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Invariant square metrics on reduced $\Sigma-$spaces

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Abstract. In this paper, we study some geometric properties of Finsler $\Sigma-$spaces with square metric. We prove that Finsler $\Sigma-$spaces with square $(\alpha, \beta)-$metrics are Riemannian.

Keywords: (α, β)-metric, square metric, $\Sigma-$space, generalized symmetric space.

1. **Introduction**

The geometry of invariant Finsler metrics on homogeneous manifolds is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers, for example see [1, 11, 2, 12, 19, 7]. A smooth manifold M with a system of diffeomorphisms $\{s_x\}_{x \in M}$ is said to be a regular $s-$manifold if

- **(1):** $s_x x = x$,
- **(2):** $s_x \circ s_y = s_{s_x y} \circ s_x$,
- **(3):** $(s_x)_{*x} - Id_x$ is invertible.

$\Sigma-$spaces and reduced $\Sigma-$spaces were first introduced by Loos as a generalization of reflection spaces and symmetric spaces [17, 14, 21]. He then proved that any $\Sigma-$space with compact Σ is a fibre bundle over a reduced $\Sigma-$space. The notion of generalized symmetric Finsler space is a natural generalization of

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generalized Riemannian symmetric spaces [10, 3, 8, 9, 15, 16, 20]. Basic properties of any reduced $\Sigma-$space M and affine, Riemannian and Finsler $\Sigma-$space was given in [17, 14, 21].

In 1929, Berwald construct an interesting family of projectively flat Finsler metrics on the unit ball $\mathbb{B}^n$ which as follows

$$F = \frac{\left(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}. \tag{1.1}$$

He showed that this class of metrics has constant flag curvature [5]. Berwald's metric can be expressed as

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{1.2}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{(1-|x|^2)^2}, \quad \beta = \frac{\langle x,y\rangle}{(1-|x|^2)^2}.$$

An Finsler metric in the form (1.2) is called a square metric.

The object of this paper is to give a formula for flag curvature of homogeneous Finsler space with square metric. The square metric belong to the class of (α,β)-metrics. An (α,β)-metric is a Finsler metric of the form $F=\alpha\phi(s), s=\frac{\beta}{\alpha}$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is induced by a Riemannian metric $\tilde{a}=a_{ij}dx^i\bigotimes dx^j$ on a connected smooth n-manifold M and $\beta=b_i(x)y^i$ is a 1-form on M [6, 13, 18, 19].

2. **Preliminary**

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M, is a non-negative function $F : TM \longrightarrow \mathbb{R}$ with the following properties [4]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM\setminus\{0\}$;
(2) $F(x,\lambda y)=\lambda F(x,y)$ for any $x\in M, y\in T_xM$ and $\lambda>0$;
(3) The $n\times n$ Hessian matrix

$$(g_{ij}) = \Big(\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j}\Big)$$

is positive definite at every point $(x,y)\in TM^0$.

The following bilinear symmetric form $g_y : T_xM\times T_xM \to R$ is positive definite

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2(x,y+su+tv)|_{s=t=0}.$$

Definition 2.1. *Let* $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ *be a Riemannian metric and* $\beta(x,y) = b_i(x)y^i$ *be a* 1*-form on an* n*-dimensional manifold* M*. Let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}$$

Now, let the function F *is defined as follows*

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\phi = \phi(s)$ *is a positive* C^∞ *function on* $-b_0, b_0$ *satisfying*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \leq b < b_0.$$

Then by lemma 1.1.2 of [6]*,* F *is a Finsler metric if* $\|\beta(x)\|_\alpha < b_0$ *for any* $x \in M$*. A Finsler metric in the form (2.1) is called an* (α,β)*-metric* [1]*.*

Let M be a smooth manifold. Suppose that $\tilde{a}$ and β are a Riemannian metric and a 1-form on M respectively. In this case we can write the square metric on M as follows:

$$F = \frac{(\alpha+\beta)^2}{\alpha} = \alpha\phi(s),$$

where $\phi(s) = 1+s^2+2s$. The Riemannian metric $\tilde{a}$ induce a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field X on M such that

$$\tilde{a}(X_x, y) = \beta(x,y).$$

Therefore we can write the square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ as follows:

$$F(x,y) = \frac{(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X_x,y))^2}{\sqrt{\tilde{a}(y,y)}}.$$

3. **Proof of Theorem**

Σ−spaces and reduced Σ−spaces were first introduced by O. Loos as a generalization of reflection spaces and symmetric spaces. First of all, we shall recall some definition and basic results about Σ−spaces. Following O. Loos we have

Definition 3.1. *Let* M *be a smooth connected manifold,* Σ *a Lie group, and* $\mu : M \times \Sigma \times M \longrightarrow M$ *a smooth map. Then the triple* (M,Σ,μ) *is a* $\Sigma-$*space if it satisfies*

- **(Σ_1):** $\mu(x,\sigma,x) = x$,
- **(Σ_2):** $\mu(x,e,y) = y$,
- **(Σ_3):** $\mu(x,\sigma,\mu(x,\tau,y)) = \mu(x,\sigma\tau,y)$
- **(Σ_5):** $\mu(x,\sigma,\mu(y,\tau,z)) = \mu(\mu(x,\sigma,y),\sigma\tau\sigma^{-1},\mu(x,\sigma,z))$

where $x,y,z \in M$*,* $\sigma,\tau \in \Sigma$ *and* e *is the identity element of* Σ*. The triple* (M,Σ,μ) *is usually just replaced by* M*.*

For a fixed point $x \in M$ we define a map $\sigma_x : M \longrightarrow M$ by $\sigma_x(y) = \mu(x, \sigma, y)$ and a map $\sigma^x : M \longrightarrow M$ by $\sigma^x(y) = \sigma_y(x)$. with respect to these maps the above conditions became

($\Sigma_1^{'}$): $\sigma_x(x) = x$,
($\Sigma_2^{'}$): $e_x = id_M$,
($\Sigma_3^{'}$): $\sigma_x\tau_x = (\sigma\tau)_x$
($\Sigma_4^{'}$): $\sigma_x\tau_y\sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y)$.

For each $x \in M$ by Σ_x we denote the image of Σ under the map $\Sigma \longrightarrow \Sigma_x$, $\sigma \longrightarrow \sigma_x$. For each $\sigma \in \Sigma$ we define (1,1) tensor field S^σ on the $\Sigma-$space M by

$$S^\sigma X_x = (\sigma_x)_* X_x \qquad \forall x \in M, X_x \in T_x M.$$

Clearly S^σ is smooth.

Definition 3.2. *A $\Sigma-$space M is a reduced $\Sigma-$space if for each $x \in M$,*

(1) *$T_x M$ is generated by the set of all $\sigma^x(X_x)$, that is*

$$T_x M = gen\{(I - S^\sigma)X_x | X_x \in T_x M, \sigma \in \Sigma\},$$

(2) *If $X_x \in T_x M$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$ then $X_x = 0$, and thus no non-zero vector in $T_x M$ is fixed by all S^σ.*

Definition 3.3. *A Finsler $\Sigma-$space, denoted by (M, Σ, F) is a reduced $\Sigma-$space together with a Finsler metric F which is invariant under Σ_p for $p \in M$.*

Theorem 3.4. *Let (M, Σ, F) be a Finsler $\Sigma-$space with square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then $(M, \Sigma, \tilde{a})$ is a Riemannian Σ space.*

Proof. Let σ_x be a diffeomorphism of (M, F) at x and let $p \in M$. Then for any $Y \in T_p M$ we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y)).$$

Then we have

$$\frac{(\sqrt{\tilde{a}(Y,Y)} + \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y,Y)}} = \frac{(\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} + \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y))^2}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}. \tag{3.1}$$

Applying the above equation to $-Y$, we get

$$\frac{(\sqrt{\tilde{a}(Y,Y)} - \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y,Y)}} = \frac{(\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} - \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y))^2}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}. \tag{3.2}$$

Subtracting equation (3.2) from (3.1) we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y). \tag{3.3}$$

On the other hand, adding equation (3.2) and (3.1), we get

$$\frac{\tilde{a}(Y,Y)+\tilde{a}(X_p,Y)^2}{\sqrt{\tilde{a}(Y,Y)}}=\frac{\tilde{a}(d\sigma_xY,d\sigma_xY)+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)^2}{\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}}. \tag{3.4}$$

By putting (3.3) in (3.4), we get

$$\left(\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}-\sqrt{\tilde{a}(Y,Y)}\right)\left(\tilde{a}(X_p,Y)^2-\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}\sqrt{\tilde{a}(Y,Y)}\right)=0. \tag{3.5}$$

Therefore from (3.5) we have

$$\tilde{a}(ds_xY,ds_xY)=\tilde{a}(Y,Y).$$

Therefore σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$. □

Theorem 3.5. *Let $(M,\Sigma,\tilde{a})$ be a Riemannian Σ space. Also suppose that F is a square Finsler metric defined by $\tilde{a}$ and a vector field X. Then (M,Σ,F) is a square metric Σ space if and only if X is σ_x-invariant for all $x\in M$.*

Proof. Let X be σ_x-invariant. Therefore for any $p\in M$, we have $X_{\sigma_x(p)}=d\sigma_xX_p$. Then for any $Y\in T_pM$ we have

$$\begin{aligned}F(\sigma_x(p),d\sigma_xY_p) &= \frac{(\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY))^2}{\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}}\\ &= \frac{(\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}+\tilde{a}(d\sigma_xX_p,d\sigma_xY))^2}{\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}}\\ &= \frac{(\sqrt{\tilde{a}(Y,Y)}+\tilde{a}(X_p,Y))^2}{\sqrt{\tilde{a}(Y,Y)}}\\ &= F(p,Y).\end{aligned}$$

Conversely, let F be a Σ_M-invariant then for any $p\in M$ and $y\in T_pM$ we have

$$F(p,Y)=F(\sigma_x(p),d\sigma_xY)$$

which implies

$$\left(\tilde{a}(Y,Y)+\tilde{a}(X_p,Y)^2+2\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X_p,Y)\right)\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)} \quad (3.6)$$
$$\left(\tilde{a}(d\sigma_xY,d\sigma_xY)+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)^2+2\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)\right)\sqrt{\tilde{a}(Y,Y)}.$$

Substituting Y with $-Y$ in (3.6), we obtain

$$\left(\tilde{a}(Y,Y)+\tilde{a}(X_p,Y)^2-2\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X_p,Y)\right)\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)} \quad (3.7)$$
$$\left(\tilde{a}(d\sigma_xY,d\sigma_xY)+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)^2-2\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)\right)\sqrt{\tilde{a}(Y,Y)}.$$

Subtracting (3.7) from (3.6) we get

$$\tilde{a}(X_p,Y)=\tilde{a}(X_{\sigma_x(p)},d\sigma_xY).$$

Therefore $(d\sigma_x)_pX_p=X_{\sigma_x(p)}$. □

Theorem 3.6. *A square metric $\Sigma-$space must be Riemannian*

Proof. Let (M, Σ, F) be a Finsler $\Sigma-$space with square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by the Riemannian meric $\tilde{a}$ and the vector field X and let σ_x be a diffeomorphism of (M, F) defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then by the theorem 3.4 $(M, \tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x, d\sigma_x Y) &= \frac{(\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} + \tilde{a}(X_x, d\sigma_x Y))^2}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}} \\ &= \frac{(\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_x, d\sigma_x Y))^2}{\sqrt{\tilde{a}(Y, Y)}} \\ &= F(x, Y). \end{aligned}$$

Therefore $\tilde{a}(X_x, d\sigma_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_x M$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of $T_x M$ without any nonzero fixed vectors. So we have $\tilde{a}(X_x, (S^\sigma - id)_x(y)) = 0$, $\forall y \in T_x M$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

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