Journal of Finsler Geometry and its Applications
Vol. 5, No. 2 (2024), pp 48-54
https://doi.org/10.22098/jfga.2024.14963.1126

On special weakly M-projective symmetric manifolds

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Abstract: The notion of a weakly symmetric and weakly projective symmetric Riemannian manifolds have been introduced by Tamassy and Binh [11],[12] and then after studied by so many authors such as De, Shaikh and Jana, Shaikh and Hui, Shaikh, Jana and Eyasmin ([1], [3], [4], [5], [6], [7], [8]). Recently, Singh and Khan [10] introduced the notion of Special weakly symmetric Riemannian manifolds and denoted such manifold by $(SWS)_n$. A.U. Khan and Q. Khan found some results On Special Weakly Projective Symmetric Manifolds [13]. And P. Verma, P. Kanaujia and S. Kishor found some results on M-Projective Curvature Tensor on (k, μ)- Contact Space Forms and Sasakian-Space-Forms ([16], [17]) . Motivated from the above, we have studied the nature of Ricci tensor R of type (1,1) in a special weakly M-projective symmetric Riemannian manifold $(SWMS)_n$ and also explored some interesting results on $(SWMS)_n$.

Keywords: M-projective curvature tensor, Curvature Tensor. Ricci tensor, Einstein manifold, Special weakly M-projective symmetric Riemannian manifold, Codazzi type.

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AMS 2020 Mathematics Subject Classification: 53A20, 58A05, 57P05

1. **Introduction**

Let (M^n, g) be a Riemannian manifold of dimension n with the Riemannian metric g and $\zeta(M)$ denote the set of differentiable vector fields on M^n. Let $K(X,Y,Z)$ be the Riemannian curvature tensor of type (1,3) for $X,Y,Z \in \zeta(M)$. A non flat Riemannian manifold (M^n, g), $(n \geq 2)$ is called a special weakly symmetric Riemannian manifold, if its curvature tensor K of type (1,3) satisfies the following condition [10].

$$\begin{aligned}(D_X K)(Y,Z,V) = 2\alpha(X)K(Y,Z,V) + \alpha(Y)K(X,Z,V) \\ +\alpha(Z)K(Y,X,V) + \alpha(V)K(Y,Z,X),\end{aligned} \tag{1.1}$$

where α is a non-zero 1-form and ρ is associated vector field such that

$$\alpha(X) = g(X,\rho) \tag{1.2}$$

for every vector field X and D denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is denoted by $(SWS)_n$. In case, the 1-form α is zero, then $(SWS)_n$ becomes locally symmetric manifold [9]. If we replace K by $\tilde{F}$ in (1.1), then it becomes

$$\begin{aligned}(D_X \tilde{F})(Y,Z,V) = 2\alpha(X)\tilde{F}(Y,Z,V) + \alpha(Y)\tilde{F}(X,Z,V) \\ +\alpha(Z)\tilde{F}(Y,X,V) + \alpha(V)\tilde{F}(Y,Z,X)\end{aligned} \tag{1.3}$$

where $\tilde{F}$ is the M-Projective curvature tensor defined by

$$\begin{aligned}\tilde{F}(Y,Z,V) = K(Y,Z,V) - \frac{1}{4n}\Big[g(Z,V)QY - g(Y,V)QZ \\ +Ric(Z,V)Y - Ric(Y,V)Z\Big]\end{aligned} \tag{1.4}$$

Here Ric is the Ricci tensor of type (0,2). Such an n-dimensional Riemannian manifold shall be called a special weakly M-projective symmetric Riemannian manifold and such a manifold is denoted by $(SWMS)_n$.
If a Riemannian manifold is Einstein, then

$$Ric(X,Y) = \lambda g(X,Y) \tag{1.5}$$

where λ is constant. From (1.5), we have

$$R(X) = \lambda X, \tag{1.6}$$

where R is the Ricci tensor of type (1,1) and is defined by [2]

$$g(R(X),Y) = Ric(X,Y). \tag{1.7}$$

Contracting (1.6) with respect to X, we get

$$r = n\lambda \tag{1.8}$$

where r is a scalar curvature.

The above results will be used in the next section.

2. **Existence of a** $(SWMS)_n$

Let (M^n, g) be a $(SWMS)_n$. Taking covariant derivative of (1.4) with respect to X and then using (1.3), we get

$$2\alpha(X)\tilde{F}(Y,Z,V)+\alpha(Y)\tilde{F}(X,Z,V)+\alpha(Z)\tilde{F}(Y,X,V)+\alpha(V)\tilde{F}(Y,Z,X)$$
$$=(D_XK)(Y,Z,V)-\frac{1}{4n}\Big[(D_XRic)(Z,V)Y-(D_XRic)(Y,V)Z\Big]. \tag{2.1}$$

By virtue of (1.4), the relation (2.1) reduces to

$$(D_XK)(Y,Z,V)-2\alpha(X)K(Y,Z,V)-\alpha(Y)K(X,Z,V)-\alpha(Z)K(Y,X,V)$$
$$-\alpha(V)K(Y,Z,X)-\frac{1}{(4n)}\Bigg[(D_XRic)(Z,V)Y-(D_XRic)(Y,V)X$$
$$-2\alpha(X)\Big\{g(Z,V)QY-g(Y,V)QZ+Ric(Z,V)Y-Ric(Y,V)Z\Big\}$$
$$-\alpha(Y)\Big\{g(Z,V)QX-g(X,V)QZ+Ric(Z,V)X-Ric(X,V)Z\Big\}$$
$$-\alpha(Z)\Big\{g(X,V)QY-g(Y,V)QX+Ric(X,V)Y-Ric(Y,V)X\Big\}$$
$$-\alpha(V)\Big\{g(Z,X)QY-g(Y,X)QZ+Ric(Z,X)Y-Ric(Y,X)Z\Big\}\Bigg]=0. \tag{2.2}$$

Permuting equation (2.2) twice with respect to X, Y, Z; and then adding the three obtained equations and using Bianchi's first and second identities; symmetric property of Ricci tensor and the skew- symmetric properties of curvature tensor, we get

$$\begin{aligned}(D_XRic)(Z,V)Y+(D_YRic)(X,V)Z+(D_ZRic)(Y,V)X\\ -(D_YRic)(Z,V)X-(D_ZRic)(X,V)Y=0.\end{aligned} \tag{2.3}$$

Theorem 2.1. *In a $(SWMS)_n$, the Ricci tensor of type (1,1) is of Codazzi type.*

Proof. Contracting (2.3) with respect to X, we get

$$(D_ZRic)(Y,V)-(D_YRic)(Z,V)=0 \tag{2.4}$$

Consequently in view of (1.7), the relation (2.4) gives

$$(D_ZR)(Y)-(D_YR)(Z)=0. \tag{2.5}$$

(2.5) shows that the Ricci tensor of type (1,1) is of Codazzi type. □

Theorem 2.2. *In a $(SWMS)_n$, the scalar curvature r is constant.*

Proof. Contracting (2.5) with respect to Y, we get

$$Z\ r=0$$

which shows that the scalar curvature r is constant. □

Theorem 2.3. *The necessary and sufficient condition for an Einstein* $(SWMS)_n$ *to be a* $(SWS)_n$ *is that*
$[2\alpha(X)QY + \lambda Y + \alpha(Y)QX + \lambda X]g(Z,V) - [2\alpha(X)QZ + \lambda Z + \alpha(Z)QX + \lambda X]g(Y,V) + [\alpha(Z)QY + \lambda Y - \alpha(Y)QZ - \lambda Z]g(X,V) + \alpha(V)[g(Z,X)QY - g(Y,X)QZ + \lambda g(Z,X)Y - \lambda g(Y,X)Z] = 0.$

Proof. By virtue of (1.5), the equation (1.4) reduces to the form

$$\tilde{F}(Y,Z,V) = K(Y,Z,V) - \frac{1}{(4n)}\Big[g(Z,V)QY - g(Y,V)QZ + \lambda\{g(Z,V)Y - g(Y,V)Z\}\Big]. \tag{2.6}$$

Taking covariant derivative of (2.6) with respect to X, we get

$$(D_X\tilde{F})(Y,Z,V) = (D_XK)(Y,Z,V). \tag{2.7}$$

Using (1.3) in (2.7), we get

$$(D_XK)(Y,Z,V) = 2\alpha(X)\tilde{F}(Y,Z,V) + \alpha(Y)\tilde{F}(X,Z,V) + \alpha(Z)\tilde{F}(Y,X,V) + \alpha(V)\tilde{F}(Y,Z,X). \tag{2.8}$$

By virtue of (2.6), the relation (2.8) reduces to the form

$$\begin{aligned}(D_XK)(Y,Z,V) = {} & 2\alpha(X)\Big[K(Y,Z,V) - \frac{1}{(4n)}\Big\{g(Z,V)QY - g(Y,V)QZ \\ & + \lambda\{g(Z,V)Y - g(Y,V)Z\}\Big\}\Big] + \alpha(Y)\Bigg[K(X,Z,V) \\ & - \frac{1}{(4n)}\Big\{g(Z,V)QX - g(X,V)QZ + \lambda\{g(Z,V)X - g(X,V)Z\}\Big\}\Bigg] \\ & + \alpha(Z)\Bigg[K(Y,X,V) - \frac{1}{(4n)}\Big\{g(X,V)QY - g(Y,V)QX \\ & + \lambda\{g(X,V)Y - g(Y,V)X\}\Big\}\Bigg] + \alpha(V)\Bigg[K(Y,Z,X) \\ & - \frac{1}{(4n)}\Big\{g(Z,X)QY - g(Y,X)QZ + \lambda\{g(Z,X)Y - g(Y,X)Z\}\Big\}\Bigg].\end{aligned} \tag{2.9}$$

This completes the proof. □

3. Manifold satisfying $\tilde{F}(Y,Z,V) = 0$

Let (M^n, g) be a M-projectively flat, that is, $\tilde{F}(Y,Z,V) = 0$, then the relation (1.4) reduces to

$$K(Y,Z,V) = \frac{1}{4n}\Big[g(Z,V)QY - g(Y,V)QZ + Ric(Z,V)Y - Ric(Y,V)Z\Big]. \tag{3.1}$$

Taking covariant derivative of (3.1) with respect to X, we have

$$(D_X K)(Y, Z, V) = \frac{1}{4n}\Big[(D_X Ric)(Z, V)Y - (D_X Ric)(Y, V)Z\Big]. \tag{3.2}$$

Permuting equation (3.2) twice with respect to X,Y,Z; and then adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}(D_X Ric)(Z, V)Y + (D_Y Ric)(X, V)Z + (D_Z Ric)(Y.V)X \\ -(D_X Ric)(Y, V)Z - (D_Y Ric)(Z, V)X - (D_Z Ric)(X, V)Y = 0.\end{aligned} \tag{3.3}$$

Theorem 3.1. *In a M-projectively flat Riemannian manifold, the Ricci tensor R of type (1,1) is of Codazzi type.*

Proof. Contracting (3.3) with respect to X, we have

$$(D_Z Ric)(Y, V) - (D_Y Ric)(Z, V) = 0. \tag{3.4}$$

Consequently in view of (1.7), the relation (3.4) gives

$$(D_Z R)(Y) - (D_Y R)(Z) = 0. \tag{3.5}$$

This completes the proof. □

Definition 3.2. *An n-dimensional Riemannian manifold is called a special weakly Ricci symmetric manifold* [10] *if the Ricci tensor Ric of type (0,2) satisfies the following condition:*

$$(D_X Ric)(Y, Z) = 2\alpha(X)Ric(Y, Z) + \alpha(Y)Ric(X, Z) + \alpha(Z)Ric(Y, X), \tag{3.6}$$

where α is a non-zero 1-form. Such a manifold is denoted by $(SWRS)_n$. Now using (3.6) *in* (3.3)*, we have*

$$\begin{aligned}\alpha(X)Ric(Z, V)Y + \alpha(Y)Ric(X, V)Z + \alpha(Z)Ric(Y, V)X \\ -\alpha(X)Ric(Y, V)Z - \alpha(Y)Ric(Z, V)X - \alpha(Z)Ric(X, V)Y = 0.\end{aligned} \tag{3.7}$$

Theorem 3.3. *In a M-projectively flat $(SWRS)_n$, 1-form α is collinear with the Ricci tensor R.*

Proof. Contracting (3.7) with respect to X, we have

$$\alpha(Z)Ric(Y, V) - \alpha(Y)Ric(Z, V) = 0. \tag{3.8}$$

which in veiw of (1.7) gives

$$\alpha(Z)g(R(Y), V) - \alpha(Y)g(R(Z), V) = 0. \tag{3.9}$$

Consequently the above relation turns into

$$\alpha(Z)R(Y) - \alpha(Y)R(Z) = 0. \tag{3.10}$$

□

Theorem 3.4. *In a $(SWMS)_n$, if a Riemannian manifold is a $(SWRS)_n$, the 1-form α is collinear with the Ricci tensor R. of type (1,1).*

Proof. Taking covariant derivative of (1.4) with respect to X, we have

$$(D_X\tilde{F})(Y,Z,V) = (D_XK)(Y,Z,V) - \frac{1}{4n}\Big[(D_XRic)(Z,V)Y - (D_XRic)(Y,V)Z\Big]. \qquad (3.11)$$

Permutating equation (3.11) twice with respect to X,Y,Z; and then adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}&(D_X\tilde{F})(Y,Z,V) + (D_Y\tilde{F})(Z,X,V) + (D_Z\tilde{F})(X,Y,V)\\ &= -\frac{1}{4n}\Big[(D_XRic)(Z,V)Y - (D_XRic)(Y,V)Z + (D_YRic)(X,V)Z\\ &\quad -(D_YRic)(Z,V)X + (D_ZRic)(Y,V)X - (D_ZRic)(X,V)Y\Big].\end{aligned} \qquad (3.12)$$

Using (1.3) and (3.6) in and taking in mind the skew- symmetric of $\tilde{F}(X,Y,Z)$, cyclic property of $\tilde{F}(X,Y,Z)$ and symmetric property of Ricci tensor of type (0,2), we have

$$\begin{aligned}&\alpha(X)Ric(Z,V)Y - \alpha(X)Ric(Y,V)Z + \alpha(Y)Ric(X,V)Z\\ &-\alpha(Y)Ric(Z,V)X + \alpha(Z)Ric(Y,V)X - \alpha(Z)Ric(X,V)Y = 0\end{aligned} \qquad (3.13)$$

Contracting (3.13) with respect to X, we have

$$(n-2)\alpha(Z)Ric(Y,V) - (n-2)\alpha(Y)Ric(Z,V) = 0, \qquad (3.14)$$

which in view of (1.7) the relation (3.14) gives

$$\alpha(Z)g(R(Y),V) - \alpha(Y)g(R(Z),V) = 0. \qquad (3.15)$$

Consequently the relation (3.15) gives

$$\alpha(Z)R(Y) - \alpha(Y)R(Z) = 0. \qquad (3.16)$$

□

Acknowledgment: The authors are grateful to Professor U.C. De for his continuous support and encouragement.

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Received: 26.04.2024
Accepted: 07.07.2024