Journal of Finsler Geometry and its Applications
Vol. 5, No. 2 (2024), pp 55-61
https://doi.org/10.22098/jfga.2024.15228.1127

Invariant square metrics on reduced $\Sigma-$spaces

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Abstract. In this paper, we study some geometric properties of Finsler $\Sigma-$spaces with square metric. We prove that Finsler $\Sigma-$spaces with square $(\alpha, \beta)-$metrics are Riemannian.

Keywords: (α, β)-metric, square metric, $\Sigma-$space, generalized symmetric space.

1. Introduction

The geometry of invariant Finsler metrics on homogeneous manifolds is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers, for example see [1, 11, 2, 12, 19, 7]. A smooth manifold M with a system of diffeomorphisms $\{s_x\}_{x\in M}$ is said to be a regular $s-$manifold if

(1): $s_x x = x$,
(2): $s_x \circ s_y = s_{s_x y} \circ s_x$,
(3): $(s_x)_{*x} - Id_x$ is invertible.

$\Sigma-$spaces and reduced $\Sigma-$spaces were first introduced by Loos as a generalization of reflection spaces and symmetric spaces [17, 14, 21]. He then proved that any $\Sigma-$space with compact Σ is a fibre bundle over a reduced $\Sigma-$space. The notion of generalized symmetric Finsler space is a natural generalization of

AMS 2020 Mathematics Subject Classification: 53C60, 53C30

generalized Riemannian symmetric spaces [10, 3, 8, 9, 15, 16, 20]. Basic properties of any reduced $\Sigma-$space M and affine, Riemannian and Finsler $\Sigma-$space was given in [17, 14, 21].

In 1929, Berwald construct an interesting family of projectively flat Finsler metrics on the unit ball $\mathbb{B}^n$ which as follows

$$F = \frac{\left(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\right)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}. \tag{1.1}$$

He showed that this class of metrics has constant flag curvature [5]. Berwald's metric can be expressed as

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{1.2}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{(1-|x|^2)^2}, \quad \beta = \frac{\langle x, y\rangle}{(1-|x|^2)^2}.$$

An Finsler metric in the form (1.2) is called a square metric.

The object of this paper is to give a formula for flag curvature of homogeneous Finsler space with square metric. The square metric belong to the class of (α, β)-metrics. An (α, β)-metric is a Finsler metric of the form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is induced by a Riemannian metric $\tilde{a} = a_{ij}dx^i \bigotimes dx^j$ on a connected smooth n-manifold M and $\beta = b_i(x)y^i$ is a 1-form on M [6, 13, 18, 19].

2. **Preliminary**

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M, is a non-negative function $F : TM \longrightarrow \mathbb{R}$ with the following properties [4]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$;
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M, y \in T_xM$ and $\lambda > 0$;
(3) The $n \times n$ Hessian matrix

$$(g_{ij}) = \Big(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\Big)$$

is positive definite at every point $(x, y) \in TM^0$.

The following bilinear symmetric form $g_y : T_xM \times T_xM \to R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}.$$

Definition 2.1. *Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta(x,y) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $-b_0, b_0$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \le b < b_0.$$

Then by lemma 1.1.2 of [6], *F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.1) is called an (α,β)-metric* [1].

Let M be a smooth manifold. Suppose that $\tilde{a}$ and β are a Riemannian metric and a 1-form on M respectively. In this case we can write the square metric on M as follows:

$$F = \frac{(\alpha+\beta)^2}{\alpha} = \alpha\phi(s),$$

where $\phi(s) = 1 + s^2 + 2s$. The Riemannian metric $\tilde{a}$ induce a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field X on M such that

$$\tilde{a}(X_x, y) = \beta(x, y).$$

Therefore we can write the square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ as follows:

$$F(x,y) = \frac{(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X_x, y))^2}{\sqrt{\tilde{a}(y,y)}}.$$

3. Proof of Theorem

$\Sigma-$spaces and reduced $\Sigma-$spaces were first introduced by O. Loos as a generalization of reflection spaces and symmetric spaces. First of all, we shall recall some definition and basic results about $\Sigma-$spaces. Following O. Loos we have

Definition 3.1. *Let M be a smooth connected manifold, Σ a Lie group, and $\mu : M \times \Sigma \times M \longrightarrow M$ a smooth map. Then the triple (M, Σ, μ) is a $\Sigma-$space if it satisfies*

(Σ_1): $\mu(x, \sigma, x) = x,$
(Σ_2): $\mu(x, e, y) = y,$
(Σ_3): $\mu(x, \sigma, \mu(x, \tau, y)) = \mu(x, \sigma\tau, y)$
(Σ_5): $\mu(x, \sigma, \mu(y, \tau, z)) = \mu(\mu(x, \sigma, y), \sigma\tau\sigma^{-1}, \mu(x, \sigma, z))$

where $x, y, z \in M$, $\sigma, \tau \in \Sigma$ and e is the identity element of Σ. The triple (M, Σ, μ) is usually just replaced by M.

For a fixed point $x \in M$ we define a map $\sigma_x : M \longrightarrow M$ by $\sigma_x(y) = \mu(x, \sigma, y)$ and a map $\sigma^x : M \longrightarrow M$ by $\sigma^x(y) = \sigma_y(x)$. with respect to these maps the above conditions became

(Σ_1'): $\sigma_x(x) = x$,
(Σ_2'): $e_x = id_M$,
(Σ_3'): $\sigma_x\tau_x = (\sigma\tau)_x$
(Σ_4'): $\sigma_x\tau_y\sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y)$.

For each $x \in M$ by Σ_x we denote the image of Σ under the map $\Sigma \longrightarrow \Sigma_x$, $\sigma \longrightarrow \sigma_x$. For each $\sigma \in \Sigma$ we define (1,1) tensor field S^σ on the $\Sigma-$space M by

$$S^\sigma X_x = (\sigma_x)_* X_x \qquad \forall x \in M, X_x \in T_xM.$$

Clearly S^σ is smooth.

Definition 3.2. *A $\Sigma-$space M is a reduced $\Sigma-$space if for each $x \in M$,*

(1) *T_xM is generated by the set of all $\sigma^x(X_x)$, that is*

$$T_xM = gen\{(I - S^\sigma)X_x | X_x \in T_xM, \sigma \in \Sigma\},$$

(2) *If $X_x \in T_xM$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$ then $X_x = 0$, and thus no non-zero vector in T_xM is fixed by all S^σ.*

Definition 3.3. *A Finsler $\Sigma-$space, denoted by (M, Σ, F) is a reduced $\Sigma-$space together with a Finsler metric F which is invariant under Σ_p for $p \in M$.*

Theorem 3.4. *Let (M, Σ, F) be a Finsler $\Sigma-$space with square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then $(M, \Sigma, \tilde{a})$ is a Riemannian Σ space.*

Proof. Let σ_x be a diffeomorphism of (M, F) at x and let $p \in M$. Then for any $Y \in T_pM$ we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y)).$$

Then we have

$$\frac{(\sqrt{\tilde{a}(Y,Y)} + \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y,Y)}} = \frac{(\sqrt{\tilde{a}(d\sigma_xY, d\sigma_xY)} + \tilde{a}(X_{\sigma_x(p)}, d\sigma_xY))^2}{\sqrt{\tilde{a}(d\sigma_xY, d\sigma_xY)}}. \tag{3.1}$$

Applying the above equation to $-Y$, we get

$$\frac{(\sqrt{\tilde{a}(Y,Y)} - \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y,Y)}} = \frac{(\sqrt{\tilde{a}(d\sigma_xY, d\sigma_xY)} - \tilde{a}(X_{\sigma_x(p)}, d\sigma_xY))^2}{\sqrt{\tilde{a}(d\sigma_xY, d\sigma_xY)}}. \tag{3.2}$$

Subtracting equation (3.2) from (3.1) we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_xY). \tag{3.3}$$

On the other hand, adding equation (3.2) and (3.1), we get

$$\frac{\tilde{a}(Y,Y)+\tilde{a}(X_p,Y)^2}{\sqrt{\tilde{a}(Y,Y)}}=\frac{\tilde{a}(d\sigma_xY,d\sigma_xY)+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)^2}{\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}}. \tag{3.4}$$

By putting (3.3) in (3.4), we get

$$\left(\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}-\sqrt{\tilde{a}(Y,Y)}\right)\left(\tilde{a}(X_p,Y)^2-\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}\sqrt{\tilde{a}(Y,Y)}\right)=0. \tag{3.5}$$

Therefore from (3.5) we have

$$\tilde{a}(ds_xY,ds_xY)=\tilde{a}(Y,Y).$$

Therefore σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$. □

Theorem 3.5. *Let $(M,\Sigma,\tilde{a})$ be a Riemannian Σ space. Also suppose that F is a square Finsler metric defined by $\tilde{a}$ and a vector field X. Then (M,Σ,F) is a square metric Σ space if and only if X is σ_x-invariant for all $x\in M$.*

Proof. Let X be σ_x-invariant. Therefore for any $p\in M$, we have $X_{\sigma_x(p)}=d\sigma_xX_p$. Then for any $Y\in T_pM$ we have

$$\begin{aligned}
F(\sigma_x(p),d\sigma_xY_p) &= \frac{(\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY))^2}{\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}}\\
&= \frac{(\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}+\tilde{a}(d\sigma_xX_p,d\sigma_xY))^2}{\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}}\\
&= \frac{(\sqrt{\tilde{a}(Y,Y)}+\tilde{a}(X_p,Y))^2}{\sqrt{\tilde{a}(Y,Y)}}\\
&= F(p,Y).
\end{aligned}$$

Conversely, let F be a Σ_M-invariant then for any $p\in M$ and $y\in T_pM$ we have

$$F(p,Y)=F(\sigma_x(p),d\sigma_xY)$$

which implies

$$\begin{aligned}
&\left(\tilde{a}(Y,Y)+\tilde{a}(X_p,Y)^2+2\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X_p,Y)\right)\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)} \qquad (3.6)\\
=&\left(\tilde{a}(d\sigma_xY,d\sigma_xY)+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)^2+2\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)\right)\sqrt{\tilde{a}(Y,Y)}.
\end{aligned}$$

Substituting Y with $-Y$ in (3.6), we obtain

$$\begin{aligned}
&\left(\tilde{a}(Y,Y)+\tilde{a}(X_p,Y)^2-2\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X_p,Y)\right)\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)} \qquad (3.7)\\
=&\left(\tilde{a}(d\sigma_xY,d\sigma_xY)+\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)^2-2\sqrt{\tilde{a}(d\sigma_xY,d\sigma_xY)}\tilde{a}(X_{\sigma_x(p)},d\sigma_xY)\right)\sqrt{\tilde{a}(Y,Y)}.
\end{aligned}$$

Subtracting (3.7) from (3.6) we get

$$\tilde{a}(X_p,Y)=\tilde{a}(X_{\sigma_x(p)},d\sigma_xY).$$

Therefore $(d\sigma_x)_pX_p=X_{\sigma_x(p)}$. □

Theorem 3.6. *A square metric $\Sigma-$space must be Riemannian*

Proof. Let (M, Σ, F) be a Finsler $\Sigma-$space with square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by the Riemannian meric $\tilde{a}$ and the vector field X and let σ_x be a diffeomorphism of (M, F) defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then by the theorem 3.4 $(M, \tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x, d\sigma_x Y) &= \frac{(\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} + \tilde{a}(X_x, d\sigma_x Y))^2}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}} \\ &= \frac{(\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_x, d\sigma_x Y))^2}{\sqrt{\tilde{a}(Y, Y)}} \\ &= F(x, Y). \end{aligned}$$

Therefore $\tilde{a}(X_x, d\sigma_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_xM$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of T_xM without any nonzero fixed vectors. So we have $\tilde{a}(X_x, (S^\sigma - id)_x(y)) = 0$, $\forall y \in T_xM$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. $\square$

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Received: 08.06.2024
Accepted: 28.06.2024