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On statistical generalized recurrent manifolds

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Abstract. In this paper, we introduce a statistical generalized recurrent manifold, which its statistical curvature tensor $\mathcal{R}^*$, satisfies the generalized recurrent condition $\nabla^*\mathcal{R}^* = \gamma\mathcal{R}^* + \theta H$. Next we prove that a statistical generalized recurrent manifold with constant statistical curvature is as same as a generalized recurrent manifold with respect to its Levi-Civita connection. Also we show that a statistical generalized recurrent manifold is neither statistical semi-symmetric, nor statistical Ricci semi-symmetric. Finally we prove that in spite of the Riemannian manifold, a statistical generalized recurrent manifold is not statistical concircular recurrent.

Keywords: Statistical manifold, statistical generalized recurrent manifold, statistical generalized concirculary recurrent, statistical semi-symmetric.

1. **Introduction**

A statistical manifold is a Riemannian (semi-Riemannian) manifold (U^n, h) which admits dual connections ∇ and ∇^* with some conditions, such that each points of that are probability distribution [1].

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U is called a Riemannian generalized recurrent manifold with respect to Levi-Civita connection $\hat{\nabla}$, if the Riemannian curvature tensor $\hat{\mathcal{R}m}$ satisfies

$$\begin{aligned}(\hat{\nabla}_E \hat{\mathcal{R}m})(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \hat{\gamma}(E)\hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\ &+ \hat{\theta}(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S}.\mathcal{I})h(\mathcal{B},\mathcal{F})], \quad (1.1)\end{aligned}$$

where $E,\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}$ are vector fields on U, and $\hat{\gamma}$ and $\hat{\theta}$ are nowhere vanishing unique 1-forms, such that there exist vector fields ρ and $\tilde{\rho}$, we have $\hat{\gamma}(E) = h(E,\rho)$ and $\hat{\theta}(E) = h(E,\tilde{\rho})$ for any $E \in \tau(U)$. For a Riemannian generalized recurrent manifold Equation (1.1) can be written as

$$\begin{aligned}(\hat{\nabla}_E \hat{\mathcal{R}})(\mathcal{S},\mathcal{B},\mathcal{I}) &= \hat{\gamma}(E)\hat{\mathcal{R}}(\mathcal{S},\mathcal{B})\mathcal{I} \\ &+ \hat{\theta}(E)[h(\mathcal{B},\mathcal{I})\mathcal{S} - h(\mathcal{S},\mathcal{I})\mathcal{B}], \quad (1.2)\end{aligned}$$

where $\hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = h(\hat{\mathcal{R}}(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F})$.

This notion was introduced by Dubey in 1979 at first and then many authors used this definition in their articles [3, 8, 10]. If $\hat{\theta}(E) = 0$ holds for all vector fields in generalized recurrent manifold, then U is reduced to be a recurrent manifold [5, 9].

In this paper, we introduce statistical generalized recurrent and statistical concirculary recurrent manifolds for (U^n, h, ∇^*) and we prove that in spite of the Riemannian case, they are not equivalent.

This paper is organized as follows. In Section 2 we review basic properties of statistical manifolds. In Section 3, we define statistical generalized recurrent, statistical concirculary recurrent, statistical semi-symmetric and statistical Ricci semi-symmetric manifolds. In section 4, we express the condition that 1-forms γ and θ can be closed. Also we prove that a statistical generalized recurrent manifold is neither statistical semi-symmetric, nor statistical Ricci semi-symmetric. Finally we show that a statistical generalized recurrent manifold is not statistical concircular recurrent.

2. **Preliminaries**

Throughout this paper, (U^n, h) denotes a smooth semi-Riemannian n dimensional manifold. We show the set of vector fields on U by $\tau(U)$.

Definition 2.1. [4] *(∇, h) is called a statistical structure on (U, h) if ∇ is an affine and torsion free connection and*

$$(\nabla_E h)(\mathcal{S},\mathcal{B}) = (\nabla_{\mathcal{S}} h)(E,\mathcal{B}), \quad (2.1)$$

holds $\forall\ \mathcal{S},\mathcal{B},E, \in \tau(U)$.

Also, (U,∇,h) is said to be a statistical manifold.

Moreover, an affine connection ∇^* is called a dual connection of ∇ with respect to h, such that

$$Eh\left(\mathcal{S},\mathcal{B}\right)=h\left(\nabla_E\mathcal{S},\mathcal{B}\right)+h\left(\mathcal{S},\nabla^*_E\mathcal{B}\right). \tag{2.2}$$

From the symmetry of h it can be verified that $(\nabla^*)^*=\nabla$ and by compatibility of $\hat{\nabla}$ with h and (2.1) it holds $\hat{\nabla}=\frac{1}{2}\left(\nabla+\nabla^*\right)$.
Also, $\nabla=\nabla^*$ if and only if ∇ is the Levi-Civita connection of the metric h.

Remark 2.2. [6] *A $(1,2)$-tensor field for a statistical structure (∇,h), is defined*

$$\mathcal{K}(E,\mathcal{S})=\nabla_E\mathcal{S}-\hat{\nabla}_E\mathcal{S}=\frac{1}{2}(\nabla_E\mathcal{S}-\nabla^*_E\mathcal{S}), \tag{2.3}$$

which $\mathcal{K}$ is symmetric and

$$h\left(\mathcal{K}(E,\mathcal{S}),\mathcal{B}\right)=h\left(\mathcal{S},\mathcal{K}(E,\mathcal{B})\right). \tag{2.4}$$

The statistical curvature tensor field with respect to ∇^* is defined in [11] as,

$$\mathcal{R}^*\left(\mathcal{S},\mathcal{B}\right)\mathcal{I}={\nabla^*}_{\mathcal{S}}{\nabla^*}_{\mathcal{B}}\mathcal{I}-{\nabla^*}_{\mathcal{B}}{\nabla^*}_{\mathcal{S}}\mathcal{I}-{\nabla^*}_{[\mathcal{S},\mathcal{B}]}\mathcal{I}, \tag{2.5}$$

and we denote

$$\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})=h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F}), \tag{2.6}$$

in which

$$h\left(\mathcal{R}^*\left(\mathcal{S},\mathcal{B}\right)\mathcal{I},\mathcal{F}\right)=-h\left(\mathcal{R}\left(\mathcal{S},\mathcal{B}\right)\mathcal{F},\mathcal{I}\right), \tag{2.7}$$

holds on statistical manifolds. The statistical curvature tensor field with respect to ∇ is defined similarly and is denoted by $\mathcal{R}$.

If there exist a real constant number a where $\mathcal{R}^*$ satisfies

$$\mathcal{R}^*\left(\mathcal{S},\mathcal{B}\right)\mathcal{I}=a\left\{h\left(\mathcal{B},\mathcal{I}\right)\mathcal{S}-h\left(\mathcal{S},\mathcal{I}\right)\mathcal{B}\right\}, \tag{2.8}$$

then (U^n,h) is said to be constant statistical curvature [2].

If U be a statistical manifold with constant statistical curvature a, then by virtue of (2.7),

$$\mathcal{R}^*=\mathcal{R}, \tag{2.9}$$

and

$$\mathcal{R}c^*(\mathcal{B},\mathcal{I})=a(n-1)h(\mathcal{B},\mathcal{I}), \tag{2.10}$$

holds for all vector fields $\mathcal{B},\mathcal{I}$, where $\mathcal{R}c^*(\mathcal{B},\mathcal{I})$ is the trace of the $\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})$ with respect to $\mathcal{S},\mathcal{F}$ and is called statistical Ricci tensor.

Remark 2.3. [7] *The condition $\mathcal{R}c=\mathcal{R}c^*$ in a statistical manifold implies*

$$\mathcal{R}c^*(\mathcal{B},\mathcal{I})=\mathcal{R}c^*(\mathcal{I},\mathcal{B}).$$

But it is not symmetric in general.

From Equation (2.9), we get $\mathcal{R}c = \mathcal{R}c^*$ for the statistical manifold with constant statistical curvature. So from Remark 2.3 $\mathcal{R}c^*$ is symmetric for the statistical manifold with constant statistical curvature.

Also from Equation (2.10), we have

$$tr_h \mathcal{R}c^* = an(n-1), \tag{2.11}$$

where $tr_h(\mathcal{R}c^*)$ is trace of the $\mathcal{R}c^*$ and is called statistical scalar curvature.

3. Statistical generalized recurrent manifold

Now we define a statistical generalized recurrent manifold.

Definition 3.1. *We say (U,h) is a statistical generalized recurrent manifold, if its statistical curvature tensor $\mathcal{R}m^*$ satisfies*

$$\begin{aligned}(\nabla^*_E \mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \gamma(E)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\ &+ \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]. \end{aligned} \tag{3.1}$$

$\forall\ \mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F},E \in \tau(U)$, where γ and θ are nowhere vanishing unique 1-forms, such that there exist vector fields ρ and $\tilde{\rho}$, in which $\gamma(E) = h(E,\rho)$ and $\theta(E) = h(E,\tilde{\rho})$, for any $E \in \tau(U)$.

If $\theta(E) = 0$ holds for all vector fields in statistical generalized recurrent manifold, then we say U is a statistical recurrent manifold. Also if $\gamma(E) = \theta(E) = 0$ holds for all vector fields in statistical generalized recurrent manifold, then we say U is a statistical locally-symmetric.

Equations (1.1) and (1.2) are equivalent for the Riemannian generalized recurrent manifold. In spite of the Riemannian generalized recurrent manifold, Equation (3.1) for the statistical generalized recurrent manifold is not equivalent to

$$\begin{aligned}(\nabla^*_E \mathcal{R}^*)(\mathcal{S},\mathcal{B},\mathcal{I}) &= \gamma(E)\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I} \\ &+ \theta(E)[h(\mathcal{B},\mathcal{I})\mathcal{S} - h(\mathcal{S},\mathcal{I})\mathcal{B}]. \end{aligned} \tag{3.2}$$

So we can state the following Remark.

Remark 3.2. *Let (U,h) be a statistical manifold in which*

$$\begin{aligned}(\nabla^*_E \mathcal{R}^*)(\mathcal{S},\mathcal{B},\mathcal{I}) &= \gamma(E)\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I} \\ &+ \theta(E)[h(\mathcal{B},\mathcal{I})\mathcal{S} - h(\mathcal{S},\mathcal{I})\mathcal{B}], \end{aligned} \tag{3.3}$$

then we have

$$(\nabla^*_E \mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = \gamma(E)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})$$

$$+ \quad \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]$$

$$+ \quad 2\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{K}(E,\mathcal{F})), \tag{3.4}$$

where $\mathcal{K}$ is the statistical $(1,2)$-tensor field in (2.3). Also if $\mathcal{K}(E,\mathcal{F}) = 0$ in Equation (3.4), the manifold reduce to be a Riemannian generalized recurrent.

Proof. Let Equation (3.3) holds for a statistical manifold. Since

$$\begin{aligned}(\nabla^*_E\mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \nabla^*_E h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F}) - h(\mathcal{R}^*(\nabla^*_E\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F})\\ &- h(\mathcal{R}^*(\mathcal{S},\nabla^*_E\mathcal{B})\mathcal{I},\mathcal{F}) - h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\nabla^*_E\mathcal{I},\mathcal{F})\\ &- h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\nabla^*_E\mathcal{F}),\end{aligned} \tag{3.5}$$

in which

$$\begin{aligned}\nabla^*_E h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F}) &= h(\nabla^*_E\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F})\\ &+ h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\nabla_E\mathcal{F}).\end{aligned} \tag{3.6}$$

By replacing (3.6) in (3.5) we infer

$$\begin{aligned}(\nabla^*_E\mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= h((\nabla^*_E\mathcal{R}^*)(\mathcal{S},\mathcal{B},\mathcal{I}),\mathcal{F})\\ &+ 2h(\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},,\mathcal{K}(E,\mathcal{F})).\end{aligned} \tag{3.7}$$

By replacing (3.3) in (3.7) we get (3.4).

Also if $\mathcal{K}(E,\mathcal{F}) = 0$ in Equation (3.4), then from (2.3) we have $\nabla = \nabla^*$. Therefore the manifold reduce to be a Riemannian generalized recurrent. □

Theorem 3.3. *If Equation (3.2) holds for a statistical manifold, then we have*

$$\begin{aligned}(\nabla^*_E\mathcal{R})(\mathcal{S},\mathcal{B},\mathcal{F}) &= \gamma(E)\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F} + \theta(E)[h(\mathcal{B},\mathcal{F})\mathcal{S} - h(\mathcal{S},\mathcal{F})\mathcal{B}]\\ &+ 2\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{K}(E,\mathcal{F}) - 2\mathcal{K}(E,\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F}).\end{aligned} \tag{3.8}$$

Proof. From (2.7) we get

$$\begin{aligned}h((\nabla^*_E\mathcal{R}^*)(\mathcal{S},\mathcal{B},\mathcal{I}),\mathcal{F}) &= h(\nabla^*_E\mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F})\\ &+ h(\mathcal{R}(\nabla^*_E\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{I}) + h(\mathcal{R}(\mathcal{S},\nabla^*_E\mathcal{B})\mathcal{F},\mathcal{I})\\ &+ h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\nabla^*_E\mathcal{I}),\end{aligned} \tag{3.9}$$

in which

$$h(\nabla^*_E \mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F}) = h(\mathcal{R}(\mathcal{S},\mathcal{B})\nabla_E \mathcal{F},\mathcal{I}) - \nabla^*_E h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{I}), \tag{3.10}$$

and

$$\nabla^*_E h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{I}) = h(\nabla^*_E \mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{I}) + h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\nabla_E \mathcal{I}). \tag{3.11}$$

From Equations (3.10) and (3.11) we infer,

$$h(\nabla^*_E \mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F}) = h(\mathcal{R}(\mathcal{S},\mathcal{B})\nabla_E \mathcal{F},\mathcal{I}) - h(\nabla^*_E \mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{I}) - h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\nabla_E \mathcal{I}). \tag{3.12}$$

Hence, by virtue of (2.3) and replacing (3.12) in (3.9), we obtain

$$h((\nabla^*_E \mathcal{R}^*)(\mathcal{S},\mathcal{B},\mathcal{I}),\mathcal{F}) = -h((\nabla^*_E \mathcal{R})(\mathcal{S},\mathcal{B},\mathcal{F}),\mathcal{I}) + 2h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{K}(E,\mathcal{F}),\mathcal{I}) - 2h(\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{K}(E,\mathcal{I}). \tag{3.13}$$

By replacing (3.2) in (3.13), we get

$$\begin{aligned} h((\nabla^*_E \mathcal{R})(\mathcal{S},\mathcal{B})\mathcal{F},\mathcal{I}) &= -\{\gamma(E)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\ &+ \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]\} \\ &+ 2\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{K}(E,\mathcal{F}),\mathcal{I}) \\ &- 2\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{F},\mathcal{K}(E,\mathcal{I})). \end{aligned} \tag{3.14}$$

Again by replacing (2.7) in (3.14), we conclude (3.8). □

Corollary 3.4. *If Equation (3.2) holds for a statistical manifold with constant statistical curvature, then we have* $\nabla = \nabla^*$.

Proof. By virtue of (2.9), and replacing (3.2) in (3.8) we conclude

$$2\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{K}(E,\mathcal{F}) - 2\mathcal{K}(E,\mathcal{R}(\mathcal{S},\mathcal{B})\mathcal{F}) = 0. \tag{3.15}$$

Hence, from (2.8), we get

$$h(\mathcal{B},\mathcal{K}(E,\mathcal{F}))\mathcal{S} - h(\mathcal{S},\mathcal{K}(E,\mathcal{F}))\mathcal{B} - \mathcal{K}(E,h(\mathcal{F},\mathcal{B})\mathcal{S}) + \mathcal{K}(E,h(\mathcal{F},\mathcal{S})\mathcal{B}) = 0. \tag{3.16}$$

Equation (2.4) implies,

$$h(\mathcal{K}(E,\mathcal{F}),\mathcal{B})\mathcal{S} = h(\mathcal{K}(E,\mathcal{F}),\mathcal{S})\mathcal{B}. \tag{3.17}$$

By replacing (3.17) in (3.16) we find

$$\mathcal{K}(E, h(\mathcal{F},\mathcal{S})\mathcal{B}) = \mathcal{K}(E, h(\mathcal{F},\mathcal{B})\mathcal{S}). \tag{3.18}$$

By account of (2.3), $\nabla = \nabla^*$. □

From Corollary 3.4, we conclude the statistical generalized recurrent manifold with constant statistical curvature is as same as the Riemannian generalized recurrent manifold with respect to its Levi-Civita connection.

Lemma 3.5. *(U,h) is statistical generalized recurrent, if and only if $\mathcal{R}m$ is statistical generalized recurrent with respect to ∇^*.*

Proof. Let (U,h) is a statistical generalized recurrent. From (2.7), we have

$$\begin{aligned}
(\nabla^*_E\mathcal{R}m)(\mathcal{S},\mathcal{B},\mathcal{F},\mathcal{I}) &= -(\nabla^*_E\mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&= -\{\gamma(E)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]\} \\
&= \gamma(E)\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{F},\mathcal{I}) \\
&+ \theta(E)[h(\mathcal{B},\mathcal{F})h(\mathcal{S},\mathcal{I}) - h(\mathcal{S},\mathcal{F})h(\mathcal{B},\mathcal{I})].
\end{aligned}$$

Therefore, $\mathcal{R}m$ is generalized recurrent with respect to ∇^*. □

Definition 3.6. *Let (U^n,h) be a statistical non-flat manifold in which $n \geq 3$. We put*

$$\begin{aligned}
\tilde{\mathcal{C}}^*(\mathcal{S},\mathcal{B})\mathcal{I} &= \mathcal{R}^*(\mathcal{S},\mathcal{B})\mathcal{I} \\
&- \frac{tr_h(\mathcal{R}c^*)}{n(n-1)}[h(\mathcal{B},\mathcal{I})\mathcal{S} - h(\mathcal{S},\mathcal{I})\mathcal{B}],
\end{aligned} \tag{3.19}$$

and we call that a statistical concircular curvature tensor field for a pair (∇^*,h) and we set

$$\tilde{\mathcal{C}r}^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = h(\tilde{\mathcal{C}}^*(\mathcal{S},\mathcal{B})\mathcal{I},\mathcal{F}). \tag{3.20}$$

Definition 3.7. *We say (U,h) is a statistical concircular recurrent manifold if for its statistical concircular curvature tensor field $\tilde{\mathcal{C}r}^*$ we have,*

$$(\nabla^*_E\tilde{\mathcal{C}r}^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = \nu(E)\tilde{\mathcal{C}r}^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}). \tag{3.21}$$

for all vector fields $\forall\ \mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F},E \in \tau(U)$, where ν is a non-vanishing 1-form such that for a vector field λ, we have $\nu(E) = h(E,\lambda)$ for any $E \in \tau(U)$.

If $\tilde{\mathcal{C}}^* = 0$ holds in Equation (3.19), then U is of constant statistical curvature. So, the statistical concircular curvature tensor act as a test of a failure of statistical manifold to be with constant statistical curvature.

Definition 3.8. *We say (U, h) is statistical semi-symmetric if*

$$\mathcal{R}^* \cdot \mathcal{R}m^* = 0, \tag{3.22}$$

where $\forall\ \mathcal{S}, \mathcal{B}, \mathcal{I}, \mathcal{F}, E, \mathcal{A} \in \tau(U)$,

$$\begin{aligned} \mathcal{R}^* \cdot \mathcal{R}m^* &= \left(\nabla^*_E \nabla^*_{\mathcal{A}} \mathcal{R}m^*\right)(\mathcal{S}, \mathcal{B}, \mathcal{I}, \mathcal{F}) - \left(\nabla^*_{\mathcal{A}} \nabla^*_E \mathcal{R}m^*\right)(\mathcal{S}, \mathcal{B}, \mathcal{I}, \mathcal{F}) \\ &- \left(\nabla^*_{[E,\mathcal{A}]} \mathcal{R}m^*\right)(\mathcal{S}, \mathcal{B}, \mathcal{I}, \mathcal{F}). \end{aligned} \tag{3.23}$$

Definition 3.9. *We say (U, h) is statistical Ricci semi-symmetric if*

$$\mathcal{R}^* \cdot \mathcal{R}c^* = 0, \tag{3.24}$$

where $\forall\ \mathcal{B}, \mathcal{I}, \mathcal{A}, E \in \tau(U)$,

$$\begin{aligned} \mathcal{R}^* \cdot \mathcal{R}c^* &= \left(\nabla^*_E \nabla^*_{\mathcal{A}} \mathcal{R}c^*\right)(\mathcal{B}, \mathcal{I}) - \left(\nabla^*_{\mathcal{A}} \nabla^*_E \mathcal{R}c^*\right)(\mathcal{B}, \mathcal{I}) \\ &- \left(\nabla^*_{[E,\mathcal{A}]} \mathcal{R}c^*\right)(\mathcal{B}, \mathcal{I}). \end{aligned} \tag{3.25}$$

4. Main results

Let U be a statistical generalized recurrent manifold. Taking contraction over $\mathcal{S}$ and $\mathcal{F}$ of Equation (3.1), we get

$$\left(\nabla^*_E \mathcal{R}c^*\right)(\mathcal{B}, \mathcal{I}) = \gamma(E)\, \mathcal{R}c^*(\mathcal{B}, \mathcal{I}) + (n-1)\theta(E) h(\mathcal{B}, \mathcal{I}). \tag{4.1}$$

Again taking contraction over $\mathcal{B}$ and $\mathcal{I}$ of Equation (4.1), we get

$$E(tr_h \mathcal{R}c^*) = \gamma(E)) tr_h \mathcal{R}c^* + n(n-1)\theta(E). \tag{4.2}$$

Theorem 4.1. *Let (U, h) be a statistical generalized recurrent manifold. 1-forms γ and θ can not be both closed, unless $\gamma(E)\theta(\mathcal{S}) = \gamma(\mathcal{S})\theta(E)$ holds on U.*

Proof. Let (U, h) be a statistical generalized recurrent. Taking covariant derivative of (4.2) we obtain

$$\begin{aligned} \mathcal{S}(E(tr_h \mathcal{R}c^*)) &= (\nabla^*_{\mathcal{S}} \gamma)(E) tr_h \mathcal{R}c^* + \gamma(E) \mathcal{S}(tr_h \mathcal{R}c^*)) \\ &+ n(n-1)(\nabla^*_{\mathcal{S}} \theta)(E) = (\nabla^*_{\mathcal{S}} \gamma)(E) tr_h \mathcal{R}c^* \\ &+ n(n-1)[\gamma(E)\theta(\mathcal{S}) + (\nabla^*_{\mathcal{S}} \theta)(E)]. \end{aligned} \tag{4.3}$$

Also,

$$E(\mathcal{S}(tr_h \mathcal{R}c^*)) = (\nabla^*_E \gamma)(\mathcal{S}) tr_h \mathcal{R}c^*$$

$$+ \quad n(n-1)[\gamma(\mathcal{S})\theta(E) + (\nabla^*_E\theta)(\mathcal{S})]. \tag{4.4}$$

So, we obtain

$$[(\nabla^*_{\mathcal{S}}\gamma)(E) - (\nabla^*_E\gamma)(\mathcal{S})]tr_h\mathcal{R}c^*$$

$$+n(n-1)[(\nabla^*_{\mathcal{S}}\theta)(E) - (\nabla^*_E\theta)(\mathcal{S}) + \gamma(E)\theta(\mathcal{S}) - \gamma(\mathcal{S})\theta(E)] = 0. \tag{4.5}$$

Hence, we get

$$[(\nabla^*_{\mathcal{S}}\gamma)(E) - (\nabla^*_E\gamma)(\mathcal{S})]tr_h\mathcal{R}c^* + n(n-1)[(\nabla^*_{\mathcal{S}}\theta)(E) - (\nabla^*_E\theta)(\mathcal{S})]$$

$$= n(1-n)[\gamma(E)\theta(\mathcal{S}) - \gamma(\mathcal{S})\theta(E)]. \tag{4.6}$$

□

Now we state a special case of Theorem 4.1 in which U is statistical generalized recurrent with constant statistical scalar curvature.

Theorem 4.2. *The 1-form γ in statistical generalized recurrent manifold with non-zero constant statistical scalar curvature a is closed if and only if the 1-form θ is closed.*

Proof. Let (U, h) be a statistical generalized recurrent manifold with constant scalar curvature. From (4.2) we obtain

$$\gamma(E)\, tr_h\mathcal{R}c^* + n(n-1)\theta(E) = 0. \tag{4.7}$$

Taking covariant derivative of (4.7) we get

$$(\nabla^*_{\mathcal{S}}\gamma)(E)tr_h\mathcal{R}c^* + n(n-1)(\nabla^*_{\mathcal{S}}\theta)(E) = 0.$$

Also,

$$(\nabla^*_E\gamma)(\mathcal{S})tr_h\mathcal{R}c^* + n(n-1)(\nabla^*_E\theta)(\mathcal{S}) = 0.$$

Hence, we obtain

$$[(\nabla^*_{\mathcal{S}}\gamma)(E) - (\nabla^*_E\gamma)(\mathcal{S})]tr_h\mathcal{R}c^* + n(n-1)[(\nabla^*_{\mathcal{S}}\theta)(E) - (\nabla^*_E\theta)(\mathcal{S})] = 0.$$

Therefore the 1-form γ is closed if and only if the 1-form θ is closed. □

Lemma 4.3. [7] *Let (U, h) be a statistical manifold. Then*

$$\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = \hat{\mathcal{R}}m(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) + h((\hat{\nabla}_{\mathcal{B}}\mathcal{K})(\mathcal{S},\mathcal{I}),\mathcal{F})$$

$$- \quad h((\hat{\nabla}_{\mathcal{S}}\mathcal{K})(\mathcal{B},\mathcal{I}),\mathcal{F}) - h([\mathcal{K}_{\mathcal{B}},\mathcal{K}_{\mathcal{S}}]\mathcal{I},\mathcal{F}), \tag{4.8}$$

$$\frac{1}{2}\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) + \frac{1}{2}\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = \hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})$$

$$+ \quad h([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F}), \tag{4.9}$$

$$\frac{1}{2}\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})-\frac{1}{2}\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = h((\hat{\nabla}_{\mathcal{S}}\mathcal{K})(\mathcal{B},\mathcal{I}),\mathcal{F}) - h((\hat{\nabla}_{\mathcal{B}}\mathcal{K})(\mathcal{S},\mathcal{I}),\mathcal{F}). \quad (4.10)$$

Lemma 4.4. *Let (U,h) be statistical generalized recurrent.*

(1) *If $(\hat{\nabla}_{\mathcal{S}}\mathcal{K})(\mathcal{B},\mathcal{I})=(\hat{\nabla}_{\mathcal{B}}\mathcal{K})(\mathcal{S},\mathcal{I})$, then*

$$\begin{aligned}(\nabla^*_E\hat{\mathcal{R}m})(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \gamma(E)\{\hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})+h([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F})\}\\ &+ \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F})-h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]\\ &+ (\nabla^*_E h)([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F}). \quad (4.11)\end{aligned}$$

(2) *If $(\hat{\nabla}_{\mathcal{B}}\mathcal{K})(\mathcal{S},\mathcal{I})-(\hat{\nabla}_{\mathcal{S}}\mathcal{K})(\mathcal{B},\mathcal{I})=[\mathcal{K}_{\mathcal{B}},\mathcal{K}_{\mathcal{S}}]\mathcal{I}$, then $\hat{\mathcal{R}m}$ is statistical generalized recurrent with respect to ∇^* and*

$$\begin{aligned}(\nabla^*_E\mathcal{R}m)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \gamma(E)\{\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})-2h([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F})\}\\ &+ \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F})-h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]\\ &+ 2(\nabla^*_E h)([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F}). \quad (4.12)\end{aligned}$$

Proof. Let U be a statistical generalized recurrent manifold. If

$$(\hat{\nabla}_{\mathcal{S}}\mathcal{K})(\mathcal{B},\mathcal{I})=(\hat{\nabla}_{\mathcal{B}}\mathcal{K})(\mathcal{S},\mathcal{I}),$$

then from (4.8) we get

$$\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})=\hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})+h([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F}). \quad (4.13)$$

Hence, we obtain

$$\begin{aligned}&(\nabla^*_E\hat{\mathcal{R}m})(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})+(\nabla^*_E h)([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F})\\ &=\gamma(E)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})+\theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F})-h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]. \quad (4.14)\end{aligned}$$

By replacing Equation (4.13) in the last equality of (4.14), and by direct computation we obtain (4.11).

If $(\hat{\nabla}_{\mathcal{B}}\mathcal{K})(\mathcal{S},\mathcal{I})-(\hat{\nabla}_{\mathcal{S}}\mathcal{K})(\mathcal{B},\mathcal{I})=[\mathcal{K}_{\mathcal{B}},\mathcal{K}_{\mathcal{S}}]\mathcal{I}$, then from (4.8) and (4.10) we get

$$\begin{aligned}\hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})=\mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})\\ &- 2h([\mathcal{K}_{\mathcal{S}},\mathcal{K}_{\mathcal{B}}]\mathcal{I},\mathcal{F}). \quad (4.15)\end{aligned}$$

Since, U is statistical generalized recurrent, so from the first equality of (4.15) we obtain

$$(\nabla^*_E\hat{\mathcal{R}m})(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = \gamma(E)\hat{\mathcal{R}m}(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})+\theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F})$$

$$- \quad h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})].$$

Also, from the last equality of (4.15) we get

$$(\nabla^*_E \mathcal{R}m)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) - 2(\nabla^*_E h)([\mathcal{K}_\mathcal{S},\mathcal{K}_\mathcal{B}]\mathcal{I},\mathcal{F})$$

$$= \gamma(E)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) + \theta(E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]. \quad (4.16)$$

By chosing $\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) = \mathcal{R}m(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) - 2h([\mathcal{K}_\mathcal{S},\mathcal{K}_\mathcal{B}]\mathcal{I},\mathcal{F})$, in the last equality of (4.16), and direct computation we obtain (4.12). □

Theorem 4.5. *Let (U,h) be statistical generalized recurrent. Then, U is not statistical semi-symmetric.*

Proof. Let U be statistical generalized recurrent. By virtue of (3.1) we obtain,

$$\begin{aligned}
(\nabla^*_\mathcal{A}\nabla^*_E\mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \mathcal{A}(\gamma(E))\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ \gamma(E)(\nabla^*_\mathcal{A}\mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ \mathcal{A}(\theta(E))[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) - h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})] \\
&+ \theta(E)[h(\mathcal{B},\mathcal{I})(\nabla^*_\mathcal{A}h)(\mathcal{S},\mathcal{F}) \\
&+ h(\mathcal{S},\mathcal{F})(\nabla^*_\mathcal{A}h)(\mathcal{B},\mathcal{I}) \\
&- h(\mathcal{S},\mathcal{I})(\nabla^*_\mathcal{A}h)(\mathcal{B},\mathcal{F}) - h(\mathcal{B},\mathcal{F})(\nabla^*_\mathcal{A}h)(\mathcal{S},\mathcal{I})] \\
&= \mathcal{A}(\gamma(E))\mathcal{R}m^*(\mathcal{A},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ \gamma(E)\gamma(\mathcal{A})\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ [\gamma(E)\theta(\mathcal{A}) + \mathcal{A}(\theta(E))][h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) \\
&- h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})] \\
&+ \theta(E)[h(\mathcal{B},\mathcal{I})(\nabla^*_\mathcal{A}h)(\mathcal{S},\mathcal{F}) \\
&+ h(\mathcal{S},\mathcal{F})(\nabla^*_\mathcal{A}h)(\mathcal{B},\mathcal{I}) \\
&- h(\mathcal{S},\mathcal{I})(\nabla^*_\mathcal{A}h)(\mathcal{B},\mathcal{F})
\end{aligned}$$

$$- \quad h(\mathcal{B},\mathcal{F})(\nabla^*_{\mathcal{A}}h)(\mathcal{S},\mathcal{I})]. \tag{4.17}$$

Also,

$$\begin{aligned}
\left(\nabla^*_E\nabla^*_{\mathcal{A}}\mathcal{R}m^*\right)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= E\left(\gamma\left(\mathcal{A}\right)\right)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ \gamma\left(\mathcal{A}\right)\gamma\left(E\right)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ [\gamma(\mathcal{A})\theta(E)+E(\theta(\mathcal{A}))][h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) \\
&- h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})] \\
&+ \theta(\mathcal{A})[h(\mathcal{B},\mathcal{I})(\nabla^*_E h)(\mathcal{S},\mathcal{F}) \\
&+ h(\mathcal{S},\mathcal{F})(\nabla^*_E h)(\mathcal{B},\mathcal{I}) \\
&- h(\mathcal{S},\mathcal{I})(\nabla^*_E h)(\mathcal{B},\mathcal{F}) \\
&- h(\mathcal{B},\mathcal{F})(\nabla^*_E h)(\mathcal{S},\mathcal{I})],
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\left(\nabla^*_{[\mathcal{A},E]}\mathcal{R}m^*\right)(\mathcal{A},\mathcal{B},\mathcal{I},\mathcal{F}) &= \gamma\left([\mathcal{A},E]\right)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ \theta([\mathcal{A},E])[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) \\
&- h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})].
\end{aligned} \tag{4.19}$$

So, by virtue of (3.23), Equations (4.17), (4.18) and (4.19), imply

$$\begin{aligned}
\left(\mathcal{R}^*\left(\mathcal{A},E\right)\cdot\mathcal{R}m^*\right)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= 2d\gamma\left(\mathcal{A},E\right)\mathcal{R}m^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) \\
&+ 2d\theta(\mathcal{A},E)[h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F})-h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]. \\
&+ [\gamma(E)\theta(\mathcal{A})-\gamma(\mathcal{A})\theta(E)][h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F}) \\
&- h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})]. \\
&+ \theta(E)[h(\mathcal{B},\mathcal{I})(\nabla^*_{\mathcal{A}}h)(\mathcal{S},\mathcal{F}) \\
&+ h(\mathcal{S},\mathcal{F})(\nabla^*_{\mathcal{A}}h)(\mathcal{B},\mathcal{I})
\end{aligned}$$

$$
\begin{aligned}
&- \quad h(\mathcal{S},\mathcal{I})(\nabla^*_{\mathcal{A}}h)(\mathcal{B},\mathcal{F}) - h(\mathcal{B},\mathcal{F})(\nabla^*_{\mathcal{A}}h)(\mathcal{S},\mathcal{I})] \\
&- \quad \theta(\mathcal{A})[h(\mathcal{B},\mathcal{I})(\nabla^*_{E}h)(\mathcal{S},\mathcal{F}) + h(\mathcal{S},\mathcal{F})(\nabla^*_{E}h)(\mathcal{B},\mathcal{I}) \\
&- \quad h(\mathcal{S},\mathcal{I})(\nabla^*_{E}h)(\mathcal{B},\mathcal{F}) \\
&- \quad h(\mathcal{B},\mathcal{F})(\nabla^*_{E}h)(\mathcal{S},\mathcal{I})].
\end{aligned}
$$

This completes the proof. □

Theorem 4.6. *Let (U^n, h) be a statistical generalized recurrent manifold. Then we have*

$$
\begin{aligned}
\left(\mathcal{R}^*\left(\mathcal{A},E\right)\cdot\mathcal{R}c^*\right)\left(\mathcal{B},\mathcal{I}\right) \quad &= \quad 2d\gamma\left(\mathcal{A},E\right)\mathcal{R}c^*(\mathcal{B},\mathcal{I}) + (2n-2)d\theta(\mathcal{A},E)h(\mathcal{B},\mathcal{I}) \\
&+ \quad (n-1)\{[\gamma(E)\theta(\mathcal{A}) - \gamma(\mathcal{A})\theta(E)][h(\mathcal{B},\mathcal{I})] \\
&+ \quad \theta(E)(\nabla^*_{\mathcal{A}}h)(\mathcal{B},\mathcal{I}) - \theta(\mathcal{A})(\nabla^*_{E}h)(\mathcal{B},\mathcal{I})\}. \qquad (4.20)
\end{aligned}
$$

$\forall\ \mathcal{B},\mathcal{I},E,\mathcal{A} \in \tau(U)$,

Proof. Let U be a statistical generalized recurrent manifold. By virtue of Equation (4.1), we obtain

$$
\begin{aligned}
\left(\nabla^*_{\mathcal{A}}\nabla^*_{E}\mathcal{R}c^*\right)\left(\mathcal{B},\mathcal{I}\right) \quad &= \quad \gamma\left(E\right)\left(\nabla^*_{\mathcal{A}}\mathcal{R}c^*\right)\left(\mathcal{B},\mathcal{I}\right) + \mathcal{A}\left(\gamma\left(E\right)\right)\mathcal{R}c^*\left(\mathcal{B},\mathcal{I}\right) \\
&+ \quad (n-1)\{\mathcal{A}\left(\theta\left(E\right)\right)h(\mathcal{B},\mathcal{I}) + \theta(E)(\nabla^*_{\mathcal{A}}h)(\mathcal{B},\mathcal{I})\} \\
&= \quad \gamma\left(E\right)\gamma\left(\mathcal{A}\right)\mathcal{R}c^*\left(\mathcal{B},\mathcal{I}\right) + \mathcal{A}\left(\gamma\left(E\right)\right)\mathcal{R}c^*\left(\mathcal{B},\mathcal{I}\right) \\
&+ \quad (n-1)\{[\gamma\left(E\right)\theta\left(\mathcal{A}\right) + \mathcal{A}\left(\theta\left(E\right)\right)]h(\mathcal{B},\mathcal{I}) \\
&+ \quad \theta(E)(\nabla^*_{\mathcal{A}}h)(\mathcal{B},\mathcal{I})\}, \qquad (4.21)
\end{aligned}
$$

also,

$$
\begin{aligned}
\left(\nabla^*_{E}\nabla^*_{\mathcal{A}}\mathcal{R}c^*\right)\left(\mathcal{B},\mathcal{I}\right) \quad &= \quad \gamma\left(\mathcal{A}\right)\gamma\left(E\right)\mathcal{R}c^*\left(\mathcal{B},\mathcal{I}\right) + E\left(\gamma\left(\mathcal{A}\right)\right)\mathcal{R}c^*\left(\mathcal{B},\mathcal{I}\right) \\
&+ \quad (n-1)\{[\gamma\left(\mathcal{A}\right)\theta\left(E\right) + E\left(\theta\left(\mathcal{A}\right)\right)]h(\mathcal{B},\mathcal{I}) \\
&+ \quad \theta(\mathcal{A})(\nabla^*_{E}h)(\mathcal{B},\mathcal{I})\}, \qquad (4.22)
\end{aligned}
$$

and

$$\left(\nabla^*_{[\mathcal{A},E]}\mathcal{R}c^*\right)(\mathcal{B},\mathcal{I}) = \gamma([\mathcal{A},E])\,\mathcal{R}c^*(\mathcal{B},\mathcal{I}) + (n-1)\theta([\mathcal{A},E])h(\mathcal{B},\mathcal{I}). \tag{4.23}$$

So, in account of (3.25) and Equations (4.21), (4.22) and (4.23), we obtain the Equation (4.20). □

Now we show that in spite of the Riemannian manifold, a statistical generalized recurrent manifold is not statistical concircular recurrent.

Theorem 4.7. *Let (U,h) be statistical generalized recurrent. Then U is not statistical concircular recurrent.*

Proof. Let U be statistical generalized recurrent. By virtue of (3.20), (3.19) and (4.2) and direct computations we obtain,

$$\begin{aligned}(\nabla^*_E\tilde{\mathcal{C}r}^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= (\nabla^*_E\mathcal{R}m^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})\\
&- \left[\frac{\gamma(E)tr_h(\mathcal{R}c^*)}{n(n-1)} + \theta(E)][h(\mathcal{B},\mathcal{I})h(\mathcal{S},\mathcal{F})\right.\\
&- \left. h(\mathcal{S},\mathcal{I})h(\mathcal{B},\mathcal{F})\right] - \frac{2tr_h(\mathcal{R}c^*)}{n(n-1)}\left\{h(\mathcal{S},\mathcal{F})\,h(\mathcal{B},\mathcal{K}(E,\mathcal{I}))\right.\\
&- h(\mathcal{B},\mathcal{F})\,h(\mathcal{S},\mathcal{K}(E,\mathcal{I})) - h(\mathcal{S},\mathcal{I})\,h(\mathcal{B},\mathcal{K}(E,\mathcal{F}))\\
&+ \left. h(\mathcal{B},\mathcal{I})\,h(\mathcal{S},\mathcal{K}(E,\mathcal{F}))\right\}.\end{aligned}$$

So, it follows from (3.1), (3.19) and (3.20),

$$\begin{aligned}(\nabla^*_E\tilde{\mathcal{C}r}^*)(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F}) &= \gamma(E)\tilde{\mathcal{C}r}^*(\mathcal{S},\mathcal{B},\mathcal{I},\mathcal{F})\\
&- \frac{2tr_h(\mathcal{R}c^*)}{n(n-1)}\left\{h(\mathcal{S},\mathcal{F})\,h(\mathcal{B},\mathcal{K}(E,\mathcal{I}))\right.\\
&- h(\mathcal{B},\mathcal{F})\,h(\mathcal{S},\mathcal{K}(E,\mathcal{I}))\\
&- h(\mathcal{S},\mathcal{I})\,h(\mathcal{B},\mathcal{K}(E,\mathcal{F}))\\
&+ \left. h(\mathcal{B},\mathcal{I})\,h(\mathcal{S},\mathcal{K}(E,\mathcal{F}))\right\}. \end{aligned} \tag{4.24}$$

By virtue of (3.21), the Equation (4.24) shows that the manifold is not statistical concircular recurrent. □

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