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Ricci semi-symmetric null hypersurfaces in a Lorentzian space form

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Abstract. In this paper, we study Ricci semi-symmetric null hypersurfaces in a Lorentzian space form. We give a necessary and sufficient condition for a screen quas-conformal null hypersurface to be Ricci semi-symmetric. We show that every screen quasi-conformal null hypersurface M of $\mathbb{R}_1^{m+2}$ such that rank$\overset{*}{A}_\xi < m$ is Ricci semi-symmetric. Next, we give a local classification of a Ricc semi-symmetric screen conformal null hypersurface of a Lorentzian space form.

Keywords: Null hypersurface, screen shape operator, screen quasi-conformal, screen conformal, screen principal curvatures, Ricci-symmetric null hypersurface.

1. Introduction

The theory of hypersurfaces, defined as submanifolds of codimension one, is one of the fundamental theories of submanifolds. As it is known, the main difference between the geometry of hypersurface in Riemannian manifold and in semi-Riemannian manifold is that in the latter case the induced metric tensor

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AMS 2020 Mathematics Subject Classification: 53C40, 53C42 , 53C50 field by the semi-Riemannian metric on the ambient space is not necessarily non-degenerate. If the induced metric tensor field is degenerate, the classical theory of Riemannian and semi-Riemannian hypersurfaces fails since the normal bundle and the tangent bundle of the hypersurface have a non zero intersection.

The existence of null hypersurfaces is one of the most remarkable features both in semi-Riemannian geometry and General Relativity [10], [13]. It has been recently developted a mathematical framework for null submanifold geometry similar to its classical Riemannian counterpart was developed in [7],[8].

In the present paper, we investigate Ricci semi-symmetric null hypersurfaces in a Lorentzian space form and is organized as follows. After the Preliminaries section, in section 3, a necessary and sufficient condition for a screen quasi-conformal null hypersurface to be Ricci semi-symmetric is obtained. We prove that every totally umbilical or totally geodesic quasi-conformal null Hypersurfaces of a $(m+2)$ dimensional Lorentzian space forms are Ricci semi-symmetric. At the end, we give a local classification of a Ricci semi-symmetric screen conformal null hypersurfaces of a Lorentzian space form.

2. Preliminaries

2.1. Null hypersurfaces.

Let $(\overline{M},\overline{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold of index ν, $(0<\nu<m+2)$. Consider a hypersurface M of $\overline{M}$ and denote by g the tensor field induced by $\overline{g}$ on M. We say that M is a null (degenerate, lightlike) hypersurface if $\mathrm{rank}(g)=m$. Then the normal vector bundle $TM^{\perp}$ intersects the tangent bundle along a nonzero differentiable distribution called the radical distribution of M and denoted by $Rad(TM)$:

$$Rad(TM): x \mapsto Rad(T_xM) = T_xM \cap T_xM^{\perp}. \tag{2.1}$$

A *screen distribution* $S(TM)$ on M is a non-degenerate vector bundle complementary to $TM^{\perp}$. A null hypersurface endowed with a specific screen distribution is denoted by the triple $(M,g,S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in the study of the geometry of lightlike hypersurfaces.

Theorem 2.1. [7] *Let $(M,g,S(TM))$ be a null hypersurface of $(\overline{M},\overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U}\subset M$, there exists a unique section N of tr(TM) on $\mathcal{U}$ satisfying*

$$\overline{g}(N,\xi)=1 \text{ and } \overline{g}(N,N)=\overline{g}(N,W)=0, \tag{2.2}$$

for all $W\in\Gamma(S(TM)_{|\mathcal{U}})$.

With this theorem we may write the following decomposition

$$T\overline{M}_{|M} = S(TM)\perp(TM^{\perp}\oplus tr(TM)) = TM\oplus tr(TM), \tag{2.3}$$

where $\perp$ denotes an orthogonal direct sum and $\oplus$ a direct sum. Throughout the paper, we denoted by $\Gamma(E)$ the $C^{\infty}(M)$-module of smooth sections of a vector bundle E over M, while $C^{\infty}(M)$ represents the algebra of a smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.

Let $(M, g, S(TM))$ be a null hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$, ∇ the induced connection on (M, g). Gauss and Weingarten formulas provide the following relations (see details in [7], section 4.2)

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.4}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$ and $V \in tr(TM)$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while h is a $\Gamma(tr(TM))$-valued symmetric $C^{\infty}(M)$-bilinear form on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. It is easy to see that ∇ is a torsion-free connection. Define a symmetric $C^{\infty}(M)$-bilinear form B and a 1-form τ on the coordinate neighborhood $\mathcal{U}\subset M$ by

$$B(X,Y) = \overline{g}(h(X,Y),\xi), \tag{2.6}$$

$$\tau(X) = \overline{g}(\nabla^t_X N, \xi) \tag{2.7}$$

for all $X, Y \in \Gamma(TM_{j\mathcal{U}})$. Then, on $\mathcal{U}$, equations (2.4) and (2.5) become,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.9}$$

respectively. It is important to stress the fact that the local second fundamental form B in Eq.(2.8) does not depend on the choice of the screen distribution and satisfies,

$$B(X,\xi) = 0, \tag{2.10}$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$. Let P be the projection morphism of TM to $S(TM)$ with respect to the decomposition (2.2). We obtain: for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$,

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X, PY), \tag{2.11}$$

$$\nabla_X U = -\overset{*}{A}_U X + \overset{*}{\nabla}{}^t{}_X U, \tag{2.12}$$

where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(TM^{\perp})$ respectively, $\overset{*}{h}$ is a $\Gamma(TM^{\perp})$-valued $C^{\infty}(M)$-bilinear form on $\Gamma(TM)\times\Gamma(S(TM))$, $\overset{*}{A}_U$ is a $\Gamma(S(TM))$-valued $C^{\infty}(M)$-linear operator on $\Gamma(S(TM))$. $\overset{*}{h}$ and $\overset{*}{A}_U$ are the second fundamental form and the shape operator of the screen distribution $S(TM)$ respectively. Define on $\mathcal{U}$ the following relations

$$C(X, PY) = \overline{g}(\overset{*}{h}(X, PY), N), \tag{2.13}$$

$$\epsilon(X) = \overline{g}(\overset{*}{\nabla}{}^t{}_X \, \xi, N). \tag{2.14}$$

One shows that

$$\epsilon(X) = -\tau(X).$$

Thus, locally (2.11) and (2.12) become

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X, PY)\xi, \tag{2.15}$$

$$\nabla_X \xi = -\overset{*}{A}_\xi X - \tau(X)\xi, \tag{2.16}$$

respectively. The linear connection $\overset{*}{\nabla}$ is a metric connection on $\Gamma(S(TM))$. But, in general, the induced connection ∇ on M is not compatible with the induced metric g. Indeed, we have:

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.17}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$, where

$$\eta(X) = \overline{g}(X, N), \tag{2.18}$$

for all $Y \in \Gamma(TM|_{\mathcal{U}})$. Finally, it is straightforward to verify that

$$B(X, Y) = g(\overset{*}{A}_\xi X, Y), \quad g(A_N Y, N) = 0, \tag{2.19}$$

$$C(X, PY) = g(A_N X, Y), \quad \overset{*}{A}_\xi \xi = 0, \tag{2.20}$$

for $X, Y \in \Gamma(TM|_{\mathcal{U}})$.

We denote the curvature tensor associated with $\overline{\nabla}$ and ∇ by $\overline{R}$ and R, respectively. Then for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$, we have ([7]) the following

$$\begin{aligned}\overline{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) \\ &- (\nabla_Y h)(X, Z),\end{aligned} \tag{2.21}$$

$$\begin{aligned}g\Big(R(X, Y)PZ, PW\Big) &= g\Big(\overset{*}{R}(X, Y)PZ, PW\Big) + C(X, PZ)B(Y, PW) \\ &\quad -C(Y, PZ)B(X, PW),\end{aligned} \tag{2.22}$$

$$\overline{g}\Big(\overline{R}(X, Y)\xi, N\Big) = C(Y, \overset{*}{A}_\xi X) - C(X, \overset{*}{A}_\xi Y) - 2d\tau(X, Y). \tag{2.23}$$

2.2. **Curvature condition of Semi-symmetric type.** Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold. We denote its curvature operator by $\overline{\mathcal{R}}(X,Y)$.

$$\overline{\mathcal{R}}(X,Y) = \overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$. Then the curvature tensor $\overline{R}$ and the Riemannian curvature tensor $\overline{\mathbf{R}}$ are defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z. \tag{2.24}$$

$$\overline{\mathbf{R}}(X,Y,Z,W) = \overline{g}(\overline{R}(X,Y)Z, W) \tag{2.25}$$

For any $(0,k)$-tensor field on $\overline{M}$, $k \geq 1$, we define a $(0,k+2)$-tensor field $\overline{\mathcal{R}} \cdot T = 0$ by

$$\begin{aligned}(\overline{\mathcal{R}} \cdot T)(X_1,\ldots,X_k,X,Y) \;\; = \;\; & -T(\overline{R}(X,Y)X_1, X_2,\ldots,X_k) \\ & -T(X_1,X_2,\ldots,\overline{R}(X,Y)X_k) \end{aligned} \tag{2.26}$$

for $X, Y, X_1, \ldots, X_k \in \Gamma(T\overline{M})$. Curvature conditions, involving the form $\overline{\mathcal{R}} \cdot T$, are called curvature conditions of semi-symmetric type [5]. A semi-Riemannian manifold $\overline{M}$ is said to be semi-symmetric if it satisfies the condition $\overline{\mathcal{R}} \cdot \overline{\mathbf{R}} = 0$. Thus, from properties of curvature tensor, we have

$$\begin{aligned}(\overline{\mathcal{R}} \cdot \; \overline{R})(U \;\;, V)W = & \overline{R}(X,Y)\overline{R}(U,V)W - \overline{R}(U,V)\overline{R}(X,Y)W \\ - \;\; & \overline{R}(\overline{R}(X,Y)U,V)W - \overline{R}(U,\overline{R}(X,Y)V)W \\ - \;\; & \overline{R}(U,V)\overline{R}(X,Y)W, \end{aligned} \tag{2.27}$$

for all $X, Y, U, V, W \in \Gamma(T\overline{M})$.

3. **Ricci Semi-symmetric null hypersurfaces in Lorentzian space forms**

Let M be a null hypersurface of a semi-Riemannian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k. We need the following proposition.

Proposition 3.1. [2] *Let $(\overline{M}(k), \overline{g})$ be a semi-Riemannian manifold of constant curvature k and M be a null hypersurface of $\overline{M}(k)$. Denote by R the curvature tensor of the induced connection ∇ on M by the Levi-civita connection $\overline{\nabla}$. For any $X, Y, Z \in \Gamma(TM)$, we have:*

(a) $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} - B(X,Z)A_NY + B(Y,Z)A_NX$;
(b) $(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X)$;
(c) $B(A_NY,X) - B(A_NX,Y) = 2d\tau(X,Y)$;
(d) $(\nabla_Y A_N)(X) - (\nabla_X A_N)(Y) + k\{\eta(X)Y - \eta(Y)X\} = \tau(Y)A_NX - \tau(X)A_NY$;
(e) $(\nabla_X \overset{*}{A}_\xi)(Y) - (\nabla_Y \overset{*}{A}_\xi)(X) = \tau(Y)\overset{*}{A}_\xi X - \tau(X)\overset{*}{A}_\xi Y - 2d\tau(X,Y)\xi$;
(f) $\nabla_X PZ = \nabla_X Z - X \cdot \eta(Z)\xi + \eta(Z)\overset{*}{A}_\xi X + \eta(Z)\tau(X)\xi$.

Now, we recall the definition of a screen conformal and screen quasi-conformal null hypersurface of a semi-Riemannian manifold $\overline{M}$ of a semi-Riemannian manifold $\overline{M}$.

Definition 3.2. *([1]). A null hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $\overline{M}$ is said to be locally screen (resp. globally) conformal if on any coordinate neighborhood $\mathcal{U}$ (resp. $\mathcal{U} = M$), the shape operators A_N and $\overset{*}{A}_\xi$ of M and its screen distribution $S(TM)$ are related by*

$$A_N = \varphi \overset{*}{A}_\xi, \tag{3.1}$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ (resp. $\mathcal{U} = M$).

We remark that $\mathcal{U}$ will be connected and maximal in the sense that there is no larger domain $\mathcal{U}' \supset \mathcal{U}$ on which Eq. (3.1) holds. It is easy to see that Eq. (3.1) is equivalent to

$$C(Y, PZ) = \varphi B(Y, Z), \tag{3.2}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$.

Definition 3.3. [11] *A null hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is locally screen quasi-conformal if the shape operators A_N and $\overset{*}{A}_\xi$ of M and $S(TM)$ satisfy*

$$A_N = \varphi \overset{*}{A}_\xi + \psi P, \tag{3.3}$$

in $\Gamma(TM)$, for some functions φ, ψ and P is the natural projection defined in section 2.

We note that there are many examples of screen conformal null hypersurfaces of semi-Riemannian manifolds see [1] and [11].

Next, we say that M is totally umbilical if there exists a smooth function ρ such that

$$B(X, Y) = \rho g(X, Y), \tag{3.4}$$

for all $X, Y \in \Gamma(TM)$, or equivalently,

$$\overset{*}{A}_\xi X = \rho P X, \tag{3.5}$$

for all $X \in \Gamma(TM)$.

M is said to be a totally geodesic null hypersurface if the second fundamental form $B = 0$ or equivalently $\overset{*}{A}_\xi = 0$.

For any null hypersurface M of an $(m+2)$-dimensional Lorentzian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k, it is known that, the induced Ricci tensor on M is symmetric. Since ξ is an eigenvector field of $\overset{*}{A}_\xi$ corresponding to the eigenvalue 0 and $\overset{*}{A}_\xi$ is $\Gamma(S(TM))$-valued real symmetric, $\overset{*}{A}_\xi$ has m orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of $\overset{*}{A}_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then, $\overset{*}{A}_\xi\ E_i = \lambda_i E_i,\quad 1 \leq i \leq m$. We call the eigenvalues λ_i the *screen principal curvatures* for all i.

We have the following Lemma

Lemma 3.4. *Let M be a screen quasi-conformal null hypersurface of a $(m+2)$ dimensional Lorentzian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k. Then, the Ricci tensor Ric of M is given by*

$$\begin{aligned} Ric(X,Y) = & - & mkg(X,Y) - \varphi g(\overset{*}{A}_\xi\ X, Y)\alpha + \varphi g(\overset{*}{A}_\xi\ X, \overset{*}{A}_\xi\ Y) \\ & - & m\psi g(\overset{*}{A}_\xi\ X, Y) + \psi g(X, \overset{*}{A}_\xi\ Y), \end{aligned} \tag{3.6}$$

where $\alpha = trace \overset{*}{A}_\xi$.

Proof. From (1) in proposition 3.1, we have

$$R(X,Y)Z = k\Big\{g(Y,Z)X - g(X,Z)Y\Big\} - B(X,Z)A_N Y + B(Y,Z)A_N X.$$

Then, we have by using equation (2.19) and (3.3):

$$\begin{aligned} R(X,Y)Z &= k\{g(Y,Z)X - g(X,Z)Y\} - \varphi g(\overset{*}{A}_\xi\ X, Z)\ \overset{*}{A}_\xi\ Y \\ &\quad + \varphi g(\overset{*}{A}_\xi\ Y, Z)\ \overset{*}{A}_\xi\ X - \psi g(\overset{*}{A}_\xi\ X, Z)PY \\ &\quad + \psi g(\overset{*}{A}_\xi\ Y, Z)PX. \end{aligned} \tag{3.7}$$

In particular, since $\overset{*}{A}_\xi\ \xi = 0$, $PE_i = E_i$ and $P\xi = 0$, we have

$$\begin{aligned} R(X,\xi)Y &= -kg(X,Y)\xi \\ R(X,E_i)Y &= k\Big\{g(E_i,Y)X - g(X,Y)E_i\Big\} - \varphi g(\overset{*}{A}_\xi\ X, Y)\ \overset{*}{A}_\xi\ E_i \\ &\quad + \varphi g(\overset{*}{A}_\xi\ E_i, Y)\ \overset{*}{A}_\xi\ X - \psi g(\overset{*}{A}_\xi\ X, Y)E_i + \psi g(\overset{*}{A}_\xi\ E_i, Y)PX. \end{aligned}$$

We have then

$$\overline{g}(R(X,\xi)Y, N) = -k\overline{g}(X,Y) = -kg(X,Y). \tag{3.8}$$

and

$$\begin{aligned} g(R(X,E_i)Y,E_i) &= k\{g(E_i,Y)g(X,E_i) - g(X,Y)g(E_i,E_i)\} \\ &\quad - \varphi g(\overset{*}{A}_\xi\ X, Y)g(\overset{*}{A}_\xi\ E_i, E_i) + \varphi g(\overset{*}{A}_\xi\ E_i, Y)g(\overset{*}{A}_\xi\ X, E_i) \\ &\quad - \psi g(\overset{*}{A}_\xi\ X, Y)g(E_i,E_i) + \psi g(\overset{*}{A}_\xi\ E_i, Y)g(PX, E_i) \\ &= k\Big\{g\big(g(X,E_i)E_i, Y\big) - g(X,Y)\Big\} - \psi g(\overset{*}{A}_\xi\ X, Y) \\ &\quad + \varphi g\Big(g(\overset{*}{A}_\xi\ X, E_i)E_i, \overset{*}{A}_\xi\ Y\Big) - \varphi g(\overset{*}{A}_\xi\ X, Y)g(\overset{*}{A}_\xi\ E_i, E_i) \\ &\quad + \psi g\Big(g(X,E_i)E_i, \overset{*}{A}_\xi\ Y\Big) \end{aligned} \tag{3.9}$$

The Ricci tensor of a null hypersurface is given by

$$Ric(X,Y) = \sum_{i=1}^{m} g(R(X,E_i)Y,E_i) + \overline{g}(R(X,\xi)Y,N) \tag{3.10}$$

Then, by using (3.8), (3.9) and (3.10), we get (3.6). □

Definition 3.5. [14] *Let M be a null hypersurface of an $(m+2)$-dimensional semi-Riemannian manifold $(\overline{M}(k),\overline{g})$. We say that M is Ricci semi-symmetric if the following condition is satisfied*

$$(\mathcal{R}(X,Y)\cdot Ric)(X_1,X_2) = 0, \tag{3.11}$$

for all $X,Y,X_1,X_2 \in \Gamma(TM)$, where Ric is the Ricci tensor of M.

Next, The following general result gives a necessary and sufficient condition for a quasi-conformal null hypersurface to be Ricci semi-symmetric.

Theorem 3.6. *Let M be a screen quasi-conformal null hypersurface of a $(m+2)$ dimensional Lorentzian manifold $(\overline{M}(k),\overline{g})$ of constant curvature k. Then, M is Ricci semi-symmetric if and only if for distinct i,j, the screen principal curvatures satisfy*

$$\psi(\lambda_i-\lambda_j)(mk-\varphi\lambda_i\lambda_j)-(k+\varphi\lambda_i\lambda_j)(\lambda_i-\lambda_j)(-\varphi\alpha+\varphi\lambda_i+\varphi\lambda_j-m\psi+\psi) = 0, \tag{3.12}$$

where $\alpha = trace \overset{*}{A}_\xi$.

Proof. Since $\{\xi,E_1,\dots,E_m\}$ a frame field of eigenvectors of $\overset{*}{A}_\xi$ such that $\{E_1,\dots,E_m\}$ is an orthonormal frame field of $S(TM)$. Then $\overset{*}{A}_\xi\, E_i = \lambda_i E_i$, $1 \le i \le m$. If i,j are distinct, we use (3.7) to get

$$R(E_i,E_j)E_j = (k+\varphi\lambda_i\lambda_j+\psi\lambda_j)E_i \quad \text{and} \quad R(E_j,E_i)E_i = (k+\varphi\lambda_i\lambda_j+\psi\lambda_i)E_j \tag{3.13}$$

By using (3.6) and (3.13), we obtain

$$\begin{aligned} Ric\Big(R(E_i,E_j)E_i,E_j\Big) =&\ Ric\Big(-R(E_j,E_i)E_i,E_j\Big) \\ =&(k+\varphi\lambda_i\lambda_j+\psi\lambda_i)(mk+\varphi\lambda_j\alpha-\varphi\lambda_j^2 \\ &+m\psi\lambda_j-\psi\lambda_j) \end{aligned} \tag{3.14}$$

$$\begin{aligned} Ric\Big(E_i,R(E_i,E_j)E_j\Big) =&(k+\varphi\lambda_i\lambda_j+\psi\lambda_j)(-mk-\varphi\lambda_i\alpha+\varphi\lambda_i^2 \\ &-m\psi\lambda_i+\psi\lambda_i) \end{aligned} \tag{3.15}$$

Then, by using (3.14) and (3.15), we

$$\begin{aligned}(\mathcal{R}(E_i,E_j)\cdot Ric)(E_i,E_j) &= -Ric\Big(R(E_i,E_j)E_i,E_j\Big)-Ric\Big(E_i,R(E_i,E_j)E_j\Big)\\ &= -(k+\varphi\lambda_i\lambda_j)(\lambda_i-\lambda_j)(-\varphi\alpha+\varphi\lambda_i+\varphi\lambda_j)\\ &\quad -m\psi+\psi+\psi(\lambda_i-\lambda_j)(mk-\varphi\lambda_i\lambda_j). \qquad (3.16)\end{aligned}$$

Thus, if M is Ricci semi-symmetric *i.e* $(\mathcal{R}(X,Y)\cdot Ric)=0$, for all X and Y, we have (3.12).

Conversely, suppose that this condition holds. It is sufficient to verify $(\mathcal{R}(E_i,E_j)\cdot Ric)=0$ for $i\neq j$. If i,j,r and s are all distinct, then

$$\begin{aligned}(\mathcal{R}(E_i,E_j)\cdot Ric)(E_r,E_s) &= (\mathcal{R}(E_i,E_j)\cdot Ric)(E_r,E_r)\\ &= (\mathcal{R}(E_i,E_j)\cdot Ric)(E_s,E_s)\\ &= 0.\end{aligned}$$

By assumption, $(\mathcal{R}(E_i,E_j)\cdot Ric)(E_i,E_j)=0$. Finally symmetry takes care of the rest. □

In the case when the screen is conformal, we have:

Corollary 3.7. *If M is a screen conformal null hypersurface of a $(m+2)$ dimensional Lorentzian manifold $(\overline{M}(k),\overline{g})$ of constant curvature k. Then, M is Ricci semi-symmetric if and only if for distinct i,j , the screen principal curvatures satisfy*

$$(k+\varphi\lambda_i\lambda_j)(\lambda_i-\lambda_j)(-\alpha+\lambda_i+\lambda_j)=0, \qquad (3.17)$$

Proof. since φ is a non-vanishing smooth function, we get (3.17) by taking $\psi=0$. □

Example 3.8. *Let $(\mathbb{R}^4_1,\overline{g})$ be a 4-dimensional semi-Euclidean space with Lorentzian signature. Consider a Monge hypersurface M of $\mathbb{R}^4_1$ given by*

$$t=\frac{1}{\sqrt{2}}\left(x+\sqrt{y^2+z^2}\right).$$

It is easy to check that M is a null hypersurface whose radical distribution $RadTM$ is spanned by

$$\xi=\partial_t+\frac{y}{\sqrt{2}\sqrt{y^2+z^2}}\partial_y+\frac{z}{\sqrt{2}\sqrt{y^2+z^2}}\partial_z+\frac{1}{\sqrt{2}}\partial_x.$$

It is readily checked that, one gets an orthonormal basis $\{E_1,E_2\}$ of $S(TM)$ given by

$$\begin{aligned}E_1 &= \frac{1}{\sqrt{y^2+z^2}}\left(-z\partial_y+y\partial_z\right);\\ E_2 &= \epsilon\frac{1}{\sqrt{2}\sqrt{y^2+z^2}}\left(\sqrt{y^2+z^2}\partial_x-y\partial_y-z\partial_z\right)\quad \epsilon=\pm.\end{aligned}$$

Then the null transversal vector bundle is given by

$$tr(TM) = Span\left\{N = -\frac{1}{2}\partial_t + \frac{y}{\sqrt{8}\sqrt{y^2+z^2}}\partial_y + \frac{z}{\sqrt{8}\sqrt{y^2+z^2}}\partial_z + \frac{1}{\sqrt{8}}\partial_x\right\}.$$

By direct computation, we obtain

$$\overline{\nabla}_{E_1}\xi = \nabla_{E_1}\xi = \frac{1}{\sqrt{2}\sqrt{y^2+z^2}}E_1 \;\; and \;\; \overline{\nabla}_{E_2}\xi = \nabla_{E_2}\xi = 0. \tag{3.18}$$

Thus, from the Weingarten formula (2.16), we have

$$\overset{*}{A}_\xi E_1 = -\frac{1}{\sqrt{2}\sqrt{y^2+z^2}}E_1, \;\; \overset{*}{A}_\xi E_2 = 0 \;\; and \;\; \tau = 0.$$

Then, M has two distinct screen principal curvatures $\lambda_1 = -\frac{1}{\sqrt{2}\sqrt{y^2+z^2}}$ and $\lambda_2 = 0$. On the other hand, we have

$$\overline{\nabla}_{E_1}N = \frac{1}{\sqrt{8}\sqrt{y^2+z^2}}E_1, \;\; \overline{\nabla}_{E_2}N = 0 \;\; and \;\; \overline{\nabla}_{\xi}N = 0. \tag{3.19}$$

Then, from the Weingarten formula (2.9), we have

$$A_N E_1 = -\frac{1}{\sqrt{8}\sqrt{y^2+z^2}}E_1 = \frac{1}{2}\overset{*}{A}_\xi E_1, \;\; A_N E_2 = 0 \;\; and \;\; A_N\xi = 0.$$

Next, any $X \in \Gamma(TM)$, is expressed by

$$X = \alpha E_1 + \beta E_2 + \gamma\xi,$$

where α, β, γ are smooth functions, and then

$$A_N X = \alpha A_N E_1 + \beta A_N E_2 + \gamma A_N \xi = \frac{1}{2}\overset{*}{A}_\xi X,$$

that is M is a screen conformal lightlike hypersurface of $\mathbb{R}^4_1$ with conformal factor $\varphi = \frac{1}{2}$. Thus, M is a screen conformal null hypersurface of $\mathbb{R}^4_1$. The two distinct screen principal curvatures satisfy Eq. (3.17), then M is Ricci semi-symmetric.

Example 3.9. *(***The null cone Λ^3_0 of $\mathbb{R}^4_1$***)*
Let $\mathbb{R}^4_1$ be the space $\mathbb{R}^4$ endowed with the semi-Euclidean metric

$$\overline{g}(u,v) = -xx' + yy' + zz' + tt',$$

where $u = (x,y,z,t)$ and $v = (x',y',z',t')$. The null cone Λ^3_0 is given by the equation $-x^2+y^2+z^2+t^2 = 0$ with $(x,y,z,t) \neq (0,0,0,0)$. It is known that Λ^3_0 is a lightlike hypersurface of $\mathbb{R}^4_1$ and the radical distribution is spanned by a global vector field

$$\xi = x\partial_x + y\partial_y + z\partial_z + t\partial_t \tag{3.20}$$

on Λ_0^3. It is easy to see that, one gets an orthonormal basis $\{E_1, E_2\}$ of $S(T\Lambda_0^3)$ given by

$$\begin{aligned} E_1 &= \left(\frac{t^2+y^2}{t^2}\right)^{\frac{1}{2}} \left(\partial_y - \frac{y}{t}\partial_t\right), \\ E_2 &= \left(\frac{t^2+y^2}{x^2}\right)^{\frac{1}{2}} \left(-\frac{yz}{t^2+y^2}\partial_y + \partial_z + -\frac{zt}{t^2+y^2}\partial_t\right). \end{aligned}$$

As ξ is a position vector field, we get for all $i = 1, 2$

$$\overline{\nabla}_{E_i}\xi = \nabla_{E_i}\xi = E_i.$$

Using (2.16), we have $\overset{*}{A}_\xi\ E_i + \tau(E_i)\xi + E_i = 0$. As $\overset{*}{A}_\xi$ is $\Gamma(S(TM))$-valued we obtain

$$\overset{*}{A}_\xi\ E_i = -E_i, \tag{3.21}$$

for all $i = 1, 2$ This proves that $\lambda_1 = \lambda_2 = -1$ and $\tau = 0$. The two distinct screen principal curvatures satisfy Eq. (3.17). Then, The null cone Λ_0^3 of $\mathbb{R}_1^4$ is Ricci semi-symmetric.

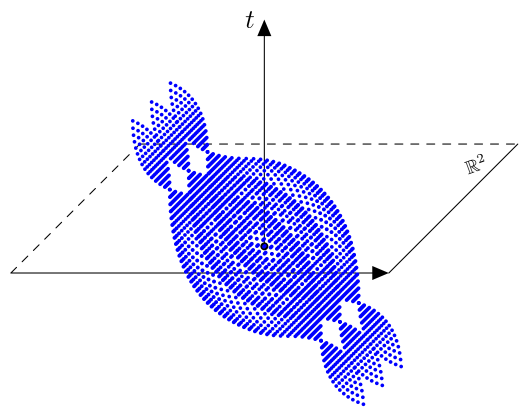

FIGURE 1. Projection of M in $\mathbb{R}^3$ for $x = -1$, $x = 0$ and $x = 1$

More generally, we have the following Proposition.

Proposition 3.10. a) *Every totally umbilical or totally geodesic quasi-conformal null Hypersurfaces of a $(m+2)$ dimensional Lorentzian space forms are Ricci semi-symmetric.*

b) *Every screen quasi-conformal null hypersurface of an $(n+2)$-dimensional Lorentzian space $\mathbb{R}_1^{n+2}$, such that at least one screen principal curvatures is zero, is Ricci semi-symmetric.*

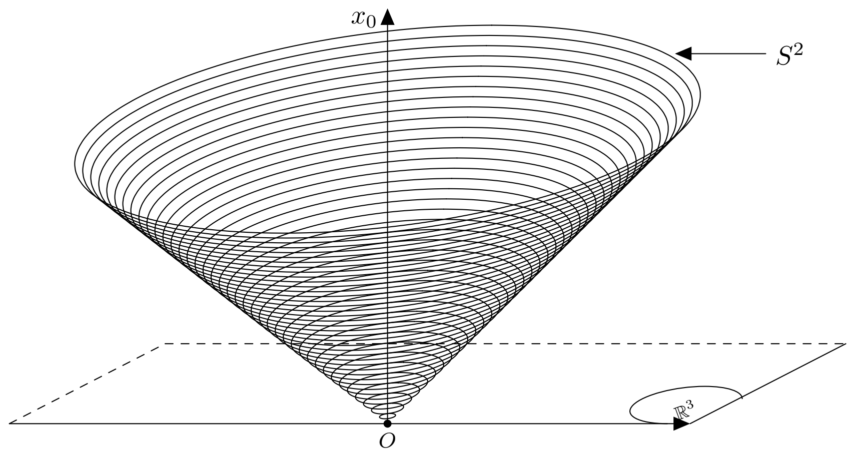

FIGURE 2. The lightcone Λ_0^3 of $\mathbb{R}_1^4$ is a stacking of spheres $S^2(x_0)$ of $\mathbb{R}^3$

Proof. a) is evident. b) By using $k = 0$ in Eq. (3.12) and assumption that there exists i_0 such that $\lambda_{i_0} = 0$, we get the result. □

This proposition shows the existence of a large class of Ricci semi-symmetric null hypersurface.

We have the following local classification theorem.

Theorem 3.11. *Let M be a screen conformal Ricci semi-symmetric null hypersurface of $(m+2)$-dimensional Lorentz manifold $(\overline{M}(k), \overline{g})$ of constant curvature k, then M is one of the following:*

1. *M is a locally null triple product manifolds locally a product of a null curve ($M = C \times M' = C \times M_\lambda \times M_\mu$), where C is a null curve, M_λ, 'M_μ are two totally umbilical spaces forms*
2. *M is totally geodesic,*
3. *locally $M = C \times L \times M_0$; where C is a null curve, L is a non-null curve and M_0 is an $(m-1)$-dimensional totally geodesic Euclidean space.*

Proof. Since for any screen conformal null hypersurface M of an $(m+2)$-dimensional Lorentzian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k, it is known that, the screen distribution $S(TM)$ is Riemannian, integrable and the induced Ricci tensor on M is symmetric [1].

Then, according to Proposition 3.4 in [7], there exists a canonical null pair $\{\xi, N\}$ satisfying (2.2) such that the corresponding 1-form τ from (2.9) vanishes. Since ξ is an eigenvector field of $\overset{*}{A}_\xi$ corresponding to the eigenvalue 0 and $\overset{*}{A}_\xi$ is $\Gamma(S(TM))$-valued real symmetric, $\overset{*}{A}_\xi$ has m orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors

$\{\xi, E_1, \ldots, E_m\}$ of $\overset{*}{A}_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then, $\overset{*}{A}_\xi E_i = \lambda_i E_i, \quad 1 \leq i \leq m$. In the following, we assume $\tau = 0$.

(1) First, suppose all screen principal curvatures are nonzero i.e rank$\overset{*}{A} = m$. We assert that it is impossible for three screen principal curvatures to be distinct. for this, Consider λ, γ and μ three distinct screen principal curvatures. Then we have the equations:

$$\begin{aligned}(k+\varphi\lambda\mu)(\lambda-\mu)(-\alpha+\lambda+\mu) &= 0\\ (k+\varphi\mu\gamma)(\mu-\gamma)(-\alpha+\mu+\gamma) &= 0\\ (k+\varphi\gamma\lambda)(\gamma-\lambda)(-\alpha+\gamma+\lambda) &= 0\end{aligned}$$

In order for these to be satisfied, two factors of the same type must vanish, for example $(k+\varphi\lambda\mu) = (k+\varphi\mu\gamma) = 0$ implies $\lambda = \gamma$ which gives a contradiction. Thus, there are at most 2 distinct screen principal curvatures, say λ and μ. By [12], the functions λ and μ have constant multiplicities and are differentiable along each leaf of $S(TM)$. Moreover, the distributions

$$T_\lambda = \{X \in \Gamma(S(TM)) | \overset{*}{A}_\xi X = \lambda X\}$$

and

$$T_\mu = \{X \in \Gamma(S(TM)) | \overset{*}{A}_\xi X = \mu X\}$$

are differentiable distributions.

Let $X, Y \in \Gamma(T_\lambda)$, we have $[X,Y] = [X,Y]_\lambda + [X,Y]_\mu$. Then

$$\begin{aligned}(\overset{*}{A}_\xi - \lambda I)[X,Y] &= (\overset{*}{A}_\xi - \lambda I[X,Y]_\lambda + (\overset{*}{A}_\xi - \lambda I[X,Y]_\mu\\ &= \overset{*}{A}_\xi [X,Y]_\lambda + \overset{*}{A}_\xi [X,Y]_\mu - \lambda[X,Y]_\lambda - \lambda[X,Y]_\mu\\ &= \lambda[X,Y]_\lambda + \mu[X,Y]_\mu - \lambda[X,Y]_\lambda - \lambda[X,Y]_\mu\\ &= (\mu-\lambda)[X,Y]_\mu\end{aligned}$$

Hence, for all $X, Y \in \Gamma(T_\lambda)$, $(\overset{*}{A}_\xi - \lambda I)[X,Y] \in \Gamma(T_\mu)$.

Since $\tau = 0$, from (e) in proposition3.1 we have

$$(\nabla_X \overset{*}{A}_\xi)(Y) - (\nabla_Y \overset{*}{A}_\xi)(X) = 0.$$

Then it follows that if $X, Y \in \Gamma(T_\lambda)$,

$$\overset{*}{A}_\xi ([X,Y]) = \overset{*}{A}_\xi (\nabla_X Y) - \overset{*}{A}_\xi (\nabla_Y X).$$

However,

$$\overset{*}{A}_\xi X = \lambda X, \quad \overset{*}{A}_\xi Y = \lambda Y$$

so that

$$\overset{*}{A}_\xi ([X,Y]) = (X \cdot \lambda)Y - (Y \cdot \lambda)X + \lambda[X,Y].$$

Thus

$$(\overset{*}{A}_\xi - \lambda I)[X,Y] = (X\cdot\lambda)Y - (Y\cdot\lambda)X.$$

The left side of the above equation lies in T_μ and the right side in T_λ, then

$$(\overset{*}{A}_\xi - \lambda)[X,Y] = 0, \qquad (X\cdot\lambda)Y - (Y\cdot\lambda)X = 0$$

implies that $[X,Y]\in T_\lambda$ which prove that T_λ is integrable. Also, since $(X\cdot\lambda)Y - (Y\cdot\lambda)X = 0$, if $\dim T_\lambda > 1$, we may choose X and Y to be linearly independent. Thus $(X\cdot\lambda) = 0$.

If we choose $X, Y \in \Gamma(T_\mu)$, by the same argument, we prove that T_μ is integrable and $(X\cdot\mu) = 0$. Hence, λ and μ are constant along the screen distribution.

By lemma 3.4 in [2], if $X\in\Gamma(T_\lambda)$, and $Y\in\Gamma(T_\mu)$, then $\nabla_X Y\in\Gamma(T_\mu)$ and $\nabla_Y X\in\Gamma(T_\lambda)$ which shows that T_λ and T_μ are parallel along their normals in $S(TM)$.

From ([1]) a conformal lightlike hypersurface M is locally a product manifold $C\times M'$, where C is a null curve and M' is a leaf of $S(TM)$. Since the leaf M' of $S(TM)$ is Riemannian and $S(TM) = T_\lambda \oplus_{orth} T_\mu$, where T_λ and T_μ are parallel distributions with respect to the induced connection $\overset{*}{\nabla}$ of M', by the decomposition theorem of de Rham ([6]) we have $M' = M_\lambda \times M_\mu$, where M_λ and M_μ are some leaves of T_λ and T_μ, respectively. It follows that $M = C\times M' = C\times M_\lambda\times M_\mu$. Let $X, Y, Z, W \in \Gamma(T_\lambda)$. Using (1) in Proposition (3.1) and equations (2.19) and (3.1) we have

$$\begin{aligned} g(R(X,Y)Z,W) &= k\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &- \varphi g(\overset{*}{A}_\xi X,Z)g(\overset{*}{A}_\xi Y,W) + \varphi g(\overset{*}{A}_\xi Y,Z)g(\overset{*}{A}_\xi X,W) \\ &= k\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad -\varphi\lambda^2 g(X,Z)g(Y,W) + \varphi\lambda^2 g(Y,Z)g(X,W) \\ &= (k+\varphi\lambda^2)g\Big(g(Y,Z)X - g(X,Z)Y, W\Big). \end{aligned} \tag{3.22}$$

Again, by using (3.1), (3.2), (2.19), (2.20), (2.22) and (3.22), we have

$$(k+\varphi\lambda^2)g\Big(g(Y,Z)X - g(X,Z)Y, W\Big) =$$

$$g(\overset{*}{R}\Big(X,Y)Z,W\Big) - \varphi\lambda^2 g\Big(g(Y,Z)X - g(X,Z)Y,W\Big).$$

Then,

$$\overset{*}{R}(X,Y)Z = (k+2\varphi\lambda^2)\{g(Y,Z)X - g(X,Z)Y\},$$

for all X, Y, Z in $\Gamma(T_\lambda)$. Thus M_λ is a Riemannian manifold of constant curvature $(k+2\varphi\lambda^2)$.

As M' is a Riemannian submanifold of codimension 2 of $\mathbb{R}_1^{m+2}$, consider in the normal bundle $TM'^{\perp}$, the vector fields

$$\zeta_1 = \frac{\varphi}{\sqrt{2|\varphi|}}\xi + \frac{1}{\sqrt{2|\varphi|}}N \quad \text{and} \quad \zeta_2 = \frac{\varphi}{\sqrt{2|\varphi|}}\xi - \frac{1}{\sqrt{2|\varphi|}}N.$$

Clearly, $\{\zeta_1, \zeta_2\}$ is an orthonormal basis , where ζ_1 and ζ_2 are spacelike and timelike respectively. Then for any $X,\ Y \in \Gamma(TM_\lambda)$, we have

$$\overline{\nabla}_X Y = \nabla^\lambda_X Y + \sum_{a=r+1}^{m+2} g_\lambda(A_{\xi^\lambda_a} X, Y)\xi^\lambda_a, \tag{3.23}$$

where g_λ, ∇^λ are the induced metric and the induced connection on M_λ respectively, ξ^λ_a are orthonormal normals to TM_λ in $\mathbb{R}_1^{m+2}$ such that $\xi^\lambda_{m+1} = \zeta_1$ and $\xi^\lambda_{m+2} = \zeta_2$, $A_{\xi^\lambda_a}$ are corresponding shape operators of ξ^λ_a. In addition,

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y + B(X,Y)N \\
&= \nabla_X Y + g(\overset{*}{A}_\xi X, Y)N \\
&= \overset{*}{\nabla}_X Y + C(X,Y)\xi + g(\overset{*}{A}_\xi X, Y)N \\
&= \overset{*}{\nabla}_X Y + g(A_N X, Y)\xi + g(\overset{*}{A}_\xi X, Y)N \\
&= \overset{*}{\nabla}_X Y + \varphi g(\overset{*}{A}_\xi X, Y)\xi + g(\overset{*}{A}_\xi X, Y)N \\
&= \nabla^\lambda_X Y + \sum_{a=r+1}^{m} g_\lambda(A'_{\xi^\lambda_a} X, Y)\xi^\lambda_a + g(\overset{*}{A}_\xi X, Y)(\varphi\xi + N) \\
&= \nabla^\lambda_X Y + \sum_{a=r+1}^{m} g_\lambda(A'_{\xi^\lambda_a} X, Y)\xi^\lambda_a + \lambda g(X,Y)(\varphi\xi + N),
\end{aligned} \tag{3.24}$$

where $A'_{\xi^\lambda_a}$ denotes the shape operator of M_λ with respect to ξ^λ_a in $S(TM)$. By Theorem **??**, M_λ is totally geodesic in $S(TM)$, and consequently the equation (3.24) can be written as follows:

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla^\lambda_X Y + \lambda g_\lambda(X,Y)(\varphi\xi + N) \\
&= \nabla^\lambda_X Y + \sqrt{2|\varphi|}\lambda g_\lambda(X,Y)\zeta_1.
\end{aligned} \tag{3.25}$$

Comparing (3.23) and (3.25), we have

$$A_{\xi^\lambda_a} X = 0, \quad \forall a \neq m+1$$

and

$$A_{\xi^\lambda_{m+1}} X = A_{\zeta_1} X = \sqrt{2|\varphi|}\lambda X.$$

Thus, M_λ is a totally umbilical submanifold of $M(k)$. Similarly, we can prove that that M_μ is a Riemannian manifold of constant curvature $(k + 2\varphi\mu^2)$ and is a totally geodesic submanifold in $M(k)$.

(2) Second, suppose that there exists a zero screen principal curvature i.e rank$\overset{*}{A}_\xi < m$. Then there exists i such that $\lambda_i = 0$. Thus, from (3.12) we have $\lambda_j = \alpha = tr\ \overset{*}{A}_\xi$ i.e $(q-1)\lambda_j = 0$, where q is the multiplicity of λ_j.

If $q \neq 1$, then $\lambda_j = 0$ for all j. Thus all screen eigenvalues are zero i.e. $\overset{*}{A}_\xi = 0$ which prove that M is totally geodesic.

(3) if $q = 1$, then the multiplicity of λ_j is one. So we have two distinct eigenvalues $\lambda = 0$ and $\mu \neq 0$. As in (1), we define two distributions

$$T_0 = \{X \in \Gamma(S(TM)) \ \ \overset{*}{A}_\xi\ X = 0\}$$

and

$$T_\mu = \{X \in \Gamma(S(TM)) \ \ \overset{*}{A}_\xi\ X = \mu X\}$$

and we prove that a leaf M_0 of T_0 is a totally geodesic $(m-1)$-dimensional Riemannian manifold of zero constant sectional curvature and the leaf L of T_μ is a curve. Thus M is a locally product $C \times L \times M_0$.

□

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