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A study of $\mathfrak{D}$-Conformal curvature tensor on (ϵ)-LP-Sasakian manifolds with the generalized symmetric metric connection

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Abstract. The present paper aims to study about (ϵ)-LP-Sasakian manifolds with the generalized symmetric metric connection. We have an example satisfying (ϵ)-LP-Sasakian manifolds with the generalized symmetric metric connection. Further, we studied $\mathfrak{D}$-conformally-flat and ξ-$\mathfrak{D}$-conformally flat curvature conditions in (ϵ)-LP-Sasakian manifolds with the generalized symmetric metric connection.

Keywords: ϵ-LP-Sasakian manifold, the generalized symmetric metric connection, $\mathfrak{D}$-Conformal curvature tensor.

1. Introduction

In 1969, T. Takahashi [1] introduced almost contact manifolds equipped with an associated pseudo-Riemannian metric. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are

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known as (ϵ)-almost contact metric manifolds and (ϵ)-Sasakian manifolds respectively (see [2], [3] and [4]). In 1989, K. Motsumoto [5] replaced the structure vector field ξ by $-\xi$ in an almost para-contact manifold and associated a Lorentzian metric with the resulting structure and gave a notion of Lorentzian para-Sasakian manifold. I. Mihai, R. Roska [7] and others [5], [6] studied Lorentzian para-Sasakian manifolds. Recently, Rajendra Prasad and Vibha Shrivastava [8] introduced the notion of Lorentzian para-Sasakian manifolds with indefinite metric. Such manifold is known to be an indefinite Lorentzian para-Sasakian manifold or (ϵ)-Lorentzian para-Sasakian manifold.

In 1982, Chuman [12] defined the concept of $\mathfrak{D}$-conformal curvature tensor. He studied $\mathfrak{D}$-conformal vector fields in para-Sasakian manifolds. $\mathfrak{D}$-conformal curvature tensor has been studied by Adati[13], Shah[11] and others[14] in different manifolds.

On a Riemannian manifold $\mathfrak{M}$, a linear connection $\tilde{\bigtriangledown}$ is called the generalized symmetric connection if its torsion tensor $\tilde{\mathcal{T}}$ is given by

$$\tilde{\mathcal{T}}(\mathcal{X},\mathcal{Y}) = \alpha\big[\eta(Y)X - \eta(X)Y\big] + \beta\big[\eta(Y)\varphi X - \eta(X)\varphi Y\big]. \tag{1.1}$$

for all vector fields X and Y on $\mathfrak{M}$, where α and β are smooth functions on $\mathfrak{M}$, φ is a (1,1)-type tensor and η is a 1-form.

Furthermore, the above-mentioned connection is said to be the generalized metric when a Riemannian metric g in $\mathfrak{M}$ is given as $\tilde{\bigtriangledown} g = 0$, otherwise, it is non-metric.

The generalized symmetric metric connection reduces to the semi-symmetric metric and the quarter-symmetric metric connection respectively according as $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (0,1)$. Thus, it can be suggested that the generalized symmetric metric connection came from the idea of the semi-symmetric and the quarter-symmetric connections. S.K. Yadav, O. Bahadir, and S.K. Chaubey [9, 10] discussed the generalized symmetric metric connection on LP-Sasakian and (ϵ)-LP-Sasakian manifolds.

In this paper, we have studied some curvature properties of $\mathfrak{D}$-conformal curvature tensor on an (ϵ)-LP-Sasakian Manifold with respect to the generalized symmetric metric connection.

2. Preliminaries

A differentiable manifold of dimension n is called an (ϵ)-Lorentzian para-Sasakian manifold if it admits a $(1,1)$-tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g, which satisfies

$$\varphi^2 X = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \qquad g(\xi,\xi) = -\epsilon, \tag{2.1}$$

$$\eta(X) = \epsilon g(X,\xi), \qquad \varphi\xi = 0, \qquad \eta(\varphi X) = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y),\ g(\varphi X, Y) = g(X, \varphi Y), \tag{2.3}$$

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \epsilon\varphi X, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(\varphi X, Y), \tag{2.6}$$

$\forall\ X,\ Y \in \mathfrak{X}(\mathfrak{M})$, where $\mathfrak{X}(\mathfrak{M})$ is the set of all smooth vector fields on $\mathfrak{M}$, ∇ denotes the operator of covariant differentiation and $\epsilon = 1\ \ or -1$ according as ξ is space-like or time-like.

On an n-dimensional (ϵ)-Lorentzian para-Sasakian manifold with structure (φ, ξ, η, g) the following results hold.

$$\mathcal{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$\mathcal{R}(\xi, X)Y = \epsilon g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$\eta(\mathcal{R}(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.9}$$

$$\mathcal{S}(\varphi X, \varphi Y) = \mathcal{S}(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.10}$$

$$\mathcal{S}(X,\xi) = (n-1)\eta(X), \tag{2.11}$$

$$\mathcal{S}(X,Y) = g(\mathcal{Q}X, Y), \tag{2.12}$$

$$\mathcal{Q}\xi = \epsilon(n-1)\xi, \tag{2.13}$$

$\forall\ X, Y, Z \in \mathfrak{X}(\mathfrak{M})$, where $\mathcal{R}$ is the curvature tensor, $\mathcal{S}$ is the Ricci tensor and $\mathcal{Q}$ is the Ricci operator.

We note that if $\epsilon = 1$ and the structure vector field ξ is space like, then an (ϵ)-Lorentzian para-Sasakian manifold is a usual Lorentzian para-Sasakian manifold.

Definition 2.1. *An (ϵ)-Lorentzian para-Sasakian manifold is called generalized η-Einstein manifold if the Ricci tensor $\mathcal{S}$ of type $(0,2)$ satisfies*

$$\mathcal{S}(X,Z) = ag(X,Z) + b\eta(Z)\eta(X) + cg(\varphi X, Z). \tag{2.14}$$

where a, b, c are scalar functions.

3. The generalized symmetric metric connection in (ϵ)-LP-Sasakian manifolds

Let $\bigtriangledown$ be the Levi-Civita connection and $\tilde{\bigtriangledown}$ be a linear connection in (ϵ)-LP-Sasakian manifold $\mathfrak{M}$. The linear connection $\tilde{\bigtriangledown}$ satisfying

$$\tilde{\bigtriangledown}_X Y = \bigtriangledown_X Y + \mathcal{H}(X,Y), \tag{3.1}$$

for all vector fields X, $Y \in \mathfrak{X}(\mathfrak{M})$, is known to be the generalized symmetric metric connection. Here $\mathcal{H}$ is $(1,2)$-type tensor such that

$$\mathcal{H}(X,Y) = \frac{1}{2}\big[\tilde{\mathcal{T}}(X,Y) + \hat{\tilde{\mathcal{T}}}(X,Y) + \hat{\tilde{\mathcal{T}}}(Y,X)\big], \tag{3.2}$$

where $\tilde{\mathcal{T}}$ is the torsion tensor of $\tilde{\bigtriangledown}$ and

$$g(\hat{\tilde{\mathcal{T}}}(X,Y),W) = g(\tilde{\mathcal{T}}(W,X),Y). \tag{3.3}$$

Given (1.1), (3.3) and (3.2), we have

$$\hat{\tilde{\mathcal{T}}}(X,Y) = \alpha\big[\eta(X)Y - g(X,Y)\xi\big] + \beta\big[\eta(X)\varphi Y - g(\varphi X,Y)\xi\big], \tag{3.4}$$

and hence

$$\mathcal{H}(X,Y) = \alpha\big[\eta(Y)X - \epsilon g(X,Y)\xi\big] + \beta\big[\eta(Y)\varphi X - \epsilon g(\varphi X,Y)\xi\big]. \tag{3.5}$$

Thus we conclude the following:

Corollary 3.1. *For an (ϵ)-LP-Sasakian manifold, the generalized symmetric metric connection $\tilde{\bigtriangledown}$ of type (α,β) is given as*

$$\tilde{\bigtriangledown}_X Y = \bigtriangledown_X Y + \alpha\big[\eta(Y)X - \epsilon g(X,Y)\xi\big] + \beta\big[\eta(Y)\varphi X - \epsilon g(\varphi X,Y)\xi\big]. \tag{3.6}$$

The generalized symmetric metric connection reduces to the semi-symmetric and the quarter-symmetric respectively when (α,β)= (1,0) and (α,β) = (0,1) respectively.

Lemma 3.2. *In (ϵ)-LP-Sasakian manifolds, the following relations are obtained with respect to the generalized symmetric metric connection*

$$\begin{aligned}(\tilde{\bigtriangledown}_X \varphi)Y &= (1-\beta\epsilon)g(X,Y)\xi + (\epsilon-\beta)\eta(Y)X - \epsilon\alpha g(X,\varphi Y)\xi \\ &\quad +2(\epsilon-\beta)\eta(X)\eta(Y)\xi - \alpha\eta(Y)\phi X,\end{aligned} \tag{3.7}$$

$$\tilde{\bigtriangledown}_X \xi = (\epsilon-\beta)\phi X - \alpha X, \tag{3.8}$$

$$(\tilde{\bigtriangledown}_X \eta)Y = (1-\epsilon\beta)g(\varphi X,Y) - \epsilon\alpha g(X,Y). \tag{3.9}$$

4. Curvature tensor of (ϵ)-LP-Sasakian manifolds with respect to the generalized symmetric metric connection

The curvature tensor $\tilde{\mathcal{R}}$ of an (ϵ)-LP-Sasakian manifold with respect to the generalized symmetric metric connection $\tilde{\bigtriangledown}$ in $\mathfrak{M}$ is defined as

$$\tilde{\mathcal{R}}(X,Y)Z = \tilde{\bigtriangledown}_X\tilde{\bigtriangledown}_Y Z - \tilde{\bigtriangledown}_Y\tilde{\bigtriangledown}_X Z - \tilde{\bigtriangledown}_{[X,Y]}Z. \tag{4.1}$$

By virtue of equations (2.1), (2.2), (2.5), (3.6) and (4.1), we obtain a relation between the curvature tensor $\tilde{\mathcal{R}}$ of the generalized symmetric metric connection $\tilde{\bigtriangledown}$ and the curvature tensor $\mathcal{R}$ of the Levi-Civita connection $\bigtriangledown$ as

$$\begin{aligned}\tilde{\mathcal{R}}(X,Y)Z &= \mathcal{R}(X,Y)Z + \alpha(\epsilon\beta-1)\big[g(\varphi Y,Z)X - g(\varphi X,Z)Y\big]\\ &+\alpha(\epsilon\beta-1)\big[g(Y,Z)\varphi X - g(X,Z)\varphi Y\big]\\ &+\beta(\epsilon\beta-2)\big[g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y\big]\\ &+\epsilon\alpha\beta\big[g(\varphi Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y)\big]\xi\\ &+(\epsilon\alpha^2+\beta)\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]\xi\\ &+\alpha\beta\big[\eta(Y)\varphi X - \eta(X)\varphi Y\big]\eta(Z)\\ &+\epsilon\alpha^2\big[g(Y,Z)X - g(X,Z)Y\big]\\ &+(\alpha^2+\epsilon\beta)\big[\eta(Y)X - \eta(X)Y\big]\eta(Z)\end{aligned} \tag{4.2}$$

where $X, Y, Z \in \chi(\mathfrak{M})$.

Taking the inner product with ξ in the above result, we have

$$\begin{aligned}g(\tilde{\mathcal{R}}(X,Y)Z,\xi) = \eta(\tilde{\mathcal{R}}(X,Y)Z) &= \big(1-\epsilon\beta\big)\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]\\ &- \epsilon\alpha\big[g(\varphi Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y)\big]\end{aligned} \tag{4.3}$$

Let $\{e_1, e_2, e_3......., e_{n-1}, \xi\}$ be a set of orthonormal basis of the tangent space at any point of the manifold. the Ricci tensor $\tilde{\mathcal{S}}$ and the scalar curvature $\tilde{\tau}$ of the manifold with the generalized symmetric metric connection are defined by

$$\tilde{\mathcal{S}}(X,Y) = \sum_{i=1}^{n}\epsilon_i g(\tilde{\mathcal{R}}(e_i,X)Y,e_i), \tag{4.4}$$

and

$$\tilde{\tau} = \sum_{i=1}^{n}\epsilon_i\tilde{\mathcal{S}}(e_i,e_i). \tag{4.5}$$

Also, we have

$$g(X,Y) = \sum_{i=1}^{n}\epsilon_i g(X,e_i)g(Y,e_i). \tag{4.6}$$

Contracting (4.2) with respect to X, we have

$$\begin{aligned}\tilde{S}(Y,Z) &= S(Y,Z) + \big[(n-2)(\epsilon\beta-1)\alpha - \epsilon\alpha\beta + \beta(\epsilon\beta-2)\psi\big]g(\varphi Y,Z)\\ &+\big[(n-2)\epsilon\alpha^2 + (1-\epsilon\beta)\beta + \alpha(\epsilon\beta-1)\psi\big]g(Y,Z)\\ &+\big[(n-2)\alpha^2 + \beta(n\epsilon-1) + \alpha\beta\psi\big]\eta(Y)\eta(Z),\end{aligned} \tag{4.7}$$

where $\psi = trace\varphi$ and have value $\psi = \sum_{i=1}^{n} \epsilon_i g(\varphi e_i, e_i)$.

Again contracting (4.7) with Y and Z, we have

$$\begin{aligned}\tilde{\tau} &= \tau + \beta(\epsilon\beta - 2)\psi^2 + \big[2(n-1)\alpha(\epsilon\beta - 1) - 2\epsilon\alpha\beta\big]\psi \\ &\quad +(n-1)(n-2)\epsilon\alpha^2 - (n-1)\epsilon\beta^2, \end{aligned} \tag{4.8}$$

where τ is the scalar curvature of $\bigtriangledown$.

$$\tilde{Q}\xi = \tag{4.9}$$

We also find the following results using the equations (4.2) and (4.7).

Lemma 4.1. *In an n-dimensional (ϵ)-LP-Sasakian manifolds with respect to the generalized symmetric metric connection, the following results hold*

$$\begin{aligned}\tilde{\mathcal{R}}(X,Y)\xi &= (1-\epsilon\beta)\big[\eta(Y)X - \eta(X)Y\big] \\ &\quad -\epsilon\alpha\big[\eta(Y)\varphi X - \eta(X)\varphi Y\big], \end{aligned} \tag{4.10}$$

$$\begin{aligned}\tilde{\mathcal{R}}(\xi,X)Y &= (1-\epsilon\beta)[\epsilon g(X,Y)\xi - \eta(Y)X] \\ &\quad -\epsilon\alpha[\epsilon g(\varphi Y,X)\xi - \eta(Y)\varphi X], \end{aligned} \tag{4.11}$$

$$\tilde{\mathcal{S}}(Y,\xi) = \mathcal{S}(Y,\xi) + \big[(1-n)\epsilon\beta - \epsilon\alpha\psi\big]\eta(Y), \tag{4.12}$$

$$\tilde{\mathcal{S}}(Y,\xi) = \big[(n-1)(1-\epsilon\beta) - \epsilon\alpha\psi\big]\eta(Y), \tag{4.13}$$

$$\tilde{\mathcal{Q}}\xi = \big[(n-1)(\epsilon - \beta) - \alpha\psi\big]\xi. \tag{4.14}$$

5. Example

Let us consider the 3-dimensional manifold $\mathfrak{M} = \big\{(x,y,z) \in R^3\big\}, z \neq 0$, with standard coordinates (x,y,z) in R^3.
Considering linear independent vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_3 = \frac{\partial}{\partial z},$$

independent at each point of $\mathfrak{M}$.
We define the Lorentzian metric as

$$\begin{aligned} g(e_1,e_1) = g(e_2,e_2) = \epsilon, \quad g(e_3,e_3) = -\epsilon, \\ g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0, \end{aligned}$$

a $(1,1)$ tensor field φ as

$$\varphi(e_1) = -e_1, \quad \varphi(e_2) = -e_2, \quad \varphi(e_3) = 0$$

and a 1-form η as

$$\eta(Z) = \epsilon g(Z,\xi),$$

then using the linearity of g and φ, for any $Z, W \in \chi(\mathfrak{M})$, we have

$$\eta(e_3) = -1,$$
$$\varphi^2(Z) = -Z + \eta(Z)e_3,$$
$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W).$$

Now by direct computation, we get

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\epsilon e_1, \quad [e_2, e_3] = -\epsilon e_2.$$

By the use of these above equations, we have

$$\begin{aligned} \triangledown_{e_1} e_1 = -\epsilon e_3, \triangledown_{e_2} e_2 = -\epsilon e_3, \triangledown_{e_3} e_3 = 0, \\ \triangledown_{e_1} e_3 = -\epsilon e_1, \triangledown_{e_2} e_3 = -\epsilon e_2, \\ \triangledown_{e_2} e_1 = \triangledown_{e_1} e_2 = \triangledown_{e_3} e_1 = \triangledown_{e_3} e_2 = 0. \end{aligned} \tag{5.1}$$

Here we can easily verify the equations (2.4), (2.5) and (2.6). Thus the manifold $\mathfrak{M}$ is an (ϵ)-LP-Sasakian manifold.

Now, the given example deals with the generalized-symmetric metric connection. So use of (3.6) and (5.1) yields

$$\begin{aligned} \tilde{\triangledown}_{e_1} e_1 = (\beta - \alpha - \epsilon)e_3, \tilde{\triangledown}_{e_2} e_2 = -\epsilon e_3, \tilde{\triangledown}_{e_3} e_3 = 0, \\ \tilde{\triangledown}_{e_1} e_3 = (\beta - \alpha - \epsilon)e_1, \tilde{\triangledown}_{e_2} e_3 = (\beta - \alpha - \epsilon)e_2, \\ \tilde{\triangledown}_{e_2} e_1 = \tilde{\triangledown}_{e_1} e_2 = \tilde{\triangledown}_{e_3} e_1 = \tilde{\triangledown}_{e_3} e_2 = 0. \end{aligned} \tag{5.2}$$

We know that

$$\mathcal{R}(X, Y)Z = \triangledown_X \triangledown_Y Z - \triangledown_Y \triangledown_X Z - \triangledown_{[X,Y]} Z. \tag{5.3}$$

Using (5.1) and (5.2), we have

$$\begin{aligned} &\mathcal{R}(e_2, e_1)e_1 = e_2, \quad \mathcal{R}(e_3, e_1)e_1 = e_3 \\ &\mathcal{R}(e_1, e_2)e_2 = e_1, \quad \mathcal{R}(e_3, e_2)e_2 = e_3, \\ &\mathcal{R}(e_1, e_3)e_3 = -e_1, \quad \mathcal{R}(e_2, e_3)e_3 = -e_2 \end{aligned} \tag{5.4}$$

and using (5.2), we get

$$\begin{aligned} &\tilde{\mathcal{R}}(e_2, e_1)e_1 = (\beta - \alpha - \epsilon)^2 e_2, \quad \tilde{\mathcal{R}}(e_3, e_1)e_1 = -\epsilon(\beta - \alpha - \epsilon)e_3 \\ &\tilde{\mathcal{R}}(e_1, e_2)e_2 = (\beta - \alpha - \epsilon)^2 e_1, \quad \tilde{\mathcal{R}}(e_3, e_2)e_2 = -\epsilon(\beta - \alpha - \epsilon)e_3, \\ &\tilde{\mathcal{R}}(e_1, e_3)e_3 = \epsilon(\beta - \alpha - \epsilon)e_1, \quad \tilde{\mathcal{R}}(e_2, e_3)e_3 = \epsilon(\beta - \alpha - \epsilon)e_2 \end{aligned} \tag{5.5}$$

Using (5.4), we obtain that

$$\mathcal{S}(e_i, e_i) = 2, i = 1, 2, \quad \mathcal{S}(e_3, e_3) = -2. \tag{5.6}$$

And using (4.4) and (5.5), we verify that

$$\tilde{\mathcal{S}}(e_i, e_i) = (\beta - \alpha - \epsilon)(\beta - \alpha - 2\epsilon), i = 1, 2, \quad \tilde{\mathcal{S}}(e_3, e_3) = 2\epsilon(\beta - \alpha - \epsilon). \tag{5.7}$$

Using (5.6) in (4.5) it is verified that $\tau = 6\epsilon$, also we find $\psi = -2$ and thus using (5.7) it is verified that $\tilde{\tau} = 2\epsilon(\beta-\alpha-\epsilon)(\beta-\alpha-3\epsilon) = 6\epsilon + 2\epsilon\beta^2 + 2\epsilon\alpha^2 - 8\beta + 8\alpha - 4\epsilon\alpha\beta$ which satisfy the equation (4.8).

Again it is verified that $(\tilde{\bigtriangledown}_X g)(Y,Z) = 0$. Hence the manifold, considered in the example, is an (ϵ)-LP-Sasakian manifold with respect to the generalized symmetric metric connection.

6. $\mathfrak{D}$-Conformal Curvature

In 1983, on an n-dimensional manifold, a tensor field $\mathfrak{B}$, given the name $\mathfrak{D}$-Conformal curvature tensor, was introduced by Chuman [12] and defined as

$$\begin{aligned}\mathfrak{B}(X,Y)Z &= \mathcal{R}(X,Y)Z \\ &\quad + \frac{1}{n-3}\big[\mathcal{S}(X,Z)Y - \mathcal{S}(Y,Z)X + g(X,Z)\mathcal{Q}Y - g(Y,Z)\mathcal{Q}X \\ &\quad + \mathcal{S}(Y,Z)\eta(X)\xi - \mathcal{S}(X,Z)Y\eta(Y)\xi + (\eta(Y)\mathcal{Q}X \\ &\quad - \eta(X)\mathcal{Q}Y)\eta(Z)\big] + \frac{\mathcal{K}}{n-3}\big[g(X,Z)\eta(Y)\xi \\ &\quad - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\big] \\ &\quad - \frac{\mathcal{K}-2}{n-3}\big[g(X,Z)Y - g(Y,Z)X\big]. \qquad (6.1)\end{aligned}$$

where

$$\mathcal{K} = \frac{\mathfrak{r} + 2(n-1)}{n-2}.$$

So, we define $\mathfrak{D}$-Conformal curvature tensor $\tilde{\mathfrak{B}}$ on (ϵ)-LP-Sasakian manifolds with the generalized symmetric metric connection as

$$\begin{aligned}\tilde{\mathfrak{B}}(X,Y)Z &= \tilde{\mathcal{R}}(X,Y)Z \\ &\quad + \frac{1}{n-3}\big[\tilde{\mathcal{S}}(Y,Z)X - \tilde{\mathcal{S}}(X,Z)Y + g(Y,Z)\tilde{\mathcal{Q}}X - g(X,Z)\tilde{\mathcal{Q}}Y \\ &\quad + \tilde{\mathcal{S}}(X,Z)\eta(Y)\xi - \tilde{\mathcal{S}}(Y,Z)\eta(X)\xi + (\eta(X)\tilde{\mathcal{Q}}Y - \eta(Y)\tilde{\mathcal{Q}}X)\eta(Z)\big] \\ &\quad + \frac{\tilde{\mathcal{K}}}{n-3}\big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X \\ &\quad - \eta(X)\eta(Z)Y\big] + \frac{\tilde{\mathcal{K}}-2}{n-3}\big[g(X,Z)Y - g(Y,Z)X\big], \qquad (6.2)\end{aligned}$$

where

$$\tilde{\mathcal{K}} = \frac{\tilde{\mathfrak{r}} + 2(n-1)}{n-2}$$

and $\tilde{\mathcal{R}}$, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{Q}}$ are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to the generalized symmetric metric connection.

7. $\mathfrak{D}$-Conformally flat (ϵ)-LP-Sasakian manifolds with the generalized symmetric metric connection

An n-dimensional (ϵ)-LP-Sasakian manifold with the generalized symmetric metric connection is said to be $\mathfrak{D}$-Conformally flat if the $\mathfrak{D}$-Conformal curvature tensor $\mathfrak{B}(X,Y)Z$ satisfies the condition

$$\tilde{\mathfrak{B}}(X,Y)Z = 0.$$

Using the above in the definition of $\mathfrak{D}$-Conformal curvature tensor given by the equation (6.2), we have

$$\begin{aligned}\tilde{\mathcal{R}}(X,Y)Z = \frac{1}{n-3}\big[&\tilde{\mathcal{S}}(Y,Z)X - \tilde{\mathcal{S}}(X,Z)Y + g(Y,Z)\tilde{\mathcal{Q}}X - g(X,Z)\tilde{\mathcal{Q}}Y \\ &+\tilde{\mathcal{S}}(X,Z)\eta(Y)\xi - \tilde{\mathcal{S}}(Y,Z)\eta(X)\xi + (\eta(X)\tilde{\mathcal{Q}}Y - \eta(Y)\tilde{\mathcal{Q}}X)\eta(Z)\big] \\ &+\frac{\tilde{\mathcal{K}}}{n-3}\big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\big] \\ &+\frac{\tilde{\mathcal{K}}-2}{n-3}\big[g(X,Z)Y - g(Y,Z)X\big]. \qquad (7.1)\end{aligned}$$

Taking the inner product with U, equation (7.1) reduces to

$$\begin{aligned}g(\tilde{\mathcal{R}}(X,Y)Z,U) = \frac{1}{n-3}\big[&\tilde{\mathcal{S}}(Y,Z)g(X,U) - \tilde{\mathcal{S}}(X,Z)g(Y,U) \\ &+g(Y,Z)g(\tilde{\mathcal{Q}}X,U) - g(X,Z)g(\tilde{\mathcal{Q}}Y,U) + \tilde{\mathcal{S}}(X,Z)\eta(Y)g(\xi,U) \\ &-\tilde{\mathcal{S}}(Y,Z)\eta(X)g(\xi,U) + \{\eta(X)g(\tilde{\mathcal{Q}}Y,U) - \eta(Y)g(\tilde{\mathcal{Q}}X,U)\}\eta(Z)\big] \\ &+\frac{\tilde{\mathcal{K}}}{n-3}\big[g(Y,Z)\eta(X)g(\xi,U) - g(X,Z)\eta(Y)g(\xi,U) \\ &+\eta(Y)\eta(Z)g(X,U) - \eta(X)\eta(Z)g(Y,U)\big] \\ &+\frac{\tilde{\mathcal{K}}-2}{n-3}\big[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\big]. \qquad (7.2)\end{aligned}$$

Putting $U=\xi$ and using (2.1), we get

$$\begin{aligned}g(\tilde{\mathcal{R}}(X,Y)Z,\xi) = {} & \frac{1}{n-3}\big[\tilde{\mathcal{S}}(Y,Z)g(X,\xi) - \tilde{\mathcal{S}}(X,Z)g(Y,\xi) + g(Y,Z)g(\tilde{\mathcal{Q}}X,\xi) \\ &-g(X,Z)g(\tilde{\mathcal{Q}}Y,\xi) - \epsilon\tilde{\mathcal{S}}(X,Z)\eta(Y) + \epsilon\tilde{\mathcal{S}}(Y,Z)\eta(X) \\ &+\{\eta(X)g(\tilde{\mathcal{Q}}Y,\xi) - \eta(Y)g(\tilde{\mathcal{Q}}X,\xi)\}\eta(Z)\big] \\ &+\frac{\tilde{\mathcal{K}}}{n-3}\big[-\epsilon g(Y,Z)\eta(X) + \epsilon g(X,Z)\eta(Y) \\ &+\eta(Y)\eta(Z)g(X,\xi) - \eta(X)\eta(Z)g(Y,\xi)\big] \\ &+\frac{\tilde{\mathcal{K}}-2}{n-3}\big[g(X,Z)g(Y,\xi) - g(Y,Z)g(X,\xi)\big]. \qquad (7.3)\end{aligned}$$

Using (2.2) and (2.12), above equation reduces to

$$\begin{aligned} g(\tilde{\mathcal{R}}(X,Y)Z,\xi) &= \frac{1}{n-3}\big[2\epsilon\eta(X)\tilde{\mathcal{S}}(Y,Z) - \epsilon\eta(Y)\tilde{\mathcal{S}}(X,Z) \\ &+\{g(Y,Z)-\eta(Y)\eta(Z)\}\tilde{\mathcal{S}}(X,\xi) \\ &-\{g(X,Z)-\eta(X)\eta(Z)\}\tilde{\mathcal{S}}(Y,\xi)\big] \\ &+\frac{\tilde{\mathcal{K}}}{n-3}\big[-\epsilon g(Y,Z)\eta(X)+\epsilon g(X,Z)\eta(Y)\big] \\ &+\frac{\tilde{\mathcal{K}}-2}{n-3}\big[g(X,Z)\epsilon\eta(Y)-g(Y,Z)\epsilon\eta(X)\big]. \end{aligned} \tag{7.4}$$

Using (4.13), we have

$$\begin{aligned} g(\tilde{\mathcal{R}}(X,Y)Z,\xi) &= \frac{1}{n-3}\big[2\epsilon\eta(X)\tilde{\mathcal{S}}(Y,Z) - 2\epsilon\eta(Y)\tilde{\mathcal{S}}(X,Z) \\ &+\big((n-1)(1-\epsilon\beta)-\epsilon\alpha\psi\big)\big\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\big\}\big] \\ &-\frac{2\epsilon(\tilde{\mathcal{K}}-1)}{n-3}\big[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\big]. \end{aligned} \tag{7.5}$$

Using (4.3) in the above equation, we get

$$\begin{aligned} &\big(1-\epsilon\beta\big)\big[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\big] \\ &-\epsilon\alpha\big[g(\varphi Y,Z)\eta(X)-g(\varphi X,Z)\eta(Y)\big]. \\ =\ &\frac{1}{n-3}\big[2\epsilon\eta(X)\tilde{\mathcal{S}}(Y,Z)-2\epsilon\eta(Y)\tilde{\mathcal{S}}(X,Z) \\ &+\big((n-1)(1-\epsilon\beta)-\epsilon\alpha\psi-2\epsilon\tilde{\mathcal{K}}+2\epsilon\big) \\ &\times\big\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\big\}\big]. \end{aligned} \tag{7.6}$$

Further, we have

$$\begin{aligned} 2\epsilon\big[\eta(X)\tilde{\mathcal{S}}(Y,Z)-\eta(Y)\tilde{\mathcal{S}}(X,Z)\big] &= -(n-3)\epsilon\alpha\big[g(\varphi Y,Z)\eta(X) \\ &-g(\varphi X,Z)\eta(Y)\big]+\big(-2(1-\epsilon\beta)+\epsilon\alpha\psi \\ &+2\epsilon\tilde{\mathcal{K}}-2\epsilon\big)\times\big[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\big]. \end{aligned} \tag{7.7}$$

Putting $Y=\xi$ and using (2.1) and (4.13), the above equation reduces to

$$\begin{aligned} 2\epsilon\tilde{\mathcal{S}}(X,Z) &= \big(-2(1-\epsilon\beta)+\epsilon\alpha\psi+2\epsilon\tilde{\mathcal{K}}-2\epsilon\big)g(X,Z) \\ &+\big(-2n\epsilon(1-\epsilon\beta)+3\alpha\psi+2\tilde{\mathcal{K}}-2\big)\eta(Z)\eta(X) \\ &-(n-3)\epsilon\alpha\ g(\varphi X,Z). \end{aligned} \tag{7.8}$$

Thus, we conclude the following:

Theorem 7.1. *A $\mathfrak{D}$-Conformally flat ϵ-LP-Sasakian manifold with the generalized symmetric metric connection is a generalized η-einstein manifold given as*

$$\tilde{\mathcal{S}}(X,Z)=ag(X,Z)+b\eta(Z)\eta(X)+cg(\varphi X,Z)$$

where

$$a = \frac{1}{2}\big(-2\epsilon(1-\epsilon\beta) + \alpha\psi + 2\tilde{\mathcal{K}} - 2\big),$$

$$b = \frac{1}{2}\big(-2n(1-\epsilon\beta) + 3\epsilon\alpha\psi + 2\epsilon\tilde{\mathcal{K}} - 2\epsilon\big)$$

and

$$c = \frac{-1}{2}(n-3)\alpha.$$

Corollary 7.2. *A $\mathfrak{D}$-Conformally flat ϵ-LP-Sasakian manifold with semi-symmetric metric connection is a generalized η-einstein manifold given as*

$$\tilde{\mathcal{S}}_1(X,Z) = a_1 g(X,Z) + b_1\eta(Z)\eta(X) + c_1 g(\varphi X, Z)$$

where

$$a_1 = \tilde{\mathcal{K}} - \epsilon - 1 + \frac{\psi}{2},$$

$$b_1 = \epsilon(\tilde{\mathcal{K}} - 1) - n + \frac{3}{2}\epsilon\psi,$$

and

$$c_1 = \frac{-1}{2}(n-3).$$

Corollary 7.3. *A $\mathfrak{D}$-Conformally flat ϵ-LP-Sasakian manifold with quarter-symmetric metric connection is an η-einstein manifold given as*

$$\tilde{\mathcal{S}}_2(X,Z) = a_2 g(X,Z) + b_2\eta(Z)\eta(X),$$

where

$$a_2 = \tilde{\mathcal{K}} - \epsilon = \frac{1}{(n-2)}\big(\tau + 2(n-1)(1-\epsilon) + \epsilon - (2-\epsilon)\psi^2\big),$$

and

$$b_2 = n(\epsilon - 1) + \epsilon(\tilde{\mathcal{K}} - 1).$$

8. ξ-$\mathfrak{D}$-Conformally flat (ϵ)-LP-Sasakian manifolds with generalized symmetric metric connection

An n-dimensional (ϵ)-LP-Sasakian manifold with generalized symmetric metric connection is said to be ξ-$\mathfrak{D}$-conformally flat if the $\mathfrak{D}$-conformal curvature tensor $\tilde{\mathfrak{B}}(X,Y)Z$ satisfies the condition

$$\tilde{\mathfrak{B}}(X,Y)\xi = 0. \tag{8.1}$$

Using the definition of $\mathfrak{D}$-Conformal curvature tensor in equation (6.2), we have

$$\begin{aligned}\tilde{\mathcal{R}}(X,Y)\xi \;=\; & \frac{1}{n-3}\big[\tilde{\mathcal{S}}(Y,\xi)X - \tilde{\mathcal{S}}(X,\xi)Y + g(Y,\xi)\tilde{\mathcal{Q}}X - g(X,\xi)\tilde{\mathcal{Q}}Y \\ & + \tilde{\mathcal{S}}(X,\xi)\eta(Y)\xi - \tilde{\mathcal{S}}(Y,\xi)\eta(X)\xi + \big(\eta(X)\tilde{\mathcal{Q}}Y - \eta(Y)\tilde{\mathcal{Q}}X\big)\eta(\xi)\big] \\ & + \frac{\tilde{\mathcal{K}}}{n-3}\big[g(Y,\xi)\eta(X)\xi - g(X,\xi)\eta(Y)\xi + \eta(Y)\eta(\xi)X \\ & - \eta(X)\eta(\xi)Y\big] + \frac{\tilde{\mathcal{K}}-2}{n-3}\big[g(X,\xi)Y - g(Y,\xi)X\big]. \end{aligned} \tag{8.2}$$

Using (2.1) and (2.2),the equation (8.2) becomes

$$\begin{aligned}\tilde{\mathcal{R}}(X,Y)\xi &= \frac{1}{n-3}\big[\tilde{\mathcal{S}}(Y,\xi)X-\tilde{\mathcal{S}}(X,\xi)Y \\ &+\tilde{\mathcal{S}}(X,\xi)\eta(Y)\xi-\tilde{\mathcal{S}}(Y,\xi)\eta(X)\xi\big] \\ &+\frac{\epsilon+1}{n-3}\big[\eta(Y)\tilde{\mathcal{Q}}X-\eta(X)\tilde{\mathcal{Q}}Y\big] \\ &+\frac{(\epsilon+1)\tilde{\mathcal{K}}-2\epsilon}{n-3}\big[\eta(X)Y-\eta(Y)X\big]. \qquad (8.3)\end{aligned}$$

Using (4.3) and (4.10), (8.3) reduces to

$$\begin{aligned}&(1+\epsilon\alpha-\epsilon\beta-\alpha^2)\big[\eta(Y)X-\eta(X)Y\big]+(\epsilon\beta-\alpha-\alpha\beta)\big[\eta(Y)\varphi X-\eta(X)\varphi Y\big] \\ &=\frac{1}{n-3}\Big[\big((n-1)(1-\alpha^2)+(\epsilon n-1)(\alpha-\beta)+\beta(1-2\epsilon)\big)\big(\eta(Y)X-\eta(X)Y\big)\Big] \\ &\quad+\frac{\epsilon+1}{n-3}\big[\eta(Y)\tilde{\mathcal{Q}}X-\eta(X)\tilde{\mathcal{Q}}Y\big]+\frac{(\epsilon+1)\tilde{\mathcal{K}}-2\epsilon}{n-3}\big[\eta(X)Y-\eta(Y)X\big].\end{aligned}$$

On simplifying , we have

$$\begin{aligned}(\epsilon+1)\big[\eta(Y)\tilde{\mathcal{Q}}X-\eta(X)\tilde{\mathcal{Q}}Y\big] &= -(n-3)\epsilon\alpha\big[\eta(Y)\varphi X-\eta(X)\varphi Y\big] \\ &+\big(-2(1-\epsilon\beta)-2\epsilon+\epsilon\alpha\psi+(1+\epsilon)\tilde{\mathcal{K}}\big) \\ &\times\big[\eta(Y)X-\eta(X)Y\big] \qquad (8.4)\end{aligned}$$

Replacing Y by ξ, we get

$$\begin{aligned}(\epsilon+1)\big[\tilde{\mathcal{Q}}X+\eta(X)\tilde{\mathcal{Q}}\xi\big] &= -(n-3)\epsilon\alpha\varphi X \\ &+\big(-2(1-\epsilon\beta)-2\epsilon+\epsilon\alpha\psi+(1+\epsilon)\tilde{\mathcal{K}}\big) \\ &\times\big[X+\eta(X)\xi\big] \qquad (8.5)\end{aligned}$$

Using (4.11), we have

$$\begin{aligned}(\epsilon+1)\tilde{\mathcal{Q}}X &= \big(-2(1-\epsilon\beta)+\epsilon\alpha\psi+(\epsilon+1)\tilde{\mathcal{K}}-2\epsilon\big)X \\ &+\big((\epsilon-2-n\epsilon)(1-\epsilon\beta)+(\epsilon+1)\alpha\psi+(\epsilon+1)\tilde{\mathcal{K}}-2\epsilon\big)\eta(X)\xi \\ &+(n-3)\epsilon\alpha\,\varphi X. \qquad (8.6)\end{aligned}$$

Taking inner product with U

$$\begin{aligned}(\epsilon+1)\tilde{\mathcal{S}}(X,U) &= \big(-2(1-\epsilon\beta)+\epsilon\alpha\psi+(\epsilon+1)\tilde{\mathcal{K}}-2\epsilon\big)g(X,U) \\ &+\epsilon\Big((\epsilon-2-n\epsilon)(1-\epsilon\beta)+(\epsilon+1)\alpha\psi \\ &+(\epsilon+1)\tilde{\mathcal{K}}-2\epsilon\Big)\eta(X)\eta(U)+(n-3)\epsilon\alpha g(\varphi X,U) \qquad (8.7)\end{aligned}$$

Thus, we can state the following:

Theorem 8.1. *A ξ-$\mathfrak{D}$-Conformally flat (ϵ)-LP-Sasakian manifold with generalized symmetric metric connection is a genaralized η-einstein manifold given as*

$$\tilde{\mathcal{S}}(X,Z)=Ag(X,Z)+B\eta(Z)\eta(X)+Cg(\varphi X,Z)$$

where

$$A = \frac{1}{1+\epsilon}\big(-2(1-\epsilon\beta) + \epsilon\alpha\psi + (\epsilon+1)\tilde{\mathcal{K}} - 2\epsilon\big),$$
$$B = \frac{\epsilon}{1+\epsilon}\big((\epsilon - 2 - n\epsilon)(1-\epsilon\beta) + (\epsilon+1)\alpha\psi + (\epsilon+1)\tilde{\mathcal{K}} - 2\epsilon\big),$$
$$C = \frac{n-3}{1+\epsilon}\epsilon\alpha.$$

Corollary 8.2. *A ξ-$\mathfrak{D}$-Conformally flat ϵ-LP-Sasakian manifold with semi-symmetric metric connection is a generalized η-einstein manifold given as*

$$\tilde{\mathcal{S}}_1(X,Z) = A_1 g(X,Z) + B_1\eta(Z)\eta(X) + C_1 g(\varphi X, Z)$$

where

$$A_1 = \tilde{\mathcal{K}} - 2 + \frac{\epsilon}{1+\epsilon}\psi, \quad B_1 = \epsilon(\tilde{\mathcal{K}} + \psi - 2) + \frac{1}{1+\epsilon}(1-n),$$

and

$$C_1 = \frac{n-3}{1+\epsilon}\epsilon.$$

Corollary 8.3. *A ξ-$\mathfrak{D}$-Conformally flat ϵ-LP-Sasakian manifold with quarter-symmetric metric connection is an η-einstein manifold given as*

$$\tilde{\mathcal{S}}_2(X,Z) = A_2 g(X,Z) + B_2\eta(Z)\eta(X),$$

where

$$A_2 = \tilde{\mathcal{K}} - \frac{2}{1+\epsilon}, \quad B_2 = \epsilon\tilde{\mathcal{K}} + \frac{\epsilon}{1+\epsilon}\big[(\epsilon-1)(1-n) - 2\big].$$

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