectively Ricci-flat.

Recently, the experienced researcher pay their attention to projective Ricci-curvature, [1], [2], [5], [10], [11].
This paper is designed to classify the Douglas type sprays of projectively PR-flat (PRic-flat), and examples are obtained.

We give the following theorem within the notion of spray framework, and then we obtain a new formula for Douglas type projectively PR-flat (PRic flat) sprays:

Theorem 1.1. *Let* $\mathbf{G}$ *be a Douglas spray on* M^n *i.e., the spray coefficients* G^i *are in the following form*

$$G^i = \hat{G}^i + P(x,y)y^i, \quad \hat{G}^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k, \tag{1.3}$$

where $\Gamma^i_{jk}(x)$ *are local functions on* M^n *and* $P(x,y)$ *is a positively homogeneous function of degree one. Then* $\mathbf{G}$ *is projectively PR-flat if and only if there is a scalar function* λ *on* M^n *such that*

$$PR^i{}_k = \big[\eta_{0\hat{|}0} - (\eta_0)^2\big]\delta^i{}_k - \big[\eta_{0\hat{|}k} - \eta_0\eta_k\big]y^i, \tag{1.4}$$

where $PR^i{}_k$ *is the Riemann curvature of* $\hat{\mathbf{G}}$*, "* $\hat{|}$ *" denotes the horizontal covariant derivative with respect to* $\hat{\mathbf{G}}$*,* $\eta_0 := \eta_i y^i$*,*

$$\eta_i := \lambda_i + \frac{1}{n+1}(\pi_i - \Gamma^m_{mi}),$$

$\eta_i := \partial_i\eta,\ \lambda_i := \partial_i\lambda$ *and* $\pi_i := \partial_i(\ln\sigma)$.

It is known that a Finsler metric on an n-dimensional manifold M is a function $F : TM \to [0,\infty)$ with the following two properties:

(a) $F(u,v)$ is C^∞ on $TM\backslash\{0\}$;
(b) the restriction $F_u := F_{|T_uM}$ is a Minkowski function on T_uM for all $u \in M$.

(α,β)-metrics forms a large class of Finsler metrics. The following form is used to define them

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha :=\ \alpha(x,y) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta := \beta(y) = b_i(x)y^i$ is a 1-form with $||\beta||_\alpha < b_0$ and $\phi(s) \in C^\infty$ is a positive function on $(-b_0, b_0)$. It is proved that $F = \alpha\phi(s)$ is a positive definite Finsler metric if and only if [3]

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0.$$

When $\phi(s) = 1 + s$, the Finsler metrics $F = \alpha + \beta$ is called Randers metrics. Now, we consider that the Randers metrics $F = \alpha + \beta$ is of Douglas type. In that case, we give the following theorem:

Theorem 1.2. *Let* $F = \alpha+\beta$ *be a Douglas type Randers metric on* M^n*. Then* F *is a projectively PR-flat if and only if there is a scalar function* g *on* M *such that*

$$PR^i{}_k = \big[g_{0\hat{|}0} - (g_0)^2\big]\delta^i{}_k - \big[g_{0\hat{|}k} - g_0\eta_k\big]y^i, \tag{1.5}$$

where $g_0 := g_{x^i} y^i$.

By the above discussion, we have the following result

Theorem 1.3. *Let* $\mathbf{G}$ *be a Douglas spray on* M^n *i.e., the spray coefficients* G^i *are in the following form*

$$G^i = \hat{G}^i + P(x,y)y^i, \quad \hat{G}^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k, \tag{1.6}$$

where $\Gamma^i_{jk}(x)$ *are local functions on* M^n *and* $P(x,y)$ *is a positively homogeneous function of degree one. Then* $\mathbf{G}$ *is projectively Ricci-flat if and only if there is a scalar function* λ *on* M^n *such that*

$$\hat{Ric} = -(n-1)\big[(\eta_0)^2 - \eta_{0\hat{|}0}\big], \tag{1.7}$$

where $\hat{Ric}$ *is the Ricci curvature of* $\hat{\mathbf{G}}$, " $\hat{|}$ " *denotes the horizontal covariant derivative with respect to* $\hat{\mathbf{G}}$, $\eta_0 := \eta_i y^i$,

$$\eta_i := \lambda_i + \frac{1}{n+1}(\pi_i - \Gamma^m_{mi}),$$

$\eta_i := \partial_i \eta$, $\lambda_i := \partial_i \lambda$ *and* $\pi_i := \partial_i(\ln\sigma)$.

Example 1.4. *Consider the metric* $F = \alpha + \beta$, *where*

$$\alpha = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}, \quad \beta = \frac{\langle x,y\rangle}{1-|x|^2}.$$

Such a metric is the famous Funk metric which is projectively flat on the unit ball $\mathbb{B}^n(1)$ *in* $\mathbb{R}^n$ *and the flag curvature* $\mathbf{K} = -1/4$. *Also, it is easy to see* $s_{ij} = 0$. *Then* F *is a projectively Ricci-flat Douglas metric.*

2. Preliminaries

Let M be a differential manifold. In a standard local coordinate system, a spray is a vector field on TM which are expressed as follows

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - G^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where $G^i = G^i(x,y)$ are local C^∞ functions on $TM \setminus \{0\}$ with $G^i(x,\lambda y) = \lambda^2 G^i(x,y), \forall \lambda > 0$.

For a spray $\mathbf{G}$, the Riemann curvature tensor $R^i{}_k$ are defined as follows.

$$R^i{}_j = 2\frac{\partial G^i}{\partial x^j} - \frac{\partial^2 G^i}{\partial x^k \partial y^j} y^k + 2G^k \frac{\partial^2 G^i}{\partial y^k \partial y^j} - \frac{\partial G^i}{\partial y^k}\frac{\partial G^k}{\partial y^j}$$

and the trace of $R^i{}_j$ is called the Ricci curvature, $Ric = R^m_{\ m}$.

One of the most widely used non-Riemannian curvatures in spray geometry is S-curvature, which is obtained by

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\big[\ln \sigma_{BH}\big], \tag{2.2}$$

where $dV_F = \sigma_F(x)dx^1 \wedge \cdots \wedge dx^n$ is the Busemann-Hausdorff volume form.

The χ-curvature can be expressed in several forms. For an arbitrary volume form dV,

$$\chi_k = \frac{1}{2}\left\{S_{.k|0} - S_{|k}\right\}. \tag{2.3}$$

The Douglas curvature tensors [2] are defined as follows

$$D_j{}^i{}_{kl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big). \tag{2.4}$$

For a Finsler metric F on an n-dimensional manifold M, the induced spray coefficients of F are obtained from the following equation

$$G^i := \frac{1}{4}g^{il}\Big\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\Big\},$$

where g^{ij} is the inverse of the fundamental tensor $g_{ij} := [\frac{1}{2}F^2]_{y^i y^j}$.

The so-called (α, β)-metrics form an important class of Finsler metrics that can be defined as $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, $\alpha := \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta := \beta(y) = b_i(x)y^i$ is a 1-form and $\phi(s) \in C^\infty$ is a positive function on some open interval. When $\phi(s) = 1 + s$, the Finsler metrics $F = \alpha + \beta$ is called Randers metrics. If $\phi(s) = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called a Kropina metric. The spray coefficients of (α, β)-metrics are given in [3]

$$G^i = {}^\alpha G^i + \alpha Q s^i_0 + \Theta(r_{00} - 2\alpha Q s_0)\frac{y^i}{\alpha} + \Psi(r_{00} - 2\alpha Q s_0)b^i,$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$

$$\Theta = \frac{(\phi - s\phi')\phi' - s\phi\phi''}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']},$$

$$\Psi = \frac{\phi''}{2\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]}.$$

H. Zhu and R. Li has proved the following useful lemma, [8]:

Lemma 2.1. *Let* $\mathbf{G}$ *be a spray on* M^n*. The followings are equivalent:*

(a) $\mathbf{G}$ *is projectively PR-flat,*

(b) *For any volume form* dV *on* M^n *there is a scalar function* λ *on* M^n *such that*

$$PR^i{}_k = \tau\delta^i{}_k - \frac{1}{2}\tau_{.k}y^i, \tag{2.5}$$

where "." denotes the vertical derivatives with respect to y *and*

$$\tau = \lambda_{0|0} - \lambda_0^2 + \frac{2}{n+1}\lambda_0 S.$$

(c) *For any volume form dV on M^n there is a scalar function λ on M^n such that*

$$Ric^i{}_k = \big[\Psi_{|0} - \Psi^2\big]\delta^i{}_k - \frac{1}{2}\big[\Psi_{|0} - \Psi^2\big]_{.k} y^i - \frac{3\chi_k}{n+1} y^i, \tag{2.6}$$

where " $|$ " is the horizontal covariant derivative with respect to $\mathbf{G}$*, $\lambda_0 := \lambda_{x^m} y^m$,*

$$\Psi := \lambda_0 - \frac{S}{n+1},$$

and $S = S_{(G,dV)}$.

It's important to remind that, Z. Shen and L. Sun has proved the following lemma, [7]:

Lemma 2.2. *Let* $\mathbf{G}$ *be a spray on M^n. The followings are equivalent:*

(a) $\mathbf{G}$ *is projectively Ricci-flat,*
(b) *For any volume form dV on M^n there is a scalar function λ on M^n such that*

$$PRic_{(G,dV)} = (n-1)\left\{\lambda_{0|0} - \lambda_0^2 + \frac{2}{n+1}\lambda_0 S\right\}, \tag{2.7}$$

(c) *For any volume form dV on M^n there is a scalar function λ on M^n such that*

$$Ric_G = (n-1)\big\{\Xi_{|0} - \Xi^2\big\}, \tag{2.8}$$

where " $|$ " is the horizontal covariant derivative with respect to $\mathbf{G}$*, $\lambda_0 := \lambda_{x^m} y^m$,*

$$\Xi := \lambda_0 - \frac{S}{n+1}$$

and $S = S_{(G,dV)}$.

Moreover, the lemma mentioned below is also crucial for us in the proof section.

Lemma 2.3. [7, 2] *Let $F = \alpha+\beta$ be a Randers metric on M^n. F is projectively Ricci-flat if and only if there is a scalar function μ on M^n such that*

$$\begin{aligned} {}^{\alpha}Ric &= 2s_{0m}s^m{}_0 + \alpha^2 s^i{}_j s^j{}_i - (n-1)[(\mu_0)^2 - \mu_{0;0}], && (2.9) \\ s^m{}_{0;m} &= -(n-1)\mu_{x^m} s^m{}_0, && (2.10) \end{aligned}$$

where ${}^{\alpha}Ric$ denotes the Ricci curvature of α.

3. Proof of Main Theorems

Proof of Theorem 1.1. Let $\mathbf{G}$ be a Douglas spray on an M^n. Then, the spray coefficients G^i satisfies (1.3). Obviously,

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

and

$$\hat{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2\hat{G}^i \frac{\partial}{\partial y^i}$$

are projectively equivalent. Consequently, the projective Riemann curvature of $(\mathbf{G}, dV)$ is given by [8]:

$$PR^i{}_k = P\hat{R}^i{}_k = \hat{R}^i{}_k + \hat{\Xi}\delta^i{}_k - \frac{1}{2}\hat{\Xi}_{.k} y^i + \frac{3\hat{\chi}_k}{n+1} y^i, \tag{3.1}$$

where

$$\hat{\Xi} := \frac{\hat{S}_{\hat{|}0}}{n+1} + \Big[\frac{\hat{S}}{n+1}\Big]^2, \tag{3.2}$$

where $\hat{S} = \hat{S}_{(\hat{\mathbf{G}}, dV)}$ is the S-curvature of $(\hat{\mathbf{G}}, dV)$, " $\hat{|}$ " denotes the horizontal covariant derivative with respect to $\hat{\mathbf{G}}$, and $\hat{\chi} = \hat{\chi}_{(\hat{\mathbf{G}}, dV)}$ is the χ-curvature of $(\hat{\mathbf{G}}, dV)$. It is possible to acquire that

$$\begin{aligned} \hat{S} &= \frac{\partial \hat{G}^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma) \\ &= \Gamma^m_{m0} - \pi_0, \end{aligned} \tag{3.3}$$

where

$$\pi_0 = \pi_m y^m, \qquad \pi_m = \frac{\partial}{\partial x^m}(\ln \sigma).$$

It is simple to see that

$$\hat{S}_{\hat{|}0} = (\Gamma^m_{m0} - \pi_0)_{\hat{|}0}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we get

$$\hat{\Xi} := \frac{(\Gamma^m_{m0} - \pi_0)_{\hat{|}0}}{n+1} + \Big[\frac{\Gamma^m_{m0} - \pi_0}{n+1}\Big]^2, \tag{3.5}$$

From (3.5), we have

$$\hat{\Xi}_{.k} := \frac{2(\Gamma^m_{m0} - \pi_0)_{\hat{|}k}}{n+1} + 2\Big[\frac{(\Gamma^m_{m0} - \pi_0)(\Gamma^m_{mk} - \pi_k)}{(n+1)^2}\Big]. \tag{3.6}$$

By (2.3) and [8, Lemma 3.1.], we obtain

$$\hat{\chi} = 0. \tag{3.7}$$

Plugging (3.5), (3.6), and (3.7) into (3.1), we get

$$\begin{aligned} PR^i{}_k &= \hat{R}^i{}_k + \Big\{ \frac{(\Gamma^m_{m0} - \pi_0)_{\hat{|}0}}{n+1} + \Big[\frac{\Gamma^m_{m0} - \pi_0}{n+1} \Big]^2 \Big\} \delta^i{}_k - \frac{1}{2} \Big\{ \frac{2(\Gamma^m_{m0} - \pi_0)_{\hat{|}k}}{n+1} \\ &\quad + 2 \Big[\frac{(\Gamma^m_{m0} - \pi_0)(\Gamma^m_{mk} - \pi_k)}{(n+1)^2} \Big] \Big\} y^i. \end{aligned} \tag{3.8}$$

By (1.6), we have

$$\begin{aligned} S &= \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} (\ln \sigma) \\ &= \hat{S} + (n+1)P. \end{aligned} \tag{3.9}$$

Besides by (1.6), we have

$$\begin{aligned} \lambda_{0|0} &= \lambda_{0\hat{|}0} - 2[Py^i]\lambda_i \\ &= \lambda_{0\hat{|}0} - 2P\lambda_0, \end{aligned} \tag{3.10}$$

where " $|$ " denotes the horizontal covariant derivative with respect to $\mathbf{G}$ and $\lambda_0 := \lambda_i y^i$. By (3.9) and (3.10), we have

$$\begin{aligned} \tau &= \lambda_{0\hat{|}0} - 2P\lambda_0 - \lambda_0^2 + \frac{2}{n+1} \lambda_0 \Big[\hat{\mathbf{S}} + (n+1)P \Big] \\ &= \lambda_{0\hat{|}0} - \lambda_0^2 + \frac{2}{n+1} \lambda_0 \Big[\Gamma^m_{m0} - \pi_0 \Big]. \end{aligned} \tag{3.11}$$

Differentiating (3.11) with respect to y^k yields

$$\tau_k = 2 \Big\{ \lambda_{0\hat{|}k} - \lambda_0 \lambda_k + \frac{1}{n+1} [\lambda_k (\Gamma^m_{m0} - \pi_0) + \lambda_0 (\Gamma^m_{mk} - \pi_k)] \Big\}. \tag{3.12}$$

Substituting (3.8)-(3.12) into (2.5), we have

$$\begin{aligned} &\hat{R}^i{}_k + \Big\{ \frac{(\Gamma^m_{m0} - \pi_0)_{\hat{|}0}}{n+1} + \Big[\frac{\Gamma^m_{m0} - \pi_0}{n+1} \Big]^2 \Big\} \delta^i{}_k - \Big\{ \frac{(\Gamma^m_{m0} - \pi_0)_{\hat{|}k}}{n+1} \\ &+ \Big[\frac{(\Gamma^m_{m0} - \pi_0)(\Gamma^m_{mk} - \pi_k)}{(n+1)^2} \Big] \Big\} y^i = \Big\{ \lambda_{0\hat{|}0} - \lambda_0^2 + \frac{2}{n+1} \lambda_0 [\Gamma^m_{m0} - \pi_0] \Big\} \delta^i{}_k \\ &- \Big\{ \lambda_{0\hat{|}k} - \lambda_0 \lambda_k + \frac{1}{n+1} \Big[\lambda_k (\Gamma^m_{m0} - \pi_0) + \lambda_0 (\Gamma^m_{mk} - \pi_k) \Big] \Big\} y^i. \end{aligned} \tag{3.13}$$

Thus,

$$\hat{R}^i{}_k = [\eta_{0\hat{|}0} - (\eta_0)^2] \delta^i{}_k - [\eta_{0\hat{|}k} - \eta_0 \eta_k] y^i, \tag{3.14}$$

where $\eta_0 := \eta_i y^i$ and $\eta_i := \lambda_i + \frac{1}{n+1}(\pi_i - \Gamma^m_{mi})$. The converse is obvious. □

Proof of Theorem 1.2. Note that a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed, i.e.

$$s_{ij} = 0. \tag{3.15}$$

Together with [8, Lemma 5.1.], we get

$${}^{\alpha}PR^i{}_k = \big[g_{0\hat{|}0} - (g_0)^2\big]\delta^i{}_k - \big[g_{0\hat{|}k} - g_0\eta_k\big]y^i, \tag{3.16}$$

where $g_0 := g_{x^i}y^i$. The sufficiency is obvious. □

The proof of Theorem 1.3 is similar to Theorem 1.1, as stated below:

Proof of Theorem 1.3. Let $\mathbf{G}$ be a Douglas spray on an M^n. Then, the spray coefficients G^i satisfies (1.6). Obviously,

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$$

and

$$\hat{\mathbf{G}} = y^i\frac{\partial}{\partial x^i} - 2\hat{G}^i\frac{\partial}{\partial y^i}$$

are projectively equivalent. Consequently, the projective Ricci curvature of $(\mathbf{G}, dV)$ is given by [7]:

$$PRic_{(\mathbf{G},dV)} = PRic_{(\hat{\mathbf{G}},dV)} = \hat{Ric} + (n-1)\left\{\frac{\hat{S}_{\hat{|}0}}{n+1} + [\frac{\hat{S}}{n+1}]^2\right\}, \tag{3.17}$$

where $\hat{S} = \hat{S}_{(\hat{\mathbf{G}},dV)}$ is the S-curvature of $(\hat{\mathbf{G}}, dV)$ and "$\hat{|}$" denotes the horizontal covariant derivative with respect to $\hat{\mathbf{G}}$. One can obtain that

$$\begin{aligned}\hat{S} &= \frac{\partial \hat{G}^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}(\ln\sigma)\\ &= \Gamma^m_{m0} - \pi_0,\end{aligned} \tag{3.18}$$

where

$$\pi_0 = \pi_m y^m, \qquad \pi_m = \frac{\partial}{\partial x^m}(\ln\sigma).$$

It is easy to see that

$$\hat{S}_{\hat{|}0} = (\Gamma^m_{m0} - \pi_0)_{\hat{|}0}. \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17), we get

$$PRic_{(\mathbf{G},dV)} = \hat{Ric} + (n-1)\Big[\frac{1}{n+1}(\Gamma^m_{m0} - \pi_0)_{\hat{|}0} + \frac{1}{(n+1)^2}(\Gamma^m_{m0} - \pi_0)^2\Big] \tag{3.20}$$

By (1.6), we have

$$\begin{aligned}S &= \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}(\ln\sigma)\\ &= \hat{S} + (n+1)P.\end{aligned} \tag{3.21}$$

Besides by (1.6), we have

$$\begin{aligned} \lambda_{0|0} &= \lambda_{0\hat{|}0} - 2[Py^i]\lambda_i \\ &= \lambda_{0\hat{|}0} - 2P\lambda_0, \end{aligned} \tag{3.22}$$

where " | " denotes the horizontal covariant derivative with respect to $\mathbf{G}$ and $\lambda_0 := \lambda_i y^i$. Substituting (3.20), (3.21) and (3.22) into (2.7), we have

$$\hat{Ric} = -(n-1)\big\{(\eta_0)^2 - \eta_{0\hat{|}0}\big\}, \tag{3.23}$$

where $\eta_0 := \eta_i y^i$ and

$$\eta_i := \lambda_i + \frac{1}{n+1}(\pi_i - \Gamma^m_{mi}).$$

The converse is obvious. □

For Randers metrics, we use the theorem 1.3. Consider the geodesic coefficients of $\mathbf{G} = \mathbf{G}_F$ of Randers metric, [4]:

$$G^i = \hat{G}^i + Py^i, \quad \hat{G}^i = {}^\alpha G^i + \alpha s^i{}_0, \tag{3.24}$$

where

$$P = \frac{r_{00} - 2\alpha s_0}{2F}.$$

Since $s_{ij} = 0$, (3.24) becomes

$$G^i = {}^\alpha G^i + Py^i, \tag{3.25}$$

where $P = \dfrac{r_{00}}{2F}$. Hence, following the proof of Theorem 1.3, we obtain

$${}^\alpha\mathbf{Ric} = -(n-1)\big[(\eta_0)^2 - \eta_{0\hat{|}0}\big], \tag{3.26}$$

where $\eta_0 := \eta_i y^i$ and

$$\begin{aligned} \eta_i &= \lambda_i + \frac{1}{n+1}(\pi_i - \Gamma^m_{mi}) \\ &= \lambda_i - \vartheta_i, \end{aligned} \tag{3.27}$$

Here,

$$\vartheta_i = \frac{1}{n+1}\frac{\partial}{\partial x^i}\Big(\ln\frac{\sigma_\alpha}{\sigma}\Big).$$

This completes the proof. □

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