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# Sacks-Uhlenbeck $\alpha-$harmonic maps from Finsler manifolds

A. Shahnavaz[a], S. M. Kazemi Torbaghan[b*] and N. Kouhestani[a]

[a] Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

[b]Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran

E-mail: amir.shahnavaz@pgs.usb.ac.ir
E-mail: m.kazemi@ub.ac.ir
E-mail: kouhestani@math.usb.ac.ir

**Abstract.** In this paper, we study the stability of Sacks-Uhlenbeck $\alpha$-harmonic maps from a Finsler manifold to a Riemannian manifold and its applications. Then we find conditions under which any non-constant $\alpha-$harmonic maps from a compact Finsler manifold to a standard unit sphere $\mathbb{S}^n(n>2)$ is unstable.

**Keywords:** Harmonic map, Finsler Geometry, $\alpha-$harmonic map.

## 1. **Introduction**

$\psi:(M,g)\longrightarrow(N,h)$ from a compact Riemannian manifold to an arbitrary Riemannian manifold is harmonic if it is a critical point of the energy functional

$$E(\psi)=\frac{1}{2}\int_M \mid d\psi\mid^2 dV. \tag{1.1}$$

Equivalently, $\psi$ solves the corresponding Euler-Lagrange equation:

$$\tau(\psi):=trace_g\nabla d\psi=0. \tag{1.2}$$

*Corresponding Author
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Noting that the section $\tau(\psi)$, as an extension of the Riemannian Laplacian operator, is called the tension field of $\psi$. Furthermore , the second order non-linear PDE, $\tau(\psi) = 0$, is typically determined. Eells and Sampson applied the heat flow technique to prove the existence of harmonic map from a compact Riemannain manifold to a compact Riemannian manifold whose sectional curvature is non-positive. The success of this technique relies on Bochner formula for maps between Riemannian manifolds. In addition, they studied some rigidity theorems for harmonic maps under stronger curvature assumptions, such as negative sectional curvature for the target manifold and non-negative Ricci curvature for the domain manifold. Due to the applications of harmonic maps in many significant physical theories such as the theory of relativity, graviational theorem, elasticity theory, etc., many scholars have done research on this topic, [7, 18].

From the point of view of calculus variations, due to the fact that the energy functional $E$ does not satisfy the well-known Palais-Smale condition, finding a harmonic map between two arbitrary Riemannian manifold is not easy when the dimension of domain manifold $dim(M) \geq 2$. Especially, when $dim(M) = 2$, it is proven that $E$ is of conformal invariance and the corresponding variational problem possesses a non-compact invariance group and represent limiting cases where the Palais-Smale condition just fails, [13]. Thus, harmonic mappings from a surface to an arbitrary Riemannian manifold are of special interest and importance. Nowadays much attention has been given to this case. Sacks and Uhlenbech proved the existence of harmonic maps from a closed surface in their pioneering paper [19] by introducing the perturbed energy functional which satisfies the Palais-Smale condition. For this purpose, they used $\alpha-$harmonic maps as the critical points of perturbed energy functional to approximate harmonic maps. More precisely, Sacks and Uhlenbeck defined the $\alpha$-energy functional as follows

$$E_\alpha(\psi) = \frac{1}{2}\int_M (1+ \mid d\psi \mid^2)^\alpha dV_g, \tag{1.3}$$

and considered $\alpha-$harmonic maps as the critical points of $E_\alpha$. Noting that, $E_\alpha$ can be regarded as a perturbation of energy functional, $E$. If there exists a subsequence of $\alpha-$harmonic maps $\{\psi_i\}$ which converges smoothly as $i \longrightarrow \infty$, then $\{\psi_i\}$ will converge to a harmonic map, [13]. Generally, there is no such smooth convergence, therefore Sacks and Uhlenbeck developed some techniques and powerful methods to study the blow-up phenomena for such a variational problem. In 2019, Karen Uhlenbeck , as the first woman, won prestigious Abel prize for her prominent works on $\alpha$- harmonic maps and their physical applications.

In recent two decades, $\alpha-$harmonic maps were investigated by many scholars. In [12] the authors studied the energy identity and necklessness for a sequence of $\alpha-$harmonic maps during blowing up when its codomain is a unit standard sphere $S^{k-1}$. Also they showed that the energy identity can be used to give an alternative proof of Perelman's result [17] that the Ricci flow from a compact orientable prime non-aspherical 3-dimensional manifold becomes extinct in finite time while in [20] it is obtained an optimal gap theorem for the $\alpha$-harmonic maps of degree -1,0 or 1 by using an energy identity in [12]. In [13], the convergence behavior of a sequence of $\alpha-$harmonic mappings $\psi_\alpha$ with $E_\alpha(\psi_\alpha) < C$, is discussed and an example which shows that the necks contain at least a geodesic of infinite length, is given. The existence and stability of $\alpha-$harmonic maps are investigated in [9]. In [11], a closed Riemannian manifold $(N, h)$ and a sequence of $\alpha-$harmonic maps from $S^2$ into $N$ with uniformly bounded energy were constructed such that the energy identity for this sequence is not true.

The notion of harmonic mappings from a Finsler manifold was first introduced by Mo, [14]. The existence of this type of harmonic maps in each homotopy class was conjectured by Professor S.S. Chern on the workshop of New Methods in Finsler Geometry in 2000. In [15], Mo and Yang solved the conjecture of Chern and proved the existence of harmonic maps in a given homotopy class from a compact Finsler manifold into a Riemannian manifold with non-positive sectional curvature. After that harmonic maps on Finsler manifolds have been studied extensively by many scholars, [8, 14, 15, 22]. For instance, the authors in [22] extended the Mo's work and stdudied the variational formulas of harmonic maps between Finsler manifolds. In [8], it is studied the conditions under which any harmonic map from an Einstein Riemannian manifold to a Finsler manifold is totally geodesic. Also it is shown that any stable harmonic maps from a Euclidean unit sphere $\mathbb{S}^n$ to any Finsler manifold is constant.

In this work, we investigate $\alpha$-harmonic maps and the stability of this type of harmonic maps as well as their practical applications by using the ideas of [3, 4, 8, 9, 14, 15, 16, 19, 23, 25]. In this regard, the notions of Sacks-Uhlenbeck $\alpha-$energy functional and $\alpha-$harmonic maps from a Finsler manifold to a Riemannian manifold are studied. Then the variational formulas of this type of energy functional is obtained. In addition, the stability of $\alpha-$harmonic maps and its applications are investigated. Finally, the criteria that cause $\alpha-$harmonic maps from a compact Finsler manifold to a Euclidean standard sphere to be unstable, are given. Sections 3 and 4 summarize our significant findings.

This paper is organized as follows. Section 2 devotes to recall some basics in Finsler geometry and introduce some terminology and notation of Finsler geometry. In section 3, the notions of $\alpha-$energy functional, $\alpha-$energy density and $\alpha-$harmonic maps are introduced. Furthermore, the Euler-Lagrange equation associated to the $\alpha-$energy functional is obtained via calculating the first variational formula of the $\alpha-$energy functional. Finally an example of $\alpha-$harmonic maps from a Finsler manifold to a Riemannian manifold is given. In the last section, the notion of stable $\alpha-$harmonic maps from a Finsler manifold to a Riemannian manifold and its applications are investigated. Additionally, we study the conditions under which any stable $\alpha-$harmonic map from a Finsler manifold to an Euclidean standard sphere $\mathbb{S}^n(n > 2)$ is constant.

## 2. **Preliminaries**

In this section, we review some basics and introduce some terminology and notations of Finsler geometry. We follow the presentation in [21], where many notions are developed from the Riemannian point of view. We refer to [23] as more exhaustive reference in Finsler geometry.

Let $M^m$ be an $m-$dimensional manifold and $\pi : TM \longrightarrow M$ be its tangent bundle, and let $(x, y)$ be a point of $TM$ with $x \in M$, $y \in T_xM$, $(x^i)$ be a local coordinate systems with the domain $V \subset M$ and $(x^i, y^i)$ be the induced standard local coordinates system on $\pi^{-1}(V)$ with $y = y^i \dfrac{\partial}{\partial x^i}$. A Finsler metric on $M$ is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties:

(i) Regularity: $F(x, y)$ is smooth on $TM \setminus \{0\}$.
(ii) Positive homogenity: $F(x, \mu y) = \mu F(x, y)$, for $\lambda > 0$,
(iii) Strong convexity: the fundamental quadratic form

$$g := g_{ij}(x, y)dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}, \tag{2.1}$$

is positive definite at every point $(x, y) \in TM \setminus \{0\}$, where $[F^2]_{y^iy^j}$ mean $\dfrac{\partial^2 F}{\partial y^i \partial y^j}$.

Locally Minkowski and Riemannian manifolds are well-known examples of Finsler manifolds. In the sequel, the Einstein summation convention is used through this paper and the following convention of index ranges shall be used

$$\begin{aligned} &1 \leq A, B, C, ... \leq 2m - 1, &&1 \leq a, b, c, ... \leq m - 1,\\ &1 \leq i, j, k, ... \leq m, &&1 \leq \beta, \gamma, \delta, ... \leq n. \end{aligned} \tag{2.2}$$

Two more significant quantities in Finsler geometry are Cartan tensor and Cartan form, denoted by $A$ and $\eta$, respectively and defined as follows

$$\begin{aligned} &A := A_{ijk}dx^i \otimes dx^j \otimes dx^k, &&A_{ijk} := \frac{F}{4}[F^2]_{y^iy^j},\\ &\eta := \eta_i dx^i, &&\eta_i := g^{jk}A_{ijk}. \end{aligned} \tag{2.3}$$

Let $SM := \cup_x S_xM$ be the projective sphere bundle of $M$. Noting that under rescaling $y \longrightarrow ty$ for $t > 0$ most geometric quantities constructed by Finsler structure are invariants, thus make sense on $SM$. The canonical projection $\rho : SM \longrightarrow M$ defind by $(x, y) \longrightarrow x$ pulls back the tangent bundle $TM$ to the $m-$dimensional vector bundle $\rho^*TM$ over $2m - 1$ dimensional manifold

$SM$. The bundle $\rho^*TM$ and its dual are called Finsler bundle and dual Finsler bundle, respectively. At any point $(x,y) \in SM$, any fiber of $\rho^*TM$ has a local basis $\{\frac{\partial}{\partial x^i}\}$ and a metric $g$ defined by (2.1). Here $\{\frac{\partial}{\partial x^i}\}$ and its dual $\{dx^i\}$ stand for the sections $(x,y,\frac{\partial}{\partial x^i}) \in \Gamma(\rho^*TM)$ and $(x,y,dx^k) \in \Gamma(\rho^*T^*M)$ , respectively. The global section $\ell(x,y) = \frac{y^j}{F}\frac{\partial}{\partial x^j} \in \rho^*TM$ is called the *distinguished section* and the *Hilbert form* of $(M,F)$ is considered as the dual of the distinguished section defined by $\omega = [F]_{y^i}dx^i$, where $[F]_{y^i}$ means $\frac{\partial F}{\partial y^i}$. Moreover, any fiber of the Riemannian vector bundle $(\rho^*TM, g)$ has an *adapted frame* $\{e_j := u^i_j\frac{\partial}{\partial x^i}\}$, i.e. $g(e_i,e_j) = \delta_{ij}$ and $e_m := \ell$. The dual of the adapted frame $\{e_j\}$ is denoted by $\{\omega^j := \vartheta^j_k dx^k\}$, where $\omega^i(e_j) = \delta^i_j$. Noting that $\omega^m = \omega$. Based on the above notations, it can be shown that $dx^j = u^j_i\omega^i$ and $\frac{\partial}{\partial x^j} = \vartheta^i_j e_i$, where $(\vartheta^i_j)$ and $(u^i_j)$ are related by $u^k_j\vartheta^i_k = \delta^i_j$. It can be found more relationships among the quadratic form of $F$, $(u^i_j)$'s and $(\vartheta^i_j)$'s in [1]. Denote the coefficients of non-linear connection on $TM$ by $N^i_j := \frac{1}{2}\frac{\partial G^i}{\partial y^j}$, where

$$G^i := \frac{1}{4}g^{ik}\Big(\frac{\partial^2 F^2}{\partial y^k \partial x^h}y^h - \frac{\partial F^2}{\partial x^k}\Big).$$

Setting

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}, \qquad \delta y^i = dy^i + N^i_j dx^j, \qquad \omega^{2m} = [F]_{y^j}\frac{\delta y^j}{F}. \tag{2.4}$$

It can be seen that $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ is the local orthonormal basis on $T_zTM$ and $\{dx^i, \delta y^i\}$ is its dual basis. Additionally, $\{\omega^k := \vartheta^k_s dx^s, \omega^{m+b} := \vartheta^b_i\frac{\delta y^i}{F}\}$ is a local basis of $T^*SM$. Noting that $\omega^{2m}$ is a dual of the vector $y^i\frac{\partial}{\partial y^i}$, then it vanishes on $SM$. According to the above notations, the vertical subbundle, horizontal subbundle, volume element and Sasaki type metric of $SM$ are denoted by $VSM, HSM, dV_{SM}$ and $G$, respectively and defined as follows (see [2])

$$\begin{aligned} &VSM := U_{x\in M}TS_xM, && HSM := \{\vartheta \in TSM, \quad \omega^{m+a}(\vartheta) = 0\}, \\ &G := \delta_{ij}\omega^i\omega^j + \delta_{ab}\omega^{m+a}\otimes\omega^{m+b}, && dV_{SM} := \omega^1\wedge\omega^2\wedge\cdots\wedge\omega^{2m-1}. \end{aligned} \tag{2.5}$$

Due to the fact that $\rho^*TM$ is isomorph with $HSM$, then $HSM$ is said to be the Finsler bundle. In the sequel, the corresponding horizontal lift of any section $Y \in \Gamma(\rho^*TM)$ is denoted by $Y^H$, [14].

The Chern connection, denoted by $\nabla^c$, is a well-known connection on $\rho^*TM$ whose connection forms are characterized by the following equations

$$dg_{ij} - g_{is}\omega_j^s - g_{js}\omega_i^s = 2A_{ijs}\frac{\delta y^s}{F}, \tag{2.6}$$

and

$$d(dx^k) - dx^j \wedge \omega_j^k = 0. \tag{2.7}$$

By means of (2.7), it can be shown that the curvature 2-forms of the Chern connection, $\Omega_i^j := d\omega_i^j - \omega_i^s \wedge \omega_s^j$, have the following structure

$$\Omega_i^j := \frac{1}{2}R^j_{ist}dx^s \wedge dx^t + P^j_{ist}dx^s \wedge \frac{\delta y^t}{F}. \tag{2.8}$$

Applying (2.8), the Landsberg curvature on $(M, F)$ is denoted by $L$ and defined as follows

$$L := L_{rst}dx^r \wedge dx^s \wedge dx^t, \qquad L_{rst} := g_{ri}\frac{y^h}{F}P^i_{hst}. \tag{2.9}$$

Considering that $L_{rst} = -\dot{A}_{rst}$, where dot denotes the covariant derivative along the Hilbert form, [22].

The divergence of any 1-form $\theta = \theta_k\omega^k \in \Gamma(\rho^*T^*M)$ is defined as follws

$$div_G\theta := trace_G D\theta, \tag{2.10}$$

where where $G$ is a Sasakian type metric of $SM$ and $D$ denotes the Levi-Civita connection on $(SM, g)$. Due to the fact that $\rho^*T^*M$ is isomomorph with $T^*SM$, it can be obtained that

$$div_G\theta = \sum_i \theta_{i|i} + \sum_{a,b}\theta_a L_{bba} = \sum_i(\nabla^c_{e_i^H}d\theta)(e_i) + \sum_{a,b}\theta_a L_{bba}, \tag{2.11}$$

where $\{e_i\}$ is an adapted frame with respect to $g$, $'' \;|''$ denotes the horizontal covariant differential with respect to the Chern connection and $L$ is the Landsberg curvature on $(M, F)$, [14].

## 3. $\alpha-$harmonic maps

Let $\psi : (M^m, F) \longrightarrow (N^n, h)$ be a smooth map from a Finsler manifold to a Riemannian manifold. Throughout this paper $(M^m, F)$ is an $m-$dimensional compact Finsler manifold, $(N^n, h)$ is an $n-$dimensional Riemannian manifold and $\alpha$ is a real constant with a value greater than one. Denote the Levi-Civita connection on $(N, h)$, the Chern connection on $\rho^*TM$ and the pull-back connection on $\rho^*(\psi^{-1}TN)$ by $\nabla^N, \nabla^c$ and $\nabla$, respectively. The $\alpha-$energy density of $\psi$ is denoted by $e_\alpha(\psi)$ and defined as follows

$$e_\alpha(\psi) := \frac{1}{2}\Big(1 + \mid d\psi \mid^2\Big)^\alpha, \tag{3.1}$$

where

$$\mid d\psi \mid^2 := trace_g h(d\psi, d\psi).$$

Here $trace_g$ stands for taking the trace with respect to $g$ (the fundamental quadratic form of $F$) at $(x, y) \in SM$. The $\alpha-$ energy density of $\psi$ with respect to a local coordinate $(x^i)$ on $M$ and $(y^\beta)$ on $N$ can be rewritten as follows

$$e_\alpha(\psi)(x, y) := \frac{1}{2}(1 + \delta^{ij} h_{\beta\gamma}(\bar{x})\psi_i^\beta \psi_j^\gamma)^\alpha \tag{3.2}$$

where $\bar{x} = \psi(x), \{e_i = u_i^j \frac{\partial}{\partial x^j}\}$ is an adapted frame with respect to $g$ at $(x, y) \in SM$ and $d\psi(e_i) = \psi_i^\beta \frac{\partial}{\partial y^\beta} \circ \psi$. The $\alpha$-energy functional of $\psi$ is denoted by $E_\alpha$ and defined as follows

$$E_\alpha(\psi) = \frac{1}{c_{m-1}} \int_M e_\alpha(\psi) dV_{SM}, \tag{3.3}$$

where $dV_{SM}$ is the canonical volume element of $SM$ defined by (2.5) and $c_{m-1}$ denotes the volume of the standard $(m-1)$ dimensional sphere.

Let $\{\psi_t : M \longrightarrow N\}$ be a smooth variation of $\psi$ such that $\psi_0 = \psi$ and

$$W = \frac{\partial \psi_t}{\partial t} \mid_{t=0} = W^\beta \frac{\partial}{\partial y^\beta} \circ \psi.$$

Applying (3.2), the $\alpha-$energy density of $\psi_t$ can be obtained as follows

$$e_\alpha(\psi_t)(x, y) = \frac{1}{2}(1 + \delta^{ij} h_{\beta\gamma}(\bar{x})\psi_{t|i}^\beta \psi_{t|j}^\gamma)^\alpha, \tag{3.4}$$

where $\bar{x} = \psi_t(x)$ and $d\psi_t(e_i) = u_i^j \frac{\partial \psi_t^\beta}{\partial x^j} \frac{\partial}{\partial y^\beta} \circ \psi_t := \psi_{t|i}^\beta \frac{\partial}{\partial y^\beta} \circ \psi$. Using (3.4), we get

$$\begin{aligned}
\frac{\partial}{\partial t} e_\alpha(\psi_t) \mid_{t=0} &= \frac{1}{2} \frac{\partial}{\partial t}(1 + \delta^{ij} h_{\beta\gamma}(\bar{x})\psi_{t|i}^\beta \psi_{t|j}^\gamma)^\alpha \mid_{t=0} \\
&= \alpha \delta^{ij} \{u_i^l \frac{\partial W^\beta}{\partial x^l} \psi_j^\gamma h_{\beta\gamma} + \frac{1}{2} \psi_i^\beta \psi_j^\gamma \frac{\partial h_{\beta\gamma}}{\partial \bar{x}^\mu} W^\mu\} \\
&\quad (1 + \delta^{mn} h_{\beta\gamma}(\bar{x})\psi_m^\beta \psi_n^\gamma)^{\alpha-1} \\
&= \sum_i \{u_i^l \frac{\delta W^\beta}{\delta x^l} \psi_i^\gamma h_{\beta\gamma} + \psi_i^\beta \psi_i^\gamma \, {}^N\Gamma_{\mu\gamma}^\sigma h_{\beta\sigma} W^\mu\} \\
&\quad (1 + \delta^{mn} h_{\beta\gamma}(\bar{x})\psi_m^\beta \psi_n^\gamma)^{\alpha-1}\} \\
&= \sum_i \alpha h(\nabla_{e_i^H} W, d\psi(e_i))(1 + \delta^{mn} h_{\gamma\beta}(\bar{x})\psi_m^\beta \psi_n^\gamma)^{\alpha-1} \\
&= \sum_i (1+ \mid d\psi \mid^2)^{\alpha-1} trace_g h(\nabla W, d\psi),
\end{aligned} \tag{3.5}$$

where $\{{}^N\Gamma_{\gamma\sigma}^\beta\}$ are the coefficients of the Levi-Civita connection on $(N, h)$. Setting

$$\theta := h(W, (1+ \mid d\psi \mid^2)^{\alpha-1} d\psi(e_i))\omega^i.$$

Due to the fact that $L_{bba} = -\dot{A}_{bba}$ and considering (2.11), we get

$$\begin{aligned} div_G\theta &= \sum_j \{(\nabla^c_{e^H_j}\theta)(e_j)\} + \sum_{a,b}(1+ \mid d\psi \mid^2)^{\alpha-1} h(W, d\psi(e_a))L_{bba} \\ &= \sum_j \{h(\nabla_{e^H_j} W, (1+ \mid d\psi \mid^2)^{\alpha-1} d\psi(e_j)) \\ &+ h(W, (\nabla_{e^H_j}(1+ \mid d\psi \mid^2)^{\alpha-1} d\psi)(e_j))\} \\ &- \sum_{a,b} h(W, (1+ \mid d\psi \mid^2)^{\alpha-1} d\psi(e_a))\dot{A}_{bba} \\ &= h\Big(W, (1+ \mid d\psi \mid^2)^{\alpha-1} trace_g \nabla d\psi + d\psi \circ \rho(grad^H(1+ \mid d\psi \mid^2)^{\alpha-1}) \\ &- (1+ \mid d\psi \mid^2)^{\alpha-1} d\psi \circ \rho(Q^H)\Big) \\ &+ \sum_j h(\nabla_{e^H_j} W, (1+ \mid d\psi \mid^2)^{\alpha-1} d\psi(e_j)), \end{aligned} \tag{3.6}$$

where $A_{bba} = A(e_b, e_b, e_a)$,

$$trace_g \nabla d\psi = g^{ij}\left(\nabla_{\frac{\partial}{\partial x^j}} d\psi(\frac{\partial}{\partial x^i}) - d\psi(\nabla^c_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i})\right)$$

and $Q$ is defined as follows

$$Q := \sum_{a,b} \dot{A}_{bba} e_a. \tag{3.7}$$

By means of (3.5) and (3.6) and using the Green's theorem, it is obtained that

$$\frac{d}{dt}E_\alpha(\psi_t) \mid_{t=0} = -\frac{1}{c_{m-1}} \int_{SM} h(\tau_\alpha(\psi), W) dV_{SM}, \tag{3.8}$$

where

$$\begin{aligned} \tau_\alpha(\psi) &= (1+ \mid d\psi \mid^2)^{\alpha-1} trace_g \nabla d\psi + d\psi \circ \rho(grad^H(1+ \mid d\psi \mid^2)^{\alpha-1}) \\ &- (1+ \mid d\psi \mid^2)^{\alpha-1} d\psi \circ \rho(Q^H), \end{aligned} \tag{3.9}$$

here $Q$ defined by (3.7), $grad^H(1+ \mid d\psi \mid^2)^{\alpha-1})$ denotes the horizontal part of $grad(1+ \mid d\psi \mid^2)^{\alpha-1}) \in \Gamma(TSM)$ and $\rho : SM \longrightarrow M$ is the canonical projection on $SM$. The section $\tau_\alpha(\psi)$ is called $\alpha-$tension field of $\psi$. By (3.8) and (3.9), the following result is obtained.

**Theorem 3.1.** *Any smooth map $\psi : (M, F) \longrightarrow (N, h)$ is $\alpha-$harmonic if and only if $\tau_\alpha(\psi) \equiv 0$.*

**Example 3.2.** *Let $\psi : (\mathbb{R}^2, F) \longrightarrow (\mathbb{R}^3, \langle , \rangle)$ be a smooth map from a locally Minkowski manifold $(\mathbb{R}^2, F)$ to the three dimensional Euclidean space $(\mathbb{R}^3, \langle , \rangle)$ defined by*

$$\psi(x) = (x^1 - 2x^2, -3x^1 + x^2, 4x^1 - 2x^2)$$

*where $x = (x^1, x^2) \in \mathbb{R}^2$. By (3.8) and (3.9) together and considering that the Landsberg curvatue of the locally Minkowski manifolds vanishes, it can be seen that $\psi$ is an $\alpha-$harmonic map.*

## 4. Stable $\alpha$-harmonic map

This section devotes to study the stability of $\alpha-$harmonic maps from a Finsler manifold to a Riemannian manifold. In this regard, first we obtain the second variation formula of the $\alpha-$energy functional by making use of the Green's theorem. Then we introduce the notion of stable $\alpha-$harmonic maps and investigate its applications. Finally the criteria that cause $\alpha-$harmonic maps from a compact Finsler manifold to a Euclidean standard sphere to be unstable, are given.

Stable Sacks-Uhlenbeck harmonic maps have various physical applications in different fields such as physics, engineering, and materials science. These applications include studying the behavior of elastic materials, analyzing the dynamics of magnetic systems, and understanding the behavior of liquid crystals. The study of stable Sacks-Uhlenbeck harmonic maps plays a crucial role in gaining insights into the physical properties and behaviors of these systems, leading to advancements in various technological and scientific applications, [20].

**Theorem 4.1.** *(The second variation formula) Let $\psi : (M, F) \longrightarrow (N, h)$ be an $\alpha-$ harmonic map and $\{\psi_{t,s} : M \longrightarrow N\}_{-\xi<s,t<\xi}$ be a 2-parameter smooth variation of $\psi$ such that $\psi_{0,0} = \psi$. Then*

$$\frac{\partial^2}{\partial t \partial s} E_\alpha(\psi)\Big|_{t=s=0}$$
$$= -\frac{1}{c_{m-1}} \int_{SM} \Big\{ \mathcal{B}_{\alpha,\psi} \langle \nabla \upsilon, d\psi \rangle \langle \nabla W, d\psi \rangle - h(\nabla_{grad^H \mathcal{A}_{\alpha,\psi}} W - \mathcal{A}_{\alpha,\psi} \nabla_{Q^H} W$$
$$+ \mathcal{A}_{\alpha,\psi}\, trace_g \nabla^2 W + \mathcal{A}_{\alpha,\psi}\, trace_g R^N(W, d\psi) d\psi, V) \Big\} dV_{SM}, \tag{4.1}$$

*where $Q$ defined by (3.7), $R^N$ is the curvature tensor on $(N, h)$, $\langle , \rangle$ denotes the inner product on $T^*M \otimes \psi^{-1}TN$ and*

$$W := \frac{\partial \psi_{t,s}}{\partial t}\Big|_{t=s=0}, \qquad V := \frac{\partial \psi_{t,s}}{\partial s}\Big|_{t=s=0},$$
$$\mathcal{A}_{\alpha,\psi} := 2\alpha(1+ \mid d\psi \mid^2)^{\alpha-1}, \qquad \mathcal{B}_{\alpha,\psi} := 4\alpha(\alpha-1)(1+ \mid d\psi \mid^2)^{\alpha-2}. \tag{4.2}$$

*Proof.* Let $\bar{M}$ be the product manifold $(-\xi, \xi) \times (-\xi, \xi) \times M$, $\bar{\rho} : S\bar{M} \longrightarrow \bar{M}$ be the natural projection on the sphere bundle $S\bar{M}$ and $\Psi : \bar{M} \longrightarrow N$ is defined by $\Psi(t, s, x) := \psi_{t,s}(x)$, and let the natural extension of $\frac{\partial}{\partial t}$ on $(-\xi, \xi)$, $\frac{\partial}{\partial s}$ on $(-\epsilon, \epsilon)$ and $\frac{\partial}{\partial x}$ on $M$ to the product manifold $(-\xi, \xi) \times (-\xi, \xi) \times M$ are denoted by $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$ and $\frac{\partial}{\partial x}$ again, respectively. The same notation $\nabla^c$ and $\nabla$ shall be used for the Chern connection on $\bar{\rho}^* T\bar{M}$ and the induced connection on $\bar{\rho}^*(\Psi^{-1}TN)$, respectively. Then, by (3.3), we have

$$\frac{\partial^2}{\partial t \partial s} E_\alpha(\psi_{t,s})\Big|_{t=s=0} = \int_{SM} \frac{\partial^2 e_\alpha(\psi_{t,s})}{\partial t \partial s} \;|_{s=t=0}\; dV_{SM}. \tag{4.3}$$

By calculating the second derivation of the $\alpha-$ energy density, we get

$$\begin{aligned}
&\int_{SM} \frac{\partial^2}{\partial t \partial s} e_\alpha(\psi_{t,s}) dV_{SM} \\
&= \int_{SM} \Big\{ (\frac{\partial \mathcal{A}_{\alpha,\psi_{t,s}}}{\partial s}) h(\nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}(e_i), d\psi_{t,s}(e_i)) \\
&+ \mathcal{A}_{\alpha,\psi_{t,s}} \Big( h(\nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}(e_i), \nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(e_i)) \\
&+ h(\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}(e_i), d\psi_{t,s}(e_i)) \Big) \Big\} dV_{SM},
\end{aligned} \tag{4.4}$$

Due to the fact that

$$\frac{\partial \mathcal{A}_{\alpha,\psi_{t,s}}}{\partial s}\Big|_{t=s=0} = \mathcal{B}_{\alpha,\psi} h(\nabla_{e_i} V, d\psi(e_i)), \tag{4.5}$$

Then the first term of the right-hand side of (4.4), can be obtained as follows

$$\frac{\partial \mathcal{A}_{\alpha,\psi_{t,s}}}{\partial s} h(\nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}(e_i), d\psi_{t,s}(e_i)) \mid \Big|_{t=s=0} = \mathcal{B}_{\alpha,\psi} \langle \nabla V, d\psi \rangle \langle \nabla W, d\psi \rangle \tag{4.6}$$

where $B_{\alpha,\psi}$ are defined by (4.2). Let

$$\eta := \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial t}), d\psi_{t,s}(\frac{\partial}{\partial s}))\omega^i.$$

By means of (2.11), we have

$$\begin{aligned}
div_G(\eta) &= \sum_i (\nabla^c_{e_i^H} \eta)(e_i) - \mathcal{A}_{\alpha,\psi_{t,s}} \sum_{a,b} h(\nabla_{e_a^H} d\psi_{t,s}(\frac{\partial}{\partial t}), d\psi_{t,s}(\frac{\partial}{\partial s})) \dot{A}_{bba} \\
&= \sum_i \Big\{ e_i^H(\mathcal{A}_{\alpha,\psi_{t,s}}) h(\nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial t}), d\psi_{t,s}(\frac{\partial}{\partial s})) \\
&+ \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{e_i^H} \nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial t}), d\psi_{t,s}(\frac{\partial}{\partial s})) \\
&+ \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial t}), \nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial s})) \Big\} \\
&- \mathcal{A}_{\alpha,\psi_{t,s}} \sum_{a,b} h(\nabla_{e_a^H} d\psi_{t,s}(\frac{\partial}{\partial t}), d\psi_{t,s}(\frac{\partial}{\partial s})) \dot{A}_{bba}.
\end{aligned} \tag{4.7}$$

Applying (4.7) and Green's theorem, the second term of the right hand side of(4.4) can be obtained as follows

$$\begin{aligned}
&\int_{SM} \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}(e_i), \nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(e_i))\Big|_{t=s=0} dV_{SM} \\
&= \int_{SM} \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial t}), \nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial s}))\Big|_{t=s=0} dV_{SM} \\
&= -\int_{SM} h(\nabla_{grad^H \mathcal{A}_{\alpha,\psi}} V + \mathcal{A}_{\alpha,\psi} trace_g \nabla^2 V - \mathcal{A}_{\alpha,\psi} \nabla_{Q^H} V, W) dV_{SM}. \quad (4.8)
\end{aligned}$$

On the other hand, let

$$\hat{\eta} := h(\nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(\frac{\partial}{\partial t}), \mathcal{A}_{\alpha,\psi_{t,s}} d\psi_{t,s}(e_i))\omega^i.$$

By means of (2.11), we have

$$\begin{aligned}
div_G(\hat{\eta}) = \sum_i \Big\{ & h(\nabla_{e_i^H} \nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(\frac{\partial}{\partial t}), \mathcal{A}_{\alpha,\psi_{t,s}} d\psi_{t,s}(e_i)) \\
& + h(\nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(\frac{\partial}{\partial t}), \nabla_{e_i^H}(\mathcal{A}_{\alpha,\psi_{t,s}} d\psi_{t,s}(e_i))) \Big\} \\
& - \mathcal{A}_{\alpha,\psi_{t,s}} \sum_{a,b} h(\nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(\frac{\partial}{\partial t}), \dot{A}_{bba} d\psi_{t,s}(e_a)). \quad (4.9)
\end{aligned}$$

By means of (4.9) and considering the Green's theorem, the last term of the right-hand side of (4.4), can be obtained as follows

$$\begin{aligned}
&\int_{SM} \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}(e_i), d\psi_{t,s}(e_i))\Big|_{t=s=0} dV_{SM} \\
&= \int_{SM} \mathcal{A}_{\alpha,\psi_{t,s}} h(\nabla_{\frac{\partial}{\partial s}} \nabla_{e_i^H} d\psi_{t,s}(\frac{\partial}{\partial t}), d\psi_{t,s}(e_i))\Big|_{t=s=0} dV_{SM} \\
&= \int_{SM} \mathcal{A}_{\alpha,\psi} trace_g h(R^N(V, d\psi)d\psi, W) dV_{SM} \\
&- \int_{SM} h((\nabla_{\frac{\partial}{\partial s}} d\psi_{t,s}(\frac{\partial}{\partial t}))\Big|_{t=s=0} \tau_\alpha(\psi)) dV_{SM} \\
&= \int_{SM} \mathcal{A}_{\alpha,\psi} trace_g h(R^N(V, d\psi)d\psi, W) dV_{SM}, \quad (4.10)
\end{aligned}$$

where we use the $\alpha-$harmonicity of $\psi$ for the last equality. Substituting (4.6),(4.8) and (4.10) in (4.4), Theorem 4.1 follows. □

Under the assumptions of theorem 4.1, setting

$$I_{\alpha,\psi}(V,W)=\frac{\partial^2}{\partial t\partial s}E_\alpha(\psi_{t,s})\Big|_{t=s=0}. \tag{4.11}$$

Then $\psi$ is said to be stable $\alpha-$harmonic if $I_{\alpha,\psi}(W,W)\geq 0$ for any compactly supported vector field $\omega$ along $\psi$. Otherwise it is called unstable. By means of Theorem 4.1, it can be obtained that

$$\begin{aligned} &I_{\alpha,\psi}(W,W)\\ &:=-\frac{1}{c_{m-1}}\int_{SM}\Big\{\mathcal{B}_{\alpha,\psi}\langle\nabla W,d\psi\rangle^2-h(\nabla_{grad^H\mathcal{A}}W-\mathcal{A}_{\alpha,\psi}\nabla_{Q^H}W\\ &+\mathcal{A}_{\alpha,\psi}trace_g\nabla^2W+\mathcal{A}_{\alpha,\psi}trace_gR^N(W,d\psi)d\psi,W)\Big\}dV_{SM}. \end{aligned} \tag{4.12}$$

Stable harmonic maps play an important role in geometry, mechanics and physics, [24]. For instance, the analysis of harmonic stability properties for planar wiggler free-electron laser(FEL) are investigated by applying the linearized Vlasov-Maxwell equations. Consider that the analysis is carried out in the Compton regime for a tenuous electron beam propagating in the $z$ direction through the constant amplitude planar wiggler magnetic field $B^0=-B_\omega cos k_0 z\hat{e}_x$, [5].

4.1. **Stability of $\alpha$-harmonic maps from a Finsler manifold to a Euclidean sphere.** In this part, we study the stability of $\alpha-$harmonic maps from a without boundary Finsler manifold to a Euclidean standard unit sphere by applying the extrinsic average variational method of Wei ([26, 27]).

Consider the unit standard sphere $\mathbb{S}^n$ as a submanifold of the Euclidean space $\mathbb{R}^{n+1}$. Denote by $\nabla^R$ and $\nabla^S$, the Levi-Civita connections on $\mathbb{R}^{n+1}$ and $\mathbb{S}^n$ , respectively. At any point $x\in\mathbb{S}^n$, any vector $\omega\in\mathbb{R}^{n+1}$ can be split as follows

$$\omega=\omega^\top+\omega^\perp, \tag{4.13}$$

where $\omega^\top$ is the tangent part to $\mathbb{S}^n$ and $\omega^\perp=\langle V,x\rangle x$ is the normal part to $\mathbb{S}^n$.

Denote by $B$ and $A^V$, the second fundamental form of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and the shape operator of $\mathbb{S}^n$ corresponding to a normal vector field $V$ respectively, which is defined as follows

$$B(X,Y)=-\langle X,Y\rangle x, \tag{4.14}$$

and

$$A^V(X)=-(\nabla^R_X V)^\top, \tag{4.15}$$

where $X,Y$ are tangent vectors of $\mathbb{S}^n$ at $x$ and $\langle,\rangle$ is the Euclidean metric on $\mathbb{R}^{n+1}$. Noting that, the the shape operator and the second fundamental form of $\mathbb{S}^n$ are satisfied the following equation

$$\langle B(X,Y),V\rangle=\langle A^V(X),Y\rangle=-\langle X,Y\rangle\langle x,V\rangle, \tag{4.16}$$

for more details see [20]. Under the notations above we have the following:

**Theorem 4.2.** *Let* $\psi : (M, F) \longrightarrow \mathbb{S}^n$ *be a* $\alpha-$*harmonic map from a compact Finsler manifold* $(M, F)$ *such that* $\mid d\psi \mid^2 < \dfrac{n-2}{2\alpha - n}$. *Then* $\psi$ *is unstable.*

*Proof.* Choose an arbitrary point $z \in SM$ and fix it. Setting $\hat{x} = \hat{\psi}(z)$ where $\hat{\psi} = \psi \circ \rho$. Here $\rho : SM \longrightarrow M$ is the canonical projection on $SM$. Let $\{\Theta_\beta\}_{\beta=1}^{n+1}$ be a constant orthonormal basis in $\mathbb{R}^{n+1}$, $R^S$ denotes the curvature tensor of $\mathbb{S}^n$ and $\{e_i\}_{i=1}^m$ be an adapted frame field on $M$. By means of (4.12), we have

$$\begin{aligned}&\sum_{\beta=1}^{n+1} I_{\alpha,\psi}(\Theta_\beta^\top, \Theta_\beta^\top)\\&:= -\frac{1}{c_{m-1}}\int_{SM}\{\mathcal{B}_{\alpha,\psi}\langle \nabla\Theta_\beta^\top, d\psi\rangle^2 - h(\nabla_{grad^H \mathcal{A}_{\alpha,\psi}}\Theta_\beta^\top - \mathcal{A}_{\alpha,\psi}\nabla_{Q^H}\Theta_\beta^\top\\&+ \mathcal{A}_{\alpha,\psi} trace_g \nabla^2\Theta_\beta^\top + \mathcal{A}_{\alpha,\psi} trace_g R^S(\Theta_\beta^\top, d\psi)d\psi, \Theta_\beta^\top)\}dV_{SM} \qquad (4.17)\end{aligned}$$

By (4.13) and (4.15), we have

$$\begin{aligned}\nabla_X \Theta_\beta^\top &= \nabla^S_{d\hat{\psi}(X)}\Theta_\beta^\top = (\nabla^R_{d\hat{\psi}(X)}\Theta_\beta^\top)^\top\\&= (\nabla^R_{d\hat{\psi}(X)}\Theta_\beta - \Theta_\beta^\perp)^\top = -(\nabla^R_{d\hat{\psi}(X)}\Theta_\beta^\perp)^\top\\&= A^{\Theta_\beta^\perp}(d\hat{\psi}(X)). \qquad (4.18)\end{aligned}$$

Making use of (4.16) and (4.18), we get

$$\begin{aligned}\langle \nabla_X \Theta_\beta^\top, d\hat{\psi}(X)\rangle &= \langle A^{\Theta_\beta^\perp}(d\hat{\psi}(X)), d\hat{\psi}(X)\rangle\\&= - \mid d\hat{\psi}(X) \mid^2 \langle \hat{x}, \Theta_\beta^\perp\rangle\\&= - \mid d\hat{\psi}(X) \mid^2 \langle \hat{x}, \Theta_\beta\rangle. \qquad (4.19)\end{aligned}$$

Applying (4.19), the first term of the right-hand side of (4.17) can be obtained as follows

$$\begin{aligned}\mathcal{B}_{\alpha,\psi}\sum_{\beta=1}^{n+1}\sum_{i=1}^{m}\langle \nabla_{e_i}\Theta_\beta^\top, d\hat{\psi}(e_i^H)\rangle^2 &= \mathcal{B}_{\alpha,\psi}\sum_{\beta=1}^{n+1}\sum_{i=1}^{m}(- \mid d\hat{\psi}(e_i^H) \mid^2 \langle \hat{x}, \Theta_\beta\rangle)^2\\&= \mathcal{B}_{\alpha,\psi}\sum_{\beta=1}^{n+1} \mid d\psi \mid^4 \langle \hat{x}, \Theta_\beta\rangle^2\\&= \mathcal{B}_{\alpha,\psi} \mid d\psi \mid^4 . \qquad (4.20)\end{aligned}$$

Applying (4.13) and (4.15), we get

$$\begin{aligned}\nabla_{grad^H \mathcal{A}_{\alpha,\psi}} \Theta_\beta^\top &= \nabla^S_{d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi})} \Theta_\beta^\top \\ &= (\nabla^R_{d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi})} \Theta_\beta^\top)^\top \\ &= (\nabla^R_{d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi})} (\Theta_\beta - \Theta_\beta^\perp))^\top \\ &= -(\nabla^R_{d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi})} \Theta_\beta^\perp)^\top \\ &= A^{\Theta_\beta^\perp}(d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi})). \end{aligned} \tag{4.21}$$

Let $\Lambda_\beta : \mathbb{S}^n \longrightarrow \mathbb{R}$ is defined by

$$\Lambda_\beta(x) := \langle \Theta_\beta, x \rangle, \qquad \forall x \in \mathbb{S}^n.$$

It can be checked that

$$A^{\Theta_\beta^\perp}(X) = -\Lambda_\beta X, \tag{4.22}$$

for every vector field $X$ on $\mathbb{S}^n$. By means of (4.16),(4.21) and (4.22), the second term of (4.17) is obtained as follows

$$\begin{aligned} -\sum_\beta \langle \nabla_{grad^H \mathcal{A}_{\alpha,\psi}} \Theta_\beta^\top, \Theta_\beta^\top \rangle &= \sum_\beta \langle -A^{\Theta_\beta^\perp}(d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi})), \Theta_\beta^\top \rangle \\ &= \sum_\beta \Lambda_\beta \circ \psi \langle d\hat{\psi}(grad^H \mathcal{A}_{\alpha,\psi}), \Theta_\beta^\top \rangle. \end{aligned} \tag{4.23}$$

Similarly,

$$-\sum_\beta \mathcal{A}_{\alpha,\psi} \langle \nabla_{Q^H} \Theta_\beta^\top, \Theta_\beta^\top \rangle = \sum_\beta \Lambda_\beta \circ \hat{\psi} \langle d\hat{\psi}(Q^H), \Theta_\beta^\top \rangle. \tag{4.24}$$

Applying (4.21) and (4.22) together and considering $\nabla_{e_i^H} \Theta_\beta^\top = A^{\Theta_\beta^\perp}(d\hat{\psi}(e_i^H))$, it can be seen that

$$\begin{aligned} \sum_i \nabla_{e_i^H} \nabla_{e_i^H} \Theta_\beta^\top &= \sum_i \nabla_{e_i^H} A^{\Theta_\beta^\perp}(d\hat{\psi}(e_i^H)) \\ &= -\sum_i \nabla_{e_i^H}(\Lambda_\beta \circ \hat{\psi}\, d\hat{\psi}(e_i^H)) \\ &= -d\hat{\psi}(grad(\Lambda_\beta \circ \hat{\psi})) - \Lambda_\beta \circ \hat{\psi} \sum_i \nabla_{e_i^H} d\psi(e_i) \\ &= -\sum_i \langle d\hat{\psi}(e_i^H), grad(\Lambda_\beta) \circ \hat{\psi}) \rangle d\hat{\psi}(e_i^H) \\ &- \Lambda_\beta \circ \hat{\psi} \sum_i \nabla_{e_i^H} d\psi(e_i) \\ &= -\sum_i \langle d\hat{\psi}(e_i^H), \Theta_\beta^\top \circ \hat{\psi} \rangle d\hat{\psi}(e_i^H) \\ &- \Lambda_\beta \circ \hat{\psi} \sum_i \nabla_{e_i^H} d\psi(e_i), \end{aligned} \tag{4.25}$$

where we use $grad\Lambda_\beta = \Theta_\beta^\top$, for the last equality. By means of (4.25), the fourth term of (4.12) is obtained as follows

$$\sum_\beta \mathcal{A}_{\alpha,\psi}\langle trace_g(\nabla^2\Theta_\beta^\top), \Theta_\beta^\top\rangle = -\mathcal{A}_{\alpha,\psi} \mid d\psi \mid^2 - \sum_\beta \Lambda_\beta \circ \hat{\psi}\langle \mathcal{A}_{\alpha,\psi} trace_g \nabla d\psi, \Theta_\beta^\top\rangle. \tag{4.26}$$

Due to the fact that the sectional curvature of $\mathbb{S}^{n+1}$ is constant, the last term of the right-hand side of (4.17) can be calculated as follows

$$\begin{aligned}
&\sum_{\beta=1}^{n+1}\sum_{i=1}^{m}\langle R^S(\Theta_\beta^\top, d\psi(e_i))d\psi(e_i), \Theta_\beta^\top\rangle \\
&= \sum_{\beta=1}^{n+1}\sum_{i=1}^{m}\Big\{\mid d\psi(e_i)\mid^2\mid \Theta_\beta^\top \mid^2 - \langle d\psi(e_i), \Theta_\beta\rangle^2\Big\} \\
&= -\mid d\psi \mid^2 + \mid d\psi \mid^2 \sum_{\beta=1}^{n+1} \mid \Theta_\beta - \Theta_\beta^\perp \mid^2 \\
&= -\mid d\psi \mid^2 + \mid d\psi \mid^2 \sum_{\beta=1}^{n+1} \mid \Theta_\beta - \langle\Theta_\beta, x\rangle x \mid^2 \\
&= -\mid d\psi \mid^2 + \mid d\psi \mid^2 \sum_{\beta=1}^{n+1} (\mid \Theta_\beta \mid^2 - \langle\Theta_\beta, x\rangle^2) \\
&= -\mid d\psi \mid^2 + \mid d\psi \mid^2 \{(n+1) - \mid x \mid^2\} \\
&= (n-1)\mid d\psi \mid^2 .
\end{aligned} \tag{4.27}$$

By substituting (4.20)- (4.27) in (4.17), we get

$$\begin{aligned}
&\sum_{\beta=1}^{n+1} I_{\alpha,\psi}(\Theta_\beta^\top, \Theta_\beta^\top) \\
&= -\frac{1}{c_{m-1}}\sum_\beta \int_{SM} \Lambda \circ \psi \langle \tau_\alpha(\psi), \Theta_\beta^\top\rangle dV_{SM} \\
&- \frac{\mathcal{E}}{c_{m-1}}\int_{SM}\{(2-n+(2\alpha-n)\mid d\psi \mid^2)\mid d\psi \mid^2\} dV_{SM},
\end{aligned} \tag{4.28}$$

where $\mathcal{E} = 2\alpha(1+ \mid d\psi \mid^2)^{\alpha-2}$. By (4.28) and using the $\alpha-$ harmonicity condition of $\psi$, it follows that

$$\sum_{\beta=1}^{n+1} I_{\alpha,\psi}(\Theta_\beta^\top, \Theta_\beta^\top) < 0. \tag{4.29}$$

Then $\psi$ is unstable and hence completes the proof. □

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