Journal of Finsler Geometry and its Applications
Vol. 5, No. 2 (2024), pp 62-69
https://doi.org/10.22098/jfga.2024.15233.1128

On the flag curvature of left invariant generalized $m-$Kropina metrics on some Lie groups

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Abstract. In this paper we study invariant Finsler spaces with generalized $m-$Kropia metrics. We give an explicit formula for the flag curvature of invariant Finsler spaces with generalized $m-$Kropina metrics on some Lie groups.

Keywords: Flag curvature, (α, β)-metric, Kropina metric, generalized Kropina metric.

1. **Introduction**

The geometry of invariant Finsler metrics on Lie groups is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers in recent years (see [2, 3, 8, 9, 10, 12, 13, 14, 15, 16, 17]). In 1972 M. Matsumoto introduced the concept of $(\alpha, \beta)-$metrics in Finsler geometry [18]. A Finsler metric of the form

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{s}$$

where $\alpha = \sqrt{\tilde{a}_{ij}y^iy^j}$ induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth $n-$manifold M and $\beta = b_i(x)y^i$ is a 1-form on M, is called

AMS 2020 Mathematics Subject Classification: 53C60, 53C30 an $(\alpha,\beta)-$metric [1]. In particular if $\phi(s) = 1/s^m$, $(m \neq 0,-1)$, then the Finsler metric

$$F = \frac{\alpha^{m+1}}{\beta^m}$$

is called generalized $m-$Kropina metric [11].

Let $\mathfrak{a}$ and $\mathfrak{r}$ be abelian Lie algebra of dimension n and 1, respectively. Let $P = (p_{ij}) \in \mathfrak{gl}(\mathfrak{n},\mathcal{R})$ be any real $(n \times n)$-matrix. The well-known homomorphism $\varphi : \mathfrak{r} \to End(\mathfrak{a})$ can be defined by

$$\varphi(\gamma)(x) = \gamma P x.$$

for $\gamma \in \mathfrak{r}$ and $x \in \mathfrak{a}$. The semi- direct product of $\mathfrak{a}$ and $\mathfrak{r}$ can be defined as follows.The underlying Linear space is the directsum $\mathfrak{a} \oplus \mathfrak{r}$ and the bracket is given by

$$[(a,\gamma),(b,\beta)] = (\varphi(\gamma)b - \varphi(\beta)a, 0)$$

we denote this Lie algebra by $\mathfrak{a} \oplus_p \mathfrak{r}$ [22, 10].

Flag curvature is the most important quantity in Finsler geometry [6, 7, 23, 19, 20, 21, 24]. In this paper we give an explicit formula for the flag curvature of invariant generalized $m-$Kropina metrics on $\mathfrak{a} \oplus_p \mathfrak{r}$.

2. **Preliminaries**

In this section we give some definitions and results of Finsler spaces (see [4, 5]).

Definition 2.1. *Let M be a C^∞ manifold of dimension n. A Finsler metric on M is a non-negative function $F : TM \to R$ with the following properties:*

1) *F is smooth on the slit tangent bundle $TM \setminus \{0\}$.*
2) *$F(x, \lambda Y) = \lambda F(x, Y)$ for any $x \in M$, and $Y \in T_xM$, $\lambda > 0$.*
3) *for a fixed $Y \in T_xM - \{0\}$, the bilinear function $g_Y : T_xM \times T_xM \to R$ defined by*

$$g_Y(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, Y + sU + tV)\mid_{s=t=0}.$$

is positive definite.

Definition 2.2. *Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x,y) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Suppose*

$$b := \|\beta(x)\|_\alpha = \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}$$

and let the function F is defined as follows

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha} \tag{2.1}$$

where $\phi := \phi(s)$ *is a positive* C^∞ *function on* $(-b_0, b_0)$ *satisfying*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

Then F *is a Finsler metric if* $\|\beta(x)\|_\alpha < b_0$ *for any* $x \in M$. *A Finsler metric in form (2.1) is called an* (α, β)*-metric. In particular,if* $\phi(s) = \frac{1}{s^m}$ $(m \neq 0, -1)$ *then the Finsler metric*

$$F = \frac{\alpha^{m+1}}{\beta^m},$$

is called generalized m*-Kropina metric. The Riemannian metric* $\tilde{a}$ *induces an inner product on any cotangent space* T_xM *such that*

$$\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x).$$

The induced inner product on T_x^*M *induced a linear isomorphism between* T_x^*M *and* T_xM. *Then the* 1*-form* β *corresponds to a vector fixed* X *on* M *such that*

$$\tilde{a}(y, X(x)) = \beta(x, y).$$

Therefore we can write generalized m*-Kropina metrics as following*

$$F(x, Y) = \frac{(\sqrt{\tilde{a}(Y, Y)})^{m+1}}{\tilde{a}(X, Y)^m}$$

Definition 2.3. *A Finsler space* (M, F) *is called a Berwald space if the Chern connection of* (M, F) *is a linear connection on* TM. *Equivalently, if each of the Chern connection coefficient* Γ^i_{jk} *in natural coordinate system, have no* y *dependence, then Finsler space* (M, F) *becomes a Berwald space.*

Definition 2.4. *Let* (M, F) *be a Finsler space. The flag curvature* $K(P, Y)$ *is a function of tangent planes* $P = span\{Y, U\} \subset T_xM$ *and direction* $Y \in T_xM - \{0\}$. *The pair* (P, Y) *is called a flag and* Y *is called the flag Pole. The flag curvature is defined by*

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y)g_y(U, U) - g_Y^2(Y, U)}$$

Definition 2.5. *Let* $\mathfrak{g}$ *be a Lie algebra and* G *is the simply connected Lie group with Lie algebra* $\mathfrak{g}$. *A Finsler metric* $F : TG \to [0, \infty)$ *will be called left invariant if*

$$F(L_{a_*}X) = F(X)\ , \forall a \in G\ , \forall X \in \mathfrak{g}$$

when L_a *is the left translation and* e *is the unit element of the Lie group.*

3. Flag curvature of left invariant generalized m-Kropina metric

Let $\mathfrak{a}$ and $\mathfrak{r}$ be Abelian Lie algebra of dimension n and 1, respectively. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a} \oplus_p \mathfrak{r}$. Put $E_i = (0, ..., 1, ..., 0)$, and let $\{E_1, ..., E_{n+1}\}$ be orthonormal basis for $\mathfrak{g} = \mathfrak{a} \oplus_p \mathfrak{r}$ with respect to a inner product $\langle , \rangle$ and equip the left invariant metric on the associated Lie group with Lie algebra $\mathfrak{g}$. for $1 \leqslant i, j \leqslant n$, we have

1) $[E_1, E_j] = 0$
2) $[E_{n+1}, E_i] = \sum_{j=1}^{n} P_{ji} E_j$
3) $[E_{n+1}, E_{n+1}] = 0$

and

$$\nabla_{E_i} E_j = \sum_{k=1}^{n+1} \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) E_k$$

where

$$[E_i, E_j] = \sum_{k=1}^{n+1} \alpha_{ijk} E_k.$$

Theorem 3.1. *Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus_p \mathfrak{r}$ and let $\{E_1, ..., E_{n+1}\}$ be orthonormal basis for $\mathfrak{g}$ with respected to the left invariant Riemannian metric $\tilde{a}$. Let F be a generalized m-Kropina metric arising by $\tilde{a}$ and the invariant vector field $X = \sum_{r=1}^{n+1} x_r E_r$ whith is of Berwald type. Then the flag curvature of F is given by*

$$K(P, E_j) = \frac{(m+1)\left(\frac{1}{4}(p_{ij} + p_{ji})^2 - p_{ii}p_{jj}\right) + m(2m+1)x_i x_j^{2m} \tilde{a}(X, A)}{(m+1)(x_j^2 + m x_i^2)},$$

$$\begin{aligned} K(P, E_{n+1}) = \frac{1}{(m+1)(x_{n+1}^2 + m x_i^2)} \Bigg\{ &(m+1) x_{n+1}^{2m+2} \sum \Big(\frac{1}{4}(p_{ji} + p_{ij})(p_{ij} - p_{ji}) \\ &- \frac{1}{2} p_{ji}(p_{ij} + p_{ji})\Big) \;\; - 2m(m+1) x_i x_{n+1}^{2m+1} \Big(\sum_{j=1}^{n} \frac{1}{4}(p_{ji} + p_{ij})(p_{(n+1)j} - p_{j(n+1)}) \\ &- \frac{1}{2} p_{ji}(p_{(n+1)j} + p_{j(n+1)})\Big) + m(2m+1) x_i x_{n+1}^{2m} \tilde{a}(X, B) \Bigg\} \end{aligned}$$

where

$$A = \sum_{k=1}^{n} \Big(\frac{-1}{2} p_{jj}(p_{ki} + p_{ik}) + \frac{1}{4}(p_{ij} + p_{ji})(p_{kj} + p_{jk})\Big) E_k$$

and

$$B = \sum_{r=1}^{n} \Big(\sum_{i-1}^{n} \frac{1}{4}(p_{ji} + p_{ij})(p_{rj} - p_{jr}) - \frac{1}{2} p_{ji}(p_{rj} + p_{jr})\Big) E_r.$$

Proof. Since F is of Berwald type, the Chern connection of F coincide with the Levi-Civita connection of $\tilde{a}$. Thus F and $\tilde{a}$ have the same curvature tensor R. So we have

$$R(E_i, E_j)E_j = \sum_{k=1}^{n} \Big(\frac{-1}{2} p_{ij}(p_{ki} + p_{ik}) + \frac{1}{4}(p_{ij} + p_{ji})(p_{kj} + p_{jk}) \Big) E_k,$$

$$R(E_i, E_{n+1})E_{n+1} = \sum_{r=1}^{n} \Big(\sum_{j=1}^{n} \frac{1}{4}(p_{ji} + p_{ij})(p_{rj} - p_{jr}) - \frac{1}{2} p_{ji}(p_{rj} + p_{jr}) \Big) E_r.$$

By using the definition $g_Y(U, V)$, we get

$$\begin{aligned} g_Y(U, V) = \frac{\langle Y, Y\rangle^{m-1}}{\langle X, Y\rangle^{2m+2}} \Big[& 2m(m+1)\langle X, Y\rangle^2 \langle U, Y\rangle\langle Y, V\rangle \\ & - 2m(m+1)\langle X, Y\rangle\langle Y, Y\rangle\langle Y, U\rangle\langle X, V\rangle \\ & - 2m(m+1)\langle X, Y\rangle\langle Y, Y\rangle\langle X, U\rangle\langle Y, V\rangle + (m+1)\langle X, Y\rangle^2\langle Y, Y\rangle\langle U, V\rangle \\ & + m(2m+1)\langle Y, Y\rangle^2\langle X, U\rangle\langle X, V\rangle \Big] \end{aligned} \tag{3.1}$$

From equation (3,1) we get the following equations

$$g_{E_j}(E_i, E_i) = \frac{1}{x_j^{2m+2}} \Big[(m+1)x_j^2 + m(2m+1)x_i^2 \Big],$$

$$g_{E_j}(E_j, E_j) = \frac{1}{x_j^{2m}},$$

$$g_{E_j}(E_i, E_j) = \frac{-m x_i}{x_j^{2m+1}}.$$

$$g_{E_j}\Big(R(E_i, E_j), E_j, E_i\Big) = \Big(\frac{1}{4}(p_{ij} + p_{ji})^2 - p_{ij}p_{jj} \Big) \frac{(m+1)}{x_j^{2m}} + \frac{x_i(2m^2 + m)}{x_j^{2m+2}} \tilde{a}(X, A)$$

where

$$\begin{aligned} A &= R(E_i, E_j)E_j \\ &= \sum_{k=1}^{n} \Big(\frac{-1}{2} p_{jj}(p_{ki} + p_{ik}) + \frac{1}{4}(p_{ij} + p_{ji})(p_{kj} + p_{jk}) \Big) E_k. \end{aligned}$$

Similarly for $j = n+1$, we get

$$g_{E_{n+1}}(E_i, E_i) = \frac{1}{x_{n+1}^{2m+2}}\Big[(m+1)x_{n+1}^2 + m(2m+1)x_i^2\Big],$$

$$g_{E_{n+1}}(E_{n+1}, E_{n+1}) = \frac{1}{x_{n+1}^{2m}},$$

$$g_{E_{n+1}}(E_i, E_{n+1}) = \frac{-mx_i}{x_{n+1}^{2m+1}}$$

and

$$\begin{aligned} & g_{E_{n+1}}\Big(R(E_i, E_{n+1})E_{n+1}, E_i\Big) \\ = (m+1)x_{n+1}^{2m+2} & \sum\Big(\frac{1}{4}(p_{ji}+p_{ij})(p_{ij}-p_{ji}) - \frac{1}{2}p_{ji}(p_{ij}+p_{ji})\Big) \\ & -2m(m+1)x_i x_{n+1}^{2m+1}\Big(\sum_{j=1}^{n}\frac{1}{4}(p_{ji}+p_{ij})(p_{(n+1)j}-p_{j(n+1)}) \\ & -\frac{1}{2}p_{ji}(p_{(n+1)j}+p_{j(n+1)})\Big) + m(2m+1)x_i x_{n+1}^{2m}\tilde{a}(X, B). \end{aligned}$$

where

$$\begin{aligned} B &= R(E_i, E_{n+1})E_{n+1} \\ &= \sum_{r=1}^{n}\Big(\sum_{i-1}^{n}\frac{1}{4}(p_{ji}+p_{ij})(p_{rj}-p_{jr}) - \frac{1}{2}p_{ji}(p_{rj}+p_{jr})\Big)E_r. \end{aligned}$$

Using the above equations in the flag curvature equation we get the required proof. □

Finally, we obtain the following.

Theorem 3.2. *Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus_p \mathfrak{r}$ and let $\{E_1, ..., E_{n+1}\}$ be orthonormal basis for $\mathfrak{g}$ with respect to the left invariant Riemannian metric $\tilde{a}$. Let F be a Kropina metric arising by $\tilde{a}$ and the invariant vector field $X = \sum_{r=1}^{n+1} x_r E_r$ with is of Berwald type. Then the flag curvature of F is given by*

$$K(p, E_j) = \frac{(p_{ij}+p_{ji})^2 - 4p_{ii}p_{jj} + 6x_i x_j^2\tilde{a}(X, A)}{4(x_j^2 + x_i^2)}.$$

$$\begin{aligned} K(P, E_{n+1}) = \frac{1}{2(x_{n+1}^2 + x_i^2)} & \{2x_{n+1}^4(\sum\frac{1}{4}(p_{ji}+p_{ij})(p_{ij}-p_{ji}) \\ & -\frac{1}{2}p_{ji}(p_{ij}+p_{ji})) - 4x_i x_{n+1}^3(\sum\frac{1}{4}(p_{ji}+p_{ij})(p_{(n+1)j}-p_{j(n+1)}) \\ & -\frac{1}{2}p_{ji}(p_{(n+1)j}+p_{j(n+1)} + 3x_i x_{n+1}^2\tilde{a}(X, B)\} \end{aligned}$$

where

$$A = \sum_{k=1}^{n} \Big(\frac{-1}{2} p_{jj}(p_{ki} + p_{ik}) + \frac{1}{4}(p_{ij} + p_{ji})(p_{kj} + p_{jk}) \Big) E_k$$

$$B = \sum_{r=1}^{n} \Big(\sum_{j=1}^{n} \frac{1}{4}(p_{ji} + p_{ij})(p_{rj} - p_{jr}) - \frac{1}{2} p_{ji}(p_{rj} + p_{jr}) \Big) E_r.$$

Acknowledgements. The author thanks Professor D. Latifi for drawing her attention to Deng's articles. The author also thanks the referees for their helpful comments and suggestions.

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Received: 09.06.2024
Accepted: 02.08.2024