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Inheritance Kulkarni-Nomizu product in generalized $\mathfrak{B}K-$fifth recurrent Finsler space by Lie - derivative

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Abstract. This paper deals with the space known as "generalized fifth recurrent Finsler space." The core idea centers around a mathematical object called the" Inheritance Kulkarni-Nomizu product" which is applied to two Ricci tensors satisfy an inheritance property. We apply the inheritance property with Kulkarni-Nomizu product of two Ricci tensosrs by using Lie - derivative in generalized fifth recurrent Finsler space. In addition, we prove that the Lie - derivative of the inheritance Kulkarni-Nomizu product of $K-$Ricci tensor and $H-$Ricci tensor vanishes simultaneously.

Keywords: Lie - derivative L_v, Inheritance Kulkarni - Nomizu product, Inheritance Ricci tensor, Generalized $\mathfrak{B}K-$fifth recurrent Finsler space.

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1. **Introduction and Preliminaries**

An inheritance Kulkarni-Nomizu product considers a new concept in Finsler geometry. Various identities on curvature inheritance in Finsler space established by Gatoto [16]. New relationship on curvature inheritance and other tensors was investigated by Ali et al. [7]. The Kulkarni-Nomizu product of two (0,2) type tensors defined by Deszcz et al. [15]. Further, AL-Qashbari and Baleedi [12] studied $K-$curvature inheritance in fifth recurrent Finsler space. Opondo [25] studied $W-$curvature inheritance in bi-recurrent Finsler space.

In the same regards, the Lie - derivative of forms and its application was investigated by authors [22, 23, 26, 28]. Several results on generalized recurrent Finsler spaces of higher orders studied by [4, 8, 11, 13, 6, 10, 24]. The relations between Ricci tensors and associate curvature tensors for various curvature tensors discussed by [1, 2, 3, 5, 9, 14, 17, 18, 19, 20, 21, 27, 29].

Let us explore a generalized $\mathfrak{B}K-$fifth recurrent Finsler space satisfying the following relations [11]

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{jk} = a_{sqlnm} K_{jk}, \tag{1.1}$$

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_{jk} = a_{sqlnm} H_{jk}, \tag{1.2}$$

and

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh} = a_{sqlnm} R_{ijkh} \tag{1.3}$$

if and only if

$$b_{sqlnm} g_{jk} - c_{sqlnm} C_{jkn} - d_{sqlnm} C_{jkl} - e_{sqlnm} C_{jkq} - 2b_{qlnm} y^r \mathfrak{B}_r C_{jks} = 0, \tag{1.4}$$

$$\begin{aligned}
&\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m \left(P^t_{jkt} + P^r_{jk}P^t_{rt} - P^t_{jtk} - P^r_{jt}P^t_{rk}\right) \\
&+a_{sqlnm}\left(-P^t_{jkt} - P^r_{jk}P^t_{rt} + P^t_{jtk} + P^r_{jt}P^t_{rk}\right) + b_{sqlnm}(n-1)\, g_{jk} \\
&-2b_{qlnm} y^r \mathfrak{B}_r (n-1)\, C_{jks} - c_{sqlnm}(n-1)\, C_{jkn} \\
&-d_{sqlnm}(n-1)\, C_{jkl} - e_{sqlnm}(n-1)\, C_{jkq} = 0
\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}
&b_{sqlnm}\left(g_{hj}g_{ik} - g_{kj}g_{ih}\right) - 2b_{qlnm} y^r \mathfrak{B}_r \left(g_{hj}C_{iks} - g_{kj}C_{ihs}\right) \\
&-c_{sqlnm}\left(g_{hj}C_{ikn} - g_{kj}C_{ihn}\right) - d_{sqlnm}\left(g_{hj}C_{ikl} - g_{kj}C_{ihl}\right) \\
&-e_{sqlnm}\left(g_{hj}C_{ikq} - g_{kj}C_{ihq}\right) + \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (C_{ijt}H^t_{kh}) \\
&-a_{sqlnm}\left(C_{ijt}H^t_{kh}\right) = 0,
\end{aligned} \tag{1.6}$$

respectively.

The Kulkarni-Nomizu product $(A \wedge U)$ of two (0,2) - type symmetric tensors A and U is defined as

$$(A \wedge U)_{ijkh} = A_{ih}U_{jk} - A_{ik}U_{jh} + A_{jk}U_{ih} - A_{jh}U_{ik}\ . \tag{1.7}$$

See [30]. The associate curvature tensors K_{ijkh}, P_{ijkh} and W_{ijkh} satisfy the following relations [30]

$$K_{ijkh} = R_{ijkh} - \frac{1}{(n-2)}(A \wedge U)_{ijkh}. \tag{1.8}$$

$$P_{ijkh} = R_{ijkh} - \frac{1}{(n-1)}\left(A_{ih}U_{jk} - A_{jh}U_{ik}\right). \tag{1.9}$$

$$W_{ijkh} = R_{ijkh} - \frac{c}{2n(n-1)}(A \wedge A)_{ijkh}, \tag{1.10}$$

where c is constant.

The non-zero covariant tensor field of fifth order a_{sqlnm} vanishes simultaneously with the vanishing of the scalar function $\alpha(x)$ by Berwald's covariant derivative of the fifth order [12]

$$L_v a_{sqlnm} = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\alpha(x) \tag{1.11}$$

if and only if

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_v K^i_{jkh}) = L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh}). \tag{1.12}$$

The $H-$Ricci tensor and $K-$Ricci tensor have an inheritance property that characterized by

$$L_v H_{jk} = \alpha(x) H_{jk} \tag{1.13}$$

$$L_v K_{jk} = \alpha(x) K_{jk}. \tag{1.14}$$

See [12].

2. Lie - Derivative of the Inheritance Kulkarni-Nomizu Product of Two Ricci - Tensors in $G\mathfrak{B}K - 5RF_n$

Definition 2.1. *The Kulkarni-Nomizu product* $(A \wedge U)$ *of two (0,2)-type symmetric tensors A and U which is defined by (1.7), is called inheritance Kulkarni-Nomizu product if the tensors A and U are satisfying the inheritance property. We denoted to the Lie - derivative of the inheritance Kulkarni-Nomizu product by* $L_v Ih(A \wedge U)_{ijkh}$.

Using (1.1) and (1.2) in (1.7), we get

$$(K \wedge H)_{ijkh} = \frac{1}{(a_{sqlnm})^2}[(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{ih})(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_{jk}) \tag{2.1}$$
$$-(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{ik})(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_{jh}) + (\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{jk})$$
$$(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_{ih}) - (\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{jh})(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_{ik})].$$

Taking the Lie - derivative of both sides in (2.1) and using $K-$Ricci tensor and $H-$Ricci tensor that have inheritance property, we get

$$L_v Ih(K\wedge H)_{ijkh} = (a_{sqlnm})^2 L_v\left(\frac{1}{(a_{sqlnm})^2}\right)\Big[K_{ih}H_{jk} - K_{ik}H_{jh} \tag{2.2}$$
$$+K_{jk}H_{ih} - K_{jh}H_{ik}\Big] + \frac{1}{(a_{sqlnm})}\Big[H_{jk}K_{ih}L_v(a_{sqlnm}) + \alpha(x)a_{sqlnm}H_{jk}K_{ih}$$
$$+K_{ih}H_{jk}L_v(a_{sqlnm}) + \alpha(x)a_{sqlnm}K_{ih}H_{jk} - H_{jh}K_{ik}L_v(a_{sqlnm})$$
$$-\alpha(x)a_{sqlnm}H_{jh}K_{ik} - K_{ik}H_{jh}L_v(a_{sqlnm}) - \alpha(x)a_{sqlnm}K_{ik}H_{jh}$$
$$+H_{ih}K_{jk}L_v(a_{sqlnm}) + \alpha(x)a_{sqlnm}H_{ih}K_{jk} + K_{jk}H_{ih}L_v(a_{sqlnm})$$
$$+\alpha(x)a_{sqlnm}K_{jk}H_{ih} - H_{ik}K_{jh}L_v(a_{sqlnm}) - \alpha(x)a_{sqlnm}H_{ik}K_{jh}$$
$$-K_{jh}H_{ik}L_v(a_{sqlnm}) - \alpha(x)a_{sqlnm}K_{jh}H_{ik}\Big].$$

Using (1.11) in (2.2), we get

$$L_v Ih(K\wedge H)_{ijkh} = (a_{sqlnm})^2 L_v\left(\frac{1}{(a_{sqlnm})^2}\right)\Big[K_{ih}H_{jk} - K_{ik}H_{jh}$$
$$+K_{jk}H_{ih} - K_{jh}H_{ik}\Big] + \frac{1}{(a_{sqlnm})}\Big[2\alpha(x)a_{sqlnm}K_{ih}H_{jk}$$
$$-2\alpha(x)a_{sqlnm}K_{ik}H_{jh} + 2\alpha(x)a_{sqlnm}K_{jk}H_{ih} - 2\alpha(x)a_{sqlnm}K_{jh}H_{ik}\Big].$$

Above equation can be written as

$$L_v Ih(K\wedge H)_{ijkh} = \left[(a_{sqlnm})^2 L_v\left(\frac{1}{(a_{sqlnm})^2}\right) + 2\alpha(x)\right]\Big[K_{ih}H_{jk}$$
$$-K_{ik}H_{jh} + K_{jk}H_{ih} - K_{jh}H_{ik}\Big]. \tag{2.3}$$

Thus, we conclude

Theorem 2.2. *In $G\mathfrak{B}K-5RF_n$, Lie - derivative of inheritance Kulkarni-Nomizu product of $K-$Ricci tensor and $H-$Ricci tensor is giving by* (2.3), *provided* (1.4), (1.5) *and* (1.12) *hold.*

Using (1.1) in (1.7), we get

$$(K\wedge K)_{ijkh} = \frac{2}{(a_{sqlnm})^2}[(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{ih})(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{jk})$$
$$-(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{ik})(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{jh})]. \tag{2.4}$$

Taking the Lie - derivative of both sides of (2.4) and using K-Ricci tensor that has inheritance property, we get

$$L_v Ih(K\wedge K)_{ijkh} = (a_{sqlnm})^2 L_v\left(\frac{2}{(a_{sqlnm})^2}\right)\Big[K_{ih}K_{jk} - K_{ik}K_{jh}\Big] \quad (2.5)$$
$$+\frac{2}{(a_{sqlnm})}\Big[K_{jk}K_{ih}L_v(a_{sqlnm}) + \alpha(x)a_{sqlnm}K_{jk}K_{ih} + K_{ih}K_{jk}L_v(a_{sqlnm})$$
$$+\alpha(x)a_{sqlnm}K_{ih}K_{jk} - K_{jh}K_{ik}L_v(a_{sqlnm}) - \alpha(x)a_{sqlnm}K_{jh}K_{ik}$$
$$-K_{ik}K_{jh}L_v(a_{sqlnm}) - \alpha(x)a_{sqlnm}K_{ik}K_{jh}\Big].$$

Using (1.11) in (2.5), we get

$$L_v Ih(K\wedge K)_{ijkh} = (a_{sqlnm})^2 L_v\left(\frac{2}{(a_{sqlnm})^2}\right)\Big[K_{ih}H_{jk} - K_{ik}H_{jh}\Big]$$
$$+\frac{2}{(a_{sqlnm})}\Big[2\alpha(x)a_{sqlnm}K_{ih}K_{jk} - 2\alpha(x)a_{sqlnm}K_{ik}K_{jh}\Big].$$

Above equation can be written as

$$L_v Ih(K\wedge K)_{ijkh} = \left[(a_{sqlnm})^2 L_v\left(\frac{2}{(a_{sqlnm})^2}\right) + 4\alpha(x)\right]\Big[K_{ih}K_{jk} - K_{ik}K_{jh}\Big] \quad (2.6)$$

Thus, we conclude

Theorem 2.3. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of inheritance Kulkarni-Nomizu product of $K-$Ricci tensor with itself is giving by (2.6), provided (1.4) and (1.12) hold.*

Taking the Lie - derivative of both sides of (1.8), using the inheritance Kulkarni-Nomizu product of $K-$Ricci tensor and $H-$Ricci tensor, we get

$$L_v K_{ijkh} = L_v R_{ijkh} - \frac{1}{(n-2)} L_v Ih(K\wedge H)_{ijkh}. \quad (2.7)$$

Using (2.3) in (2.7), we get

$$L_v K_{ijkh} = L_v R_{ijkh} - \frac{1}{(n-2)}\left[(a_{sqlnm})^2 L_v\left(\frac{1}{(a_{sqlnm})^2}\right) + 2\alpha(x)\right] \quad (2.8)$$
$$\Big[K_{ih}H_{jk} - K_{ik}H_{jh} + K_{jk}H_{ih} - K_{jh}H_{ik}\Big].$$

Thus, we conclude

Corollary 2.4. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of the associate curvature tensor K_{ijkh} of the curvature tensor K^i_{jkh} is giving by (2.8) if $H-$Ricci tensor and $K-$Ricci tensor have an inheritance property, provided (1.4), (1.5) and (1.12) hold.*

Transvecting (2.7) by a_{sqlnm}, we get

$$a_{sqlnm}(L_v K_{ijkh}) = a_{sqlnm}(L_v R_{ijkh}) - \frac{a_{sqlnm}}{(n-2)} L_v Ih(K \wedge H)_{ijkh}.$$

Taking the Lie - derivative of both sides of (1.3) and using the result in above equation, we get

$$\begin{aligned} a_{sqlnm}(L_v K_{ijkh}) &= L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh}) - (L_v a_{sqlnm}) R_{ijkh} \\ &\quad - \frac{a_{sqlnm}}{(n-2)} L_v Ih(K \wedge H)_{ijkh}. \end{aligned}$$

Above equation can be written as

$$L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh}) = a_{sqlnm}(L_v K_{ijkh}) \tag{2.9}$$

if and only if

$$L_v Ih(K \wedge H)_{ijkh} = \frac{2-n}{a_{sqlnm}} (L_v a_{sqlnm}) R_{ijkh}. \tag{2.10}$$

Thus, we conclude

Theorem 2.5. *In $G\mathfrak{B}K-5RF_n$, Lie- derivatives of associate curvature tensor K_{ijkh} and Berwald's covariant derivative of the fifth order for associate curvature tensor R_{ijkh} are codirectional if and only if the Lie- derivative of inheritance Kulkarni-Nomizu product of $K-$Ricci tensor and $H-$Ricci tensor is giving by* (2.10)*, provided* (1.6) *holds.*

Taking the Lie - derivative of both sides of (1.9) and using $K-$Ricci tensor and $H-$Ricci tensor that have inheritance property, we get

$$L_v P_{ijkh} = L_v R_{ijkh} - \frac{1}{(n-1)} L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}). \tag{2.11}$$

Using (2.3) in (2.11), we get

$$\begin{aligned} L_v P_{ijkh} = L_v R_{ijkh} - \frac{1}{(n-1)} L_v Ih \Bigg[& \frac{1}{(a_{sqlnm})^2 L_v\left(\frac{1}{(a_{sqlnm})^2}\right) + 2\alpha(x)} \\ & [L_v Ih(K \wedge H)_{ijkh} + \left[(a_{sqlnm})^2 L_v \left(\frac{1}{(a_{sqlnm})^2}\right)\right. \\ & \left. + 2\alpha(x)\right] (K_{ik}H_{jh} - K_{jk}H_{ih})] \Bigg]. \end{aligned} \tag{2.12}$$

Thus, we conclude the following.

Corollary 2.6. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of the associate curvature tensor P_{ijkh} of the curvature tensor P^i_{jkh} is giving by* (2.12) *if $H-$Ricci tensor and $K-$Ricci tensor have an inheritance property, provided* (1.4)*,* (1.5) *and* (1.12) *hold.*

Transvecting (2.11) by a_{sqlnm}, we get

$$a_{sqlnm}(L_v P_{ijkh}) = a_{sqlnm}(L_v R_{ijkh}) - \frac{a_{sqlnm}}{(n-1)} L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}).$$

Taking the Lie - derivative of both sides of (1.3) and using the result in above equation, we get

$$\begin{aligned} a_{sqlnm}(L_v P_{ijkh}) &= L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh}) - (L_v a_{sqlnm})R_{ijkh} \\ &\quad - \frac{a_{sqlnm}}{(n-1)} L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}). \end{aligned}$$

Above equation can be written as

$$L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh}) = a_{sqlnm}(L_v P_{ijkh}) \tag{2.13}$$

if and only if

$$L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}) = \frac{1-n}{a_{sqlnm}}(L_v a_{sqlnm})R_{ijkh}. \tag{2.14}$$

Thus, we conclude

Theorem 2.7. *In $G\mathfrak{B}K-5RF_n$, Lie- derivatives of associate curvature tensor P_{ijkh} and Berwald's covariant derivative of the fifth order for associate curvature tensor R_{ijkh} are codirectional if and only if the Lie - derivative of inheritance tensor $(K_{ih}H_{jk} - K_{jh}H_{ik})$ is giving by (2.14), provided (1.6) holds.*

Taking the Lie - derivative of both sides of (1.10) and using the inheritance Kulkarni-Nomizu product of $K-$Ricci tensor with itself, we get

$$L_v W_{ijkh} = L_v R_{ijkh} - \frac{c}{2n(n-1)} L_v Ih(K \wedge K)_{ijkh}. \tag{2.15}$$

Using (2.6) in (2.15), we get

$$\begin{aligned} L_v W_{ijkh} &= L_v R_{ijkh} - \frac{c}{2n(n-1)} \Big[(a_{sqlnm})^2 L_v\left(\frac{2}{(a_{sqlnm})^2}\right) \\ &\quad +4\alpha(x)\Big]\Big[K_{ih}K_{jk} - K_{ik}K_{jh}\Big]. \end{aligned} \tag{2.16}$$

Thus, we conclude

Corollary 2.8. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of the associate curvature tensor W_{ijkh} of the curvature tensor W^i_{jkh} is giving by (2.16) if $K-$Ricci tensor has an inheritance property, provided (1.4) and (1.12) hold.*

Transvecting (2.15) by a_{sqlnm}, we get

$$a_{sqlnm}(L_v W_{ijkh}) = a_{sqlnm}(L_v R_{ijkh}) - \frac{c\ a_{sqlnm}}{2n(n-1)} L_v Ih(K \wedge K)_{ijkh}.$$

Taking the Lie - derivative of both sides of (1.3) and using the result in above equation, we get

$$a_{sqlnm}(L_v W_{ijkh}) = L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh}) - (L_v a_{sqlnm})R_{ijkh} - \frac{c\ a_{sqlnm}}{2n(n-1)} L_v Ih(K \wedge K)_{ijkh}.$$

Above equation can be written as

$$L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh}) = a_{sqlnm}(L_v W_{ijkh}) \tag{2.17}$$

if and only if

$$L_v Ih(K \wedge K)_{ijkh} = \frac{2n(1-n)}{c\ a_{sqlnm}}(L_v a_{sqlnm})R_{ijkh}. \tag{2.18}$$

Thus, we conclude

Theorem 2.9. *In $G\mathfrak{B}K-5RF_n$, Lie- derivatives of associate curvature tensor W_{ijkh} and Berwald's covariant derivative of the fifth order for associate curvature tensor R_{ijkh} are codirectional if and only if the Lie - derivative of the inheritance Kulkarni-Nomizu product of $K-$Ricci tensor with itself is giving by* (2.18)*, provided* (1.6) *holds.*

From (2.7), we get

$$L_v K_{ijkh} = L_v R_{ijkh}$$

if and only if

$$L_v Ih(K \wedge H)_{ijkh} = 0.$$

Thus, we conclude

Corollary 2.10. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of the associate curvature tensor K_{ijkh} and associate curvature tensor R_{ijkh} are equal if and only if the Lie - derivative of the inheritance inheritance Kulkarni-Nomizu product of $K-$Ricci tensor and $H-$Ricci tensor is equal zero.*

From (2.15), we get

$$L_v W_{ijkh} = L_v R_{ijkh}$$

if and only if

$$L_v Ih(K \wedge K)_{ijkh} = 0.$$

Thus, we conclude

Corollary 2.11. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of the associate curvature tensor W_{ijkh} and associate curvature tensor R_{ijkh} are equal if and only if the Lie - derivative of the inheritance Kulkarni-Nomizu product of $K-$Ricci tensor with itself is equal zero.*

From (2.11), we get

$$L_v P_{ijkh} = L_v R_{ijkh}$$

if and only if

$$L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}) = 0.$$

Thus, we conclude

Corollary 2.12. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of the associate curvature tensor P_{ijkh} and associate curvature tensor R_{ijkh} are equal if and only if the Lie - derivative of the inheritance tensor $(K_{ih}H_{jk} - K_{jh}H_{ik})$ is equal zero.*

In view of (2.3) and (2.6), we get

$$L_v Ih(K \wedge H)_{ijkh} = L_v Ih(K \wedge K)_{ijkh} = 0 \tag{2.19}$$

if and only if

$$(a_{sqlnm})^2 L_v \left(\frac{1}{(a_{sqlnm})^2} \right) = -2\alpha(x). \tag{2.20}$$

Thus, we conclude

Corollary 2.13. *In $G\mathfrak{B}K-5RF_n$, the Lie - derivative of Inheritance Kulkarni-Nomizu product of $K-$Ricci tensor and $H-$Ricci tensor vanishes simultaneously with the vanishing of the Lie - derivative of inheritance Kulkarni-Nomizu product for $K-$Ricci tensor with itself if and only if (2.20) holds.*

In view of (2.7) with (2.11), (2.11) with (2.15) and (2.15) with (2.7), respectively, we get

$$L_v K_{ijkh} = L_v P_{ijkh}, \tag{2.21}$$

$$L_v P_{ijkh} = L_v W_{ijkh} \tag{2.22}$$

and

$$L_v K_{ijkh} = L_v W_{ijkh} \tag{2.23}$$

if and only if

$$L_v Ih(K \wedge H)_{ijkh} = \frac{n-2}{n-1} L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}), \tag{2.24}$$

$$L_v Ih(K \wedge K)_{ijkh} = \frac{2n}{c} L_v Ih(K_{ih}H_{jk} - K_{jh}H_{ik}) \tag{2.25}$$

and

$$L_v Ih(K \wedge H)_{ijkh} = \frac{c\ (n-2)}{2n(n-1)} L_v Ih(K \wedge K)_{ijkh}, \tag{2.26}$$

respectively. Thus, we conclude

Corollary 2.14. *In $G\mathfrak{B}K - 5RF_n$, the Lie - derivatives of the associate curvature tensor K_{ijkh}, associate curvature tensor P_{ijkh} and associate curvature tensor W_{ijkh} are equivalent if and only if inheritance $K-$Ricci tensor and inheritance $H-$Ricci tensor satisfying* (2.24), (2.25) *and* (2.26) *respectively.*

3. Conclusions

We established new identities by using the Lie - derivative of Inheritance Kulkarni-Nomizu product which applied two Ricci tensors. Specifically, we demonstrated that under important conditions, we obtained equivalence between three associate curvature tensors when the inheritance $K-$Ricci tensor and inheritance $H-$Ricci tensor satisfying certain relations in $G\mathfrak{B}K - 5RF_n$.

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