


Special projective algebra of exponential metrics of isotropic S -curvature

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Abstract. Exponential metrics are popular Finsler metrics. Let F be an exponential (α, β) -metric of isotropic S -curvature on manifold M . In this paper, a Lie sub-algebra of projective vector fields of a Finsler metric F is introduced and denoted by $SP(F)$. We classify $SP(F)$ of isotropic S -curvature as a certain Lie sub-algebra of the Killing algebra $K(M, \alpha)$.

Keywords: Projective vector field, Exponential Finsler metric, S -curvature.

1. Introduction

The projective Finsler metrics are smooth solutions to the historical Hilbert's fourth problem. The projective vector fields are a way to characterize the projective metrics. The collection of all projective vector fields on a Finsler space is a finite-dimensional Lie algebra with respect to the usual Lie bracket, called the projective algebra denoted by $p(M, F)$. The collection of all projective

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vector fields on a Finsler space $p(M, F)$ is a finite-dimensional Lie algebra with respect to the usual Lie bracket, called the projective algebra. A specific Lie sub-algebra of projective algebra of Finsler spaces, called the special projective algebra and denoted by $SP(F)$.

In [8], Rafie-Rad studied on the projective vector fields on the class of Randers metrics and introduced Lie sub-algebra of projective vector fields of a Finsler metric. In [4], B. Rezaei and M.Rafie-Rad studied the projective algebra of some (α, β) -metrics of isotropic S -curvature. In [10], the auther show that if the Matsumoto metric admits a projective vector field, then this is a conformal vector field with to Riemannian metric α or F has vanishing S -curvature.

In this paper, we characterize the special projective vector field V on manifold M with exponential metric of isotropic S -curvature. We prove the following theorem:

Theorem 1.1. *Let $(M, F = \alpha e^{\beta/\alpha})$ be exponential metric of isotropic S -curvature on a manifold and $b := \|\beta\|_\alpha$ is constant. Then one of the following statements holds:*

- (a) β is parallel with respect to α and the projective algebra $p(M, F)$ of F is coincides with the projective algebra $p(M, \alpha)$ of α .
- (b) Every special projective vector field V on (M, F) is an Killing vector field on (M, α) and $\mathcal{L}_{\hat{V}}\beta = 0$.

2. Preliminaries

Let F be a Finsler metric on an n -dimensional manifold M . It induces a spray G on TM . In local coordinates in TM , it is expressed by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ $\lambda > 0$. Assume the following conventions:

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}, \quad G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}.$$

The local functions G^i_{jk} give rise to a torison-free connection in π^*TM called the berwald connection which is this paper, see [5].

Let

$$\alpha(\mathbf{y}) := \sqrt{g_{ij}(x)y^i y^j}, \quad \mathbf{y} = y^i \frac{\partial}{\partial x^i} \Big|_x \in T_x M.$$

α is a family of Euclidean norms on tangent spaces. Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ a 1-form on a manifold M . An (α, β) -metric is a scalar function F on TM defined by $F := \alpha \phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity such that F is a positive

definite Finsler metric. A special (α, β) -metric defined by $\phi(s) = e^s$ is called exponential metric.

Denote the Levi-Civita connection of α by ∇ and define $b_{i|j}$ by $(b_{i|j})\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$, $\theta_i^j := \Gamma_{ik}^j dx^k$.

In order to study the geometric properties of (α, β) -metrics, one needs a formula for the spray coefficients of an (α, β) -metrics. Let

$$\begin{aligned} r_{ij} &= (\nabla_j b_i + \nabla_i b_j)/2, & s_{ij} &= (\nabla_j b_i - \nabla_i b_j)/2, & r^i_j &:= a^{ik} r_{kj}, \\ r_{\circ\circ} &:= r_{ij} y^i y^j, & r_{i\circ} &:= r_{ij} y^j, & s^i_j &:= a^{ik} s_{kj}, \\ s_j &:= b^i s_{ij}, & s_\circ &:= s_i y^i, & s_{i\circ} &:= s_{ij} y^j. \end{aligned}$$

The spray coefficients G^i of F and G_α^i of α are related as follows:

$$G^i = G_\alpha^i + \alpha Q s^i_\circ + \alpha^{-1} \Theta \{r_{\circ\circ} - 2\alpha Q s_\circ\} y^i + \Psi \{r_{\circ\circ} - 2\alpha Q s_\circ\} b^i, \quad (2.1)$$

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, & \Theta &= \frac{\phi\phi' - s(\phi\phi'' - \phi'\phi')}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}, \\ \Psi &= \frac{\phi''}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}. \end{aligned}$$

There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form on manifold, i.e., $dV_F = \sigma_F(x) dx^1 \cdots dx^n$, which is defined by

$$\tau(\mathbf{y}) := \ln \left[\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)} \right],$$

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left\{(y^i) \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1\right\}}. \quad (2.2)$$

For a vector $\mathbf{y} \in T_x M$. Let $c(t)$, $-\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. Define

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt} \left[\tau(\dot{c}(t)) \right] \Big|_{t=0}.$$

We say S -curvature is isotropic if there exists a scalar function $c(x)$ on M such that $S(x, y) = (n+1)c(x)F(x, y)$, and constant S -curvature if $c(x) = \text{constant}$, see [2, 6, 7].

Let $G^i(x, y)$ denote the geodesic coefficients of F in the same local coordinate system. By the definition of the S -curvature, we have

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[\ln \sigma_F(x) \right], \quad (2.3)$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \Big|_x \in T_x M$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [5]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$. To prove the Theorem 1.1, we need the following theorem which is proved in [3].

Theorem 2.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a (α, β) -metric on a manifold of dimension n and $b := \|\beta\|_\alpha$ is constant. Suppose that*

$$\phi \neq k_1\sqrt{1 + k_2s^2} + k_3s,$$

for any constant $k_1 > 0$, k_2 and k_3 . Then F is of isotropic S -curvature, $S = (n + 1)cF$, if and only if one of the following holds:

(i) β satisfies

$$r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j), s_j = 0 \quad (2.4)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n + 1)k \frac{\phi\Delta^2}{b^2 - s^2}$$

where k is a constant. In this case, $S = (n + 1)cF$ with $c = k\varepsilon$.

(ii) β satisfies

$$r_{ij} = 0, s_j = 0 \quad (2.5)$$

In this case, $S = 0$, regardless of the choice of a particular ϕ .

3. Projective vector fields on Finsler spaces

Every vector field X on M induces naturally a transformation under the following infinitesimal coordinate transformations on TM , $(x^i, y^i) \rightarrow (\bar{x}^i, \bar{y}^i)$ given by

$$\bar{x}^i = x^i + V^i dt, \quad \bar{y}^i = y^i + y^k \frac{\partial V^i}{\partial x^k} dt.$$

This leads us to the notion of the complete lift \hat{V} (see [9]) of V to a vector field on TM_0 given by

$$\hat{V} = V^i \frac{\partial}{\partial x^i} + y^k \frac{\partial V^i}{\partial x^k} \frac{\partial}{\partial y^i}.$$

Almost any geometric object in Finsler geometry depends on the both points and velocities, hence the Lie derivatives of such geometric objects should rather be regarded with respect to \hat{V} . For computational use, it is known $\mathcal{L}_{\hat{V}}y^i = 0$, $\mathcal{L}_{\hat{V}}dx^i = 0$ and the differential operators $\mathcal{L}_{\hat{V}}$, $\frac{\partial}{\partial x^i}$, exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute as well. The vector field V is called a projective vector field, if there is a function P on TM_0 such that

$$\mathcal{L}_{\hat{V}}G^i_k = P\delta^i_k + P_k y^i,$$

where $P_k = P_{,k}$, see [1]. Thereby, given any appropriate t , the local flow $\{\phi_t\}$ associated to V is projective transformation. If V is a projective vector field, then [1]:

$$\begin{aligned}\mathcal{L}_{\hat{V}}G^i &= Py^i, \\ \mathcal{L}_{\hat{V}}G^i_{jk} &= \delta^i_j P_k + \delta^i_k P_j + y^i P_{k,j}, \\ \mathcal{L}_{\hat{V}}G^i_{jkl} &= \delta^i_j P_{k,l} + \delta^i_k P_{j,l} + \delta^i_l P_{k,j} + y^i P_{k,j,l}, \\ 2\mathcal{L}_{\hat{V}}\mathbf{E}_{jl} &= (n+1)P_{j,l}.\end{aligned}$$

On the Riemannian spaces, given any projective vector field V the function $P = P(x, y)$ is linear with respect to y . A projective vector field V is called a *special projective vector field* if $\mathcal{L}_{\hat{V}}\mathbf{E} = 0$, equivalently, $P(x, y) = P_i(x)y^i$.

Remark 3.1. *On a weakly-Berwald space, every projective vector field is special.*

4. Proof of Theorem 1.1

Let $F = \alpha e^s$, $s := \beta/\alpha$ be exponential Finsler metric of isotropic S -curvature on a manifold M and $b := \|\beta\|_\alpha$ is constant. According to theorem 2.1, F is of isotropic S -curvature, $S = (n+1)cF$, if and only if β satisfies $r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j)$, $s_j = 0$ or $r_{ij} = 0$, $s_j = 0$. Plugging $r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j)$, $s_j = 0$ in (2.1) the geodesic coefficients of F can be calculated by

$$G^i = G^i_\alpha + \frac{\alpha^2}{\alpha - \beta} s^i_\circ + \frac{\varepsilon(b^2 \alpha^3 - \beta^2 \alpha) y^i}{2b^2 \alpha^2 - 2\beta^2 - 2\beta \alpha + 2\alpha^2} e^s + \frac{\varepsilon(b^2 \alpha^3 - \beta^2 \alpha) b^i}{2b^2 \alpha^2 - 2\beta^2 - 2\beta \alpha + 2\alpha^2} \alpha. \quad (4.1)$$

Assuming $s^i_\circ = 0$, equation (4.1) can be seen as follows:

$$G^i = G^i_\alpha + \frac{\varepsilon(b^2 \alpha^3 - \beta^2 \alpha) y^i}{2b^2 \alpha^2 - 2\beta^2 - 2\beta \alpha + 2\alpha^2} e^s + \frac{\varepsilon(b^2 \alpha^3 - \beta^2 \alpha) b^i}{2b^2 \alpha^2 - 2\beta^2 - 2\beta \alpha + 2\alpha^2} \alpha. \quad (4.2)$$

Let us suppose that V is a projective vector field on (M, F) . By assuming, V is a special projective field, that is to exists a function P of the form $P(x, y) = P_k(x)y^k$ on M such that

$$\mathcal{L}_{\hat{V}}G^i = Py^i.$$

If $s^i_\circ = 0$, by (4.2) we can write this equation as follows

$$\mathcal{L}_{\hat{V}}(G^i_\alpha + \frac{\varepsilon(b^2 \alpha^3 - \beta^2 \alpha) y^i}{2b^2 \alpha^2 - 2\beta^2 - 2\beta \alpha + 2\alpha^2} e^s + \frac{\varepsilon(b^2 \alpha^3 - \beta^2 \alpha) b^i}{2b^2 \alpha^2 - 2\beta^2 - 2\beta \alpha + 2\alpha^2} \alpha) = Py^i.$$

Therefore, Equation mentioned above is equivalent to the following equality

$$\begin{aligned}
0 &= -Py^i + \mathcal{L}_{\hat{\nu}}G_{\alpha}^i + \frac{\varepsilon(e^s y^i + \alpha b^i)}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \mathcal{L}_{\hat{\nu}}(b^2\alpha^3 - \beta^2\alpha) \\
&+ \frac{\varepsilon(e^s y^i + \alpha b^i)(b^2\alpha^3 - \beta^2\alpha)}{(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2} \mathcal{L}_{\hat{\nu}}(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2) \\
&+ \frac{\varepsilon(b^2\alpha^3 - \beta^2\alpha)}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \mathcal{L}_{\hat{\nu}}(e^s y^i + \alpha b^i).
\end{aligned}$$

Let us denote

$$t_{\infty} = \mathcal{L}_{\hat{\nu}}\alpha^2.$$

By simplifying above equation and multiplying both sides of this very equation by $\alpha^3(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2$, we can rewrite (4.3) as follows:

$$K(x, y)\alpha + R(x, y)e^s = 0 \quad (4.3)$$

where

$$\begin{aligned}
K(x, y) &= \alpha^8(2b^i\varepsilon\mathcal{L}_{\hat{\nu}}b^2 + 2b^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^4\varepsilon\mathcal{L}_{\hat{\nu}}b^i) \\
&+ \alpha^7(2b^2b^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 2\beta\varepsilon b^i\mathcal{L}_{\hat{\nu}}b^2 - 2\beta\varepsilon b^2\mathcal{L}_{\hat{\nu}}b^i) \\
&+ \alpha^6(-4Py^i + 4\mathcal{L}_{\hat{\nu}}G_{\alpha}^i - 8b^2Py^i + 8b^2\mathcal{L}_{\hat{\nu}}G_{\alpha}^i - 4b^4Py^i \\
&+ 4b^4\mathcal{L}_{\hat{\nu}}G_{\alpha}^i - 2\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i - 4b^2\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^4b^i\varepsilon t_{\infty} \\
&+ 2b^2\varepsilon t_{\infty}b^i - 4b^i\beta\varepsilon\mathcal{L}_{\hat{\nu}}\beta) \\
&+ \alpha^5(8\beta Py^i - 8\beta\mathcal{L}_{\hat{\nu}}G_{\alpha}^i + 8\beta b^2Py^i - 8\beta b^2\mathcal{L}_{\hat{\nu}}G_{\alpha}^i \\
&+ 2b^i\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 3\beta b^2b^i\varepsilon t_{\infty} + 2\beta^3\varepsilon\mathcal{L}_{\hat{\nu}}b^i) \\
&+ \alpha^4(4\beta^2Py^i - 4\beta^2\mathcal{L}_{\hat{\nu}}G_{\alpha}^i + 8b^2\beta^2Py^i \\
&- 8b^2\beta^2\mathcal{L}_{\hat{\nu}}G_{\alpha}^i + 2\beta^4\varepsilon\mathcal{L}_{\hat{\nu}}b^i - 4b^i\beta^2b^2\varepsilon t_{\infty}) \\
&+ \alpha^3(-8\beta^3Py^i + 8\beta^3\mathcal{L}_{\hat{\nu}}G_{\alpha}^i + b^i\beta^3\varepsilon t_{\infty}) \\
&+ \alpha^2(-4\beta^4Py^i + 4\beta^4\mathcal{L}_{\hat{\nu}}G_{\alpha}^i + 2b^i\varepsilon\beta^4 t_{\infty}). \\
R(x, y) &= \alpha^8(2y^i\varepsilon\mathcal{L}_{\hat{\nu}}b^2) + \alpha^7(2y^i b^4\varepsilon\mathcal{L}_{\hat{\nu}}\beta + 4b^2y^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 2y^i\varepsilon\beta\mathcal{L}_{\hat{\nu}}b^2) \\
&+ \alpha^6(-2\beta b^2\varepsilon y^i\mathcal{L}_{\hat{\nu}}\beta + b^4\varepsilon y^i t_{\infty} + b^2\varepsilon y^i t_{\infty} - 4y^i\beta\varepsilon\mathcal{L}_{\hat{\nu}}\beta) \\
&+ \alpha^5(-b^4\beta\varepsilon y^i t_{\infty} - 4b^2y^i\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 3\beta b^2y^i\varepsilon t_{\infty}) \\
&+ \alpha^4(-b^2\beta^2\varepsilon y^i t_{\infty} + 2y^i\beta^3\varepsilon\mathcal{L}_{\hat{\nu}}\beta + \beta^2y^i\varepsilon t_{\infty}) \\
&+ \alpha^3(2\beta^4y^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta + 2b^2\beta^3\varepsilon y^i t_{\infty} + y^i\beta^3\varepsilon t_{\infty}) \\
&+ \alpha^1(-\beta^5\varepsilon y^i t_{\infty}).
\end{aligned}$$

By changing all the terms y to $-y$ in (4.3) we obtain $R(x, y) = K(x, y) = 0$. Equation $R(x) = 0$ is equivalent to following polynomial equation:

$$a_8\alpha^8 + a_7\alpha^7 + a_6\alpha^6 + a_5\alpha^5 + a_4\alpha^4 + a_3\alpha^3 + a_1\alpha^1 = 0 \quad (4.4)$$

where

$$\begin{aligned}
a_8 &= 2y^i \varepsilon \mathcal{L}_{\hat{\nu}} b^2, \\
a_7 &= 2y^i b^4 \varepsilon \mathcal{L}_{\hat{\nu}} \beta + 4b^2 y^i \varepsilon \mathcal{L}_{\hat{\nu}} \beta - 2y^i \varepsilon \beta \mathcal{L}_{\hat{\nu}} b^2, \\
a_6 &= -2\beta b^2 \varepsilon y^i \mathcal{L}_{\hat{\nu}} \beta + b^4 \varepsilon y^i t_{\infty} + b^2 \varepsilon y^i t_{\infty} - 4y^i \beta \varepsilon \mathcal{L}_{\hat{\nu}} \beta, \\
a_5 &= -b^4 \beta \varepsilon y^i t_{\infty} - 4b^2 y^i \beta^2 \varepsilon \mathcal{L}_{\hat{\nu}} \beta - 3\beta b^2 y^i \varepsilon t_{\infty}, \\
a_4 &= -b^2 \beta^2 \varepsilon y^i t_{\infty} + 2y^i \beta^3 \varepsilon \mathcal{L}_{\hat{\nu}} \beta + \beta^2 y^i \varepsilon t_{\infty}, \\
a_3 &= 2\beta^4 y^i \varepsilon \mathcal{L}_{\hat{\nu}} \beta + 2b^2 \beta^3 \varepsilon y^i t_{\infty} + y^i \beta^3 \varepsilon t_{\infty}, \\
a_1 &= -\beta^5 \varepsilon y^i t_{\infty}.
\end{aligned}$$

From above equation, we can get two fundamental equations

$$\begin{aligned}
a_8 \alpha^8 + a_6 \alpha^6 + a_4 \alpha^4 &= 0, \\
a_7 \alpha^6 + a_5 \alpha^4 + a_3 \alpha^2 + a_1 \alpha^0 &= 0.
\end{aligned} \tag{4.5}$$

From (4.5), we can see that a_1 has the factor α^2 and then

$$t_{\infty} = c^i(x) \alpha^2$$

for some scalar function $c^i(x)$ on M .

By the equation mentioned above we can conclude that the coefficient a_4 must be divided by α^2 , hence there is a class of homogenous of degree one functions $g^i = g^i(y)$ on M such that,

$$-b^2 \varepsilon y^i t_{\infty} + 2y^i \beta \varepsilon \mathcal{L}_{\hat{\nu}} \beta + y^i \varepsilon t_{\infty} = g^i(y) \alpha^2 \tag{4.6}$$

Replacing this quantity $t_{\infty} = c^i(x) \alpha^2$ into (4.6) and taking into account the non-degeneracy of $\varepsilon, \beta \neq 0$ we conclude that

$$\mathcal{L}_{\hat{\nu}} \beta = 0.$$

Plugging the quantities $t_{\infty} = c^i(x) \alpha^2$, $\mathcal{L}_{\hat{\nu}} \beta = 0$ in $R(x) = 0$ and sorting again by α , we can get the following equation

$$m_8 \alpha^8 + m_7 \alpha^7 + m_6 \alpha^6 + m_5 \alpha^5 + m_3 \alpha^3 = 0. \tag{4.7}$$

where

$$\begin{aligned}
m_8 &= 2\varepsilon y^i \mathcal{L}_{\hat{\nu}} b^2 + \varepsilon b^4 y^i c^i(x) + \varepsilon b^2 y^i c^i(x), \\
m_7 &= -2y^i \beta \varepsilon \mathcal{L}_{\hat{\nu}} b^2 - \varepsilon \beta y^i b^4 c^i(x) - 3\varepsilon \beta y^i b^2 c^i(x), \\
m_6 &= \beta^2 \varepsilon y^i c^i(x) - \varepsilon \beta^2 y^i b^2 c^i(x), \\
m_5 &= \beta^3 \varepsilon c^i(x) y^i + 2b^2 \beta^3 \varepsilon c^i(x) y^i, \\
m_3 &= -\beta^5 \varepsilon y^i c^i(x).
\end{aligned}$$

From above equation, we can get two fundamental equations

$$\begin{aligned}
m_8 \alpha^6 + m_6 \alpha^4 &= 0, \\
m_7 \alpha^4 + m_5 \alpha^2 + m_3 \alpha^0 &= 0.
\end{aligned} \tag{4.8}$$

From (4.8), we see that m_3 has the factor α^2 and taking into account the non-degeneracy of $\varepsilon, \beta \neq 0$ we conclude that

$$c^i(x) = 0, \quad \text{for any index } i.$$

Therefore $t_{\infty} = 0$.

If we assume that $s_{\circ}^i \neq 0$, by (4.3) we can write equation (4.1) as follows

$$\begin{aligned} \mathcal{L}_{\hat{V}}(G_{\alpha}^i + \frac{\alpha^2}{\alpha - \beta} s_{\circ}^i + \frac{\varepsilon(b^2\alpha^3 - \beta^2\alpha)y^i}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} e^s \\ + \frac{\varepsilon(b^2\alpha^3 - \beta^2\alpha)b^i}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \alpha) = Py^i. \end{aligned}$$

Therefore, Equation mentioned above is equivalent to the following equality

$$\begin{aligned} 0 &= \mathcal{L}_{\hat{V}}G_{\alpha}^i - Py^i + \left(\frac{t_{\infty}}{\alpha - \beta} - \frac{\alpha t_{\infty}}{2(\alpha - \beta)^2} + \frac{\alpha^2 \mathcal{L}_{\hat{V}}\beta}{(\alpha - \beta)^2}\right) s_{\circ}^i \\ &+ \frac{\alpha^2}{\alpha - \beta} \mathcal{L}_{\hat{V}}s_{\circ}^i + \frac{\varepsilon(e^s y^i + \alpha b^i)}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \mathcal{L}_{\hat{V}}(b^2\alpha^3 - \beta^2\alpha) \\ &+ \frac{\varepsilon(e^s y^i + \alpha b^i)(b^2\alpha^3 - \beta^2\alpha)}{(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2} \mathcal{L}_{\hat{V}}(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2) \\ &+ \frac{\varepsilon(b^2\alpha^3 - \beta^2\alpha)}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \mathcal{L}_{\hat{V}}(e^s y^i + \alpha b^i). \end{aligned} \quad (4.9)$$

By simplifying above equation and multiplying both sides of this very equation by $\alpha^2(\alpha - \beta)^2(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2$, we can rewrite (4.9) as follows:

$$L(x, y)\alpha + D(x, y)e^s = 0 \quad (4.10)$$

where

$$\begin{aligned}
L(x, y) = & \alpha^9(2b^4\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^i\varepsilon\mathcal{L}_{\hat{\nu}}b^2) \\
& + \alpha^8(4\mathcal{L}_{\hat{\nu}}s_o^i + 4b^4\mathcal{L}_{\hat{\nu}}s_o^i + 8b^2\mathcal{L}_{\hat{\nu}}s_o^i - 4b^4\beta\varepsilon\mathcal{L}_{\hat{\nu}}b^i \\
& + 2b^2b^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 6b^2\beta\varepsilon\mathcal{L}_{\hat{\nu}}b^i - 6b^i\beta\varepsilon\mathcal{L}_{\hat{\nu}}b^2) \\
& + \alpha^7(-4Py^i + 4\mathcal{L}_{\hat{\nu}}G_\alpha^i - 4b^4\beta\mathcal{L}_{\hat{\nu}}s_o^i - 2\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i \\
& + 4b^4s_o^i\mathcal{L}_{\hat{\nu}}\beta - 16b^2\beta\mathcal{L}_{\hat{\nu}}s_o^i + 8b^2s_o^i\mathcal{L}_{\hat{\nu}}\beta + 6b^i\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}b^2 \\
& - 4b^i\beta\varepsilon\mathcal{L}_{\hat{\nu}}\beta + 2b^4\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^2\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^2b^i\varepsilon t_{oo} \\
& + 2b^4b^i\varepsilon t_{oo} + 8b^2\mathcal{L}_{\hat{\nu}}G_\alpha^i - 8b^2Py^i + 4b^4\mathcal{L}_{\hat{\nu}}G_\alpha^i \\
& - 4b^4Py^i + 4s_o^i\mathcal{L}_{\hat{\nu}}\beta - 12\beta\mathcal{L}_{\hat{\nu}}s_o^i - 4b^2b^i\beta\varepsilon\mathcal{L}_{\hat{\nu}}\beta) \\
& + \alpha^6(6\beta^3\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 2b^4s_o^it_{oo} + 4b^2s_o^it_{oo} - 8\beta b^4\mathcal{L}_{\hat{\nu}}G_\alpha^i \\
& + 8\beta b^4Py^i - 24\beta b^2\mathcal{L}_{\hat{\nu}}G_\alpha^i + 24\beta b^2Py^i - 8\beta s_o^i\mathcal{L}_{\hat{\nu}}\beta \\
& + 6b^2\beta^3\varepsilon\mathcal{L}_{\hat{\nu}}b^i - 2b^i\beta^3\varepsilon\mathcal{L}_{\hat{\nu}}b^2 + 10b^i\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}\beta \\
& - 8b^2\beta s_o^i\mathcal{L}_{\hat{\nu}}\beta - 16\beta\mathcal{L}_{\hat{\nu}}G_\alpha^i + 16\beta Py^i + 4\beta^2\mathcal{L}_{\hat{\nu}}s_o^i \\
& + 2s_o^it_{oo} - 4\beta b^it_{oo}b^4\varepsilon - 7\beta b^2\varepsilon b^it_{oo} + 2b^ib^2\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}\beta) \\
& + \alpha^5(-4b^4\beta^2Py^i - 16b^2\beta^2Py^i + 8b^2\beta^3\mathcal{L}_{\hat{\nu}}s_o^i + 16b^2\beta^2\mathcal{L}_{\hat{\nu}}G_\alpha^i \\
& - 8s_o^it_{oo}\beta + 4b^4\beta^2\mathcal{L}_{\hat{\nu}}G_\alpha^i - 4\beta^2s_o^i\mathcal{L}_{\hat{\nu}}\beta - 4\beta^4\varepsilon\mathcal{L}_{\hat{\nu}}b^i \\
& - 4\beta b^4s_o^it_{oo} - 12\beta b^2s_o^it_{oo} - 8\beta^2b^2s_o^i\mathcal{L}_{\hat{\nu}}\beta - 8b^i\beta^3\varepsilon\mathcal{L}_{\hat{\nu}}\beta \\
& - 4b^2\beta^4\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 12\beta^3\mathcal{L}_{\hat{\nu}}s_o^i - 16\beta^2Py^i + 16\beta^2\mathcal{L}_{\hat{\nu}}G_\alpha^i \\
& + 2b^ib^4\beta^2t_{oo}\varepsilon + 4b^ib^2\beta^2\varepsilon t_{oo}) \\
& + \alpha^4(-2\beta^5\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 8b^2\beta^3\mathcal{L}_{\hat{\nu}}G_\alpha^i - 8b^2\beta^3Py^i + 8\beta^3s_o^i\mathcal{L}_{\hat{\nu}}\beta \\
& + 6\beta^2t_{oo}s_o^i + 5b^ib^2\beta^3\varepsilon t_{oo} - 8\beta^3Py^i - 4\beta^4\mathcal{L}_{\hat{\nu}}s_o^i + 8\beta^3\mathcal{L}_{\hat{\nu}}G_\alpha^i \\
& + \beta^3b^i\varepsilon t_{oo} + 4\beta^2b^2s_o^it_{oo} + 2\beta^4b^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta) \\
& + \alpha^3(2\beta^6\varepsilon\mathcal{L}_{\hat{\nu}}b^i + 8b^2\beta^4Py^i - 8b^2\beta^4\mathcal{L}_{\hat{\nu}}G_\alpha^i + 16\beta^4Py^i \\
& + 8s_o^it_{oo}b^2\beta^3 - 4b^it_{oo}b^2\beta^4\varepsilon) \\
& + \alpha^2(-6\beta^4s_o^it_{oo} - 3\beta^5b^it_{oo}\varepsilon) \\
& + \alpha^1(-4\beta^6Py^i + 4\beta^6\mathcal{L}_{\hat{\nu}}G_\alpha^i - 4\beta^5s_o^it_{oo} + 2\beta^6b^it_{oo}\varepsilon).
\end{aligned}$$

and

$$\begin{aligned}
D(x, y) = & \alpha^9(2y^i\varepsilon\mathcal{L}_{\hat{\nu}}b^2) \\
& + \alpha^8(-6\beta y^i\varepsilon\mathcal{L}_{\hat{\nu}}b^2 + 2\varepsilon b^4y^i\mathcal{L}_{\hat{\nu}}\beta + 4b^2y^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta) \\
& + \alpha^7(-10b^2\beta\varepsilon y^i\mathcal{L}_{\hat{\nu}}\beta - 4b^4\beta\varepsilon y^i\mathcal{L}_{\hat{\nu}}\beta + b^4y^i\varepsilon t_{oo} + b^2y^i\varepsilon t_{oo} \\
& - 7\beta b^2\varepsilon b^it_{oo} + 2b^ib^2\beta^2\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 4\beta\varepsilon y^i\mathcal{L}_{\hat{\nu}}\beta + 6\beta^2\varepsilon y^i\mathcal{L}_{\hat{\nu}}b^2) \\
& + \alpha^6(4b^2\beta^2y^i\varepsilon\mathcal{L}_{\hat{\nu}}\beta - 5b^2\beta\varepsilon y^it_{oo} - 3b^4\beta\varepsilon y^it_{oo} + 2b^4\beta^2\varepsilon y^i\mathcal{L}_{\hat{\nu}}\beta \\
& - 2\beta^3\varepsilon y^i\mathcal{L}_{\hat{\nu}}b^2 + 8\beta^2\varepsilon y^i\mathcal{L}_{\hat{\nu}}\beta)
\end{aligned}$$

$$\begin{aligned}
& +\alpha^5(-2\beta^3\varepsilon y^i \mathcal{L}_{\hat{\nabla}}\beta + \beta^2\varepsilon y^i t_{\infty} + 3b^4\beta^2 y^i \varepsilon t_{\infty} + 6b^2\beta^3\varepsilon y^i \mathcal{L}_{\hat{\nabla}}\beta \\
& + 6y^i b^2\beta^3\varepsilon t_{\infty}) \\
& +\alpha^4(b^2\beta^3 y^i \varepsilon t_{\infty} - b^4\beta^3\varepsilon y^i t_{\infty} - 4b^2\beta^4 y^i \varepsilon \mathcal{L}_{\hat{\nabla}}\beta - \varepsilon\beta^3 y^i t_{\infty} \\
& - 2\beta^4 y^i \varepsilon \mathcal{L}_{\hat{\nabla}}\beta) \\
& +\alpha^3(-5\beta^4 b^2\varepsilon y^i t_{\infty} - \beta^4\varepsilon y^i t_{\infty} - 2y^i \beta^5\varepsilon \mathcal{L}_{\hat{\nabla}}\beta) \\
& +\alpha^2(2\beta^5 b^2\varepsilon y^i t_{\infty} + 2\beta^6\varepsilon y^i \mathcal{L}_{\hat{\nabla}}\beta) \\
& +\alpha^1(2\beta^6\varepsilon y^i t_{\infty}) \\
& +\alpha^0(-\beta^7\varepsilon y^i t_{\infty}).
\end{aligned}$$

By changing all the terms y to $-y$ in (4.10) we obtain $L(x, y) = D(x, y) = 0$. From equation $D(x) = 0$, we can get two fundamental equations

$$\begin{aligned}
a_9\alpha^8 + a_7\alpha^6 + a_5\alpha^4 + a_3\alpha^2 + a_1\alpha^0 &= 0, \\
a_8\alpha^8 + a_6\alpha^6 + a_4\alpha^4 + a_2\alpha^2 + a_0\alpha^0 &= 0.
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
a_9 &= 2y^i \varepsilon \mathcal{L}_{\hat{\nabla}} b^2, \\
a_8 &= -6\beta y^i \varepsilon \mathcal{L}_{\hat{\nabla}} b^2 + 2\varepsilon b^4 y^i \mathcal{L}_{\hat{\nabla}} \beta + 4b^2 y^i \varepsilon \mathcal{L}_{\hat{\nabla}} \beta, \\
a_7 &= -10b^2 \beta \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta - 4b^4 \beta \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta + b^4 y^i \varepsilon t_{\infty} + b^2 y^i \varepsilon t_{\infty}, \\
& \quad 7\beta b^2 \varepsilon b^i t_{\infty} + 2b^i b^2 \beta^2 \varepsilon \mathcal{L}_{\hat{\nabla}} \beta - 4\beta \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta + 6\beta^2 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} b^2, \\
a_6 &= 4b^2 \beta^2 y^i \varepsilon \mathcal{L}_{\hat{\nabla}} \beta - 5b^2 \beta \varepsilon y^i t_{\infty} - 3b^4 \beta \varepsilon y^i t_{\infty} + 2b^4 \beta^2 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta, \\
& \quad 2\beta^3 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} b^2 + 8\beta^2 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta, \\
a_5 &= -2\beta^3 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta + \beta^2 \varepsilon y^i t_{\infty} + 3b^4 \beta^2 y^i \varepsilon t_{\infty} + 6b^2 \beta^3 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta \\
& \quad + 6y^i b^2 \beta^3 \varepsilon t_{\infty}, \\
a_4 &= b^2 \beta^3 y^i \varepsilon t_{\infty} - b^4 \beta^3 \varepsilon y^i t_{\infty} - 4b^2 \beta^4 y^i \varepsilon \mathcal{L}_{\hat{\nabla}} \beta - \varepsilon \beta^3 y^i t_{\infty} - 2\beta^4 y^i \varepsilon \mathcal{L}_{\hat{\nabla}} \beta, \\
a_3 &= -5\beta^4 b^2 \varepsilon y^i t_{\infty} - \beta^4 \varepsilon y^i t_{\infty} - 2y^i \beta^5 \varepsilon \mathcal{L}_{\hat{\nabla}} \beta, \\
a_2 &= 2\beta^5 b^2 \varepsilon y^i t_{\infty} + 2\beta^6 \varepsilon y^i \mathcal{L}_{\hat{\nabla}} \beta, \\
a_1 &= 2\beta^6 \varepsilon y^i t_{\infty}, \\
a_0 &= -\beta^7 \varepsilon y^i t_{\infty}.
\end{aligned}$$

From (4.11), we see that a_0 has the factor α^2 and then $t_{\infty} = c^i(x)\alpha^2$ for some scalar function $c^i(x)$ on M .

Replacing this quantity $t_{\infty} = c^i(x)\alpha^2$ into (4.9) and sorting sorting again by α , we have equation

$$\bar{L}(x, y)\alpha + \bar{D}(x, y)e^s = 0 \tag{4.12}$$

By similar computations we can conclude $\bar{L}(x, y) = \bar{D}(x, y) = 0$. Equation $\bar{D}(x, y) = 0$ is as

$$\bar{m}_9\alpha^9 + \bar{m}_8\alpha^8 + \bar{m}_7\alpha^7 + \bar{m}_6\alpha^6 + \bar{m}_5\alpha^5 + \bar{m}_4\alpha^4 + \bar{m}_3\alpha^3 + \bar{m}_2\alpha^2 = 0. \quad (4.13)$$

where

$$\begin{aligned} \bar{m}_3 &= -2\beta^5\varepsilon y^i \mathcal{L}_{\hat{V}}\beta + 2\beta^6\varepsilon y^i c^i(x), \\ \bar{m}_2 &= +2\beta^6\varepsilon y^i \mathcal{L}_{\hat{V}}\beta - \beta^7\varepsilon y^i c^i(x). \end{aligned}$$

From (4.13), we have two fundamental equation

$$\bar{m}_9\alpha^6 + \bar{m}_7\alpha^4 + \bar{m}_5\alpha^2 + \bar{m}_3\alpha^0 = 0,$$

$$\bar{m}_8\alpha^6 + \bar{m}_6\alpha^4 + \bar{m}_4\alpha^2 + \bar{m}_2\alpha^0 = 0.$$

By the equations mentioned above we conclude that \bar{m}_2, \bar{m}_3 must be divided by α^2 , therefore there are two scalar function $q^i(x), g^i(x)$ on M where

$$-2\varepsilon y^i \mathcal{L}_{\hat{V}}\beta + 2\beta\varepsilon y^i c^i(x) = q^i(x)\alpha^2, \quad (4.14)$$

$$2\varepsilon y^i \mathcal{L}_{\hat{V}}\beta - \beta\varepsilon y^i c^i(x) = g^i(x)\alpha^2. \quad (4.15)$$

Let us compute the terms given by (4.14) and (4.15),

$$\beta\varepsilon y^i c^i(x) = (q^i(x) + g^i(x))\alpha^2. \quad (4.16)$$

Taking into account the non-degeneracy of $\varepsilon, \beta \neq 0$ yields

$$c^i(x) = 0,$$

therefore

$$t_{oo} = 0.$$

Plugging $c^i(x) = 0$ in (4.14) follows that

$$\mathcal{L}_{\hat{V}}\beta = 0.$$

Now, let us assume β satisfies

$$r_{oo} = 0, \quad s_o = 0.$$

In this case, $\mathbf{S} = 0$. Substituting $r_{oo} = 0$ and $s_o = 0$ in (2.1), the spray coefficients of F can be calculated by $G^i = G_\alpha^i + \alpha Q s_o^i$, i.e.

$$G^i = G_\alpha^i + \frac{\alpha^2}{\alpha - \beta} s_o^i. \quad (4.17)$$

Suppose that $s_o^i = 0$, so we observe

$$G^i = G_\alpha^i.$$

In this case one can see that the projective algebra $p(M, F)$ of F is coincides with the projective algebra $p(M, \alpha)$ of α and this proves (a).

If $s_o^i \neq 0$ and V be a projective vector field on (M, F) . From remark 3.1, V is a special projective vector field on M , so

$$\mathcal{L}_{\hat{V}}G^i = P y^i.$$

where $P(x, y) = P_k(x)y^k$. From (4.17)

$$\mathcal{L}_{\hat{V}}G^i = \mathcal{L}_{\hat{V}}(G_\alpha^i + \frac{\alpha^2}{\alpha - \beta}s_\circ^i) = \mathcal{L}_{\hat{V}}G_\alpha^i + \mathcal{L}_{\hat{V}}(\frac{\alpha^2}{\alpha - \beta}s_\circ^i) = Py^i.$$

Therefore

$$\mathcal{L}_{\hat{V}}G^i = \mathcal{L}_{\hat{V}}G_\alpha^i + \frac{t_{\circ\circ}}{\alpha - \beta}s_\circ^i - \frac{1}{2} \frac{\alpha t_{\circ\circ}}{(\alpha - \beta)^2}s_\circ^i + \frac{\alpha^2 \mathcal{L}_{\hat{V}}\beta}{(\alpha - \beta)^2}s_\circ^i + \frac{\alpha^2}{\alpha - \beta} \mathcal{L}_{\hat{V}}s_\circ^i. \quad (4.18)$$

By replacing y^i in (4.18) with $-y^i$ we have:

$$\mathcal{L}_{\hat{V}}G^i = \mathcal{L}_{\hat{V}}G_\alpha^i - \frac{t_{\circ\circ}}{\alpha + \beta}s_\circ^i + \frac{1}{2} \frac{\alpha t_{\circ\circ}}{(\alpha + \beta)^2}s_\circ^i + \frac{\alpha^2 \mathcal{L}_{\hat{V}}\beta}{(\alpha + \beta)^2}s_\circ^i - \frac{\alpha^2}{\alpha + \beta} \mathcal{L}_{\hat{V}}s_\circ^i \quad (4.19)$$

Let us compute the terms given by (4.18), (4.19)

$$\alpha t_{\circ\circ} s_\circ^i (\alpha^2 - 3\beta^2) + 4\alpha^3 \beta s_\circ^i \mathcal{L}_{\hat{V}}\beta + 2\alpha^3 \mathcal{L}_{\hat{V}}s_\circ^i (\alpha^2 - \beta^2) = 0. \quad (4.20)$$

Eq. (4.20) is equivalent to following polynimal equation:

$$a_1 + \alpha^2 a_3 + \alpha^4 a_5 = 0. \quad (4.21)$$

where

$$\begin{aligned} a_1 &= -3\beta^2 s_\circ^i t_{\circ\circ}, \\ a_3 &= s_\circ^i t_{\circ\circ} - 2\beta^2 \mathcal{L}_{\hat{V}}s_\circ^i + 4\beta s_\circ^i \mathcal{L}_{\hat{V}}\beta, \\ a_5 &= 2\mathcal{L}_{\hat{V}}s_\circ^i. \end{aligned}$$

we see that a_1 has the factor α^2 and then

$$t_{\circ\circ} = c^i(x)\alpha^2$$

for some scalar function $c^i(x)$ on M . Plugging it in (4.21), changes it into the following equation

$$\alpha^2 a_5 + a_3 + a_1 = 0. \quad (4.22)$$

where

$$\begin{aligned} a_5 &= 2\mathcal{L}_{\hat{V}}s_\circ^i, \\ a_3 &= s_\circ^i c^i(x)\alpha^2 - 2\beta^2 \mathcal{L}_{\hat{V}}s_\circ^i + 4\beta s_\circ^i \mathcal{L}_{\hat{V}}\beta, \\ a_1 &= -3\beta^2 s_\circ^i c^i(x). \end{aligned}$$

From which it follows that α^2 must divide $a_1 + a_3$, hence there is a class of functions $\mu^i = \mu^i(x)$ on M such that,

$$-3\beta^2 s_\circ^i c^i(x) + s_\circ^i c^i(x)\alpha^2 - 2\beta^2 \mathcal{L}_{\hat{V}}s_\circ^i + 4\beta s_\circ^i \mathcal{L}_{\hat{V}}\beta = \mu^i(x)\alpha^2 \quad (4.23)$$

Converting the two sides of (4.23) with y_i and taking the facts that $y_i = a_{ij}y^j$, $y_i s_\circ^i = 0$ and $\mathcal{L}_{\hat{V}}y_i = 0$, Eq.(4.23) reads as $\mu^i(x)y_i\alpha^2 = 0$.

After a derivation with respect to y^k , we have

$$2\mu^i(x)a_{ik} = 0, \quad \mu^i = 0.$$

Plugging $\mu^i = 0$ in (4.23) and then (4.22) follows that

$$\mathcal{L}_{\hat{V}} s_{\circ}^i = 0$$

and thus,

$$-3\beta^2 s_{\circ}^i c^i(x) + s_{\circ}^i c^i(x) \alpha^2 + 4\beta s_{\circ}^i \mathcal{L}_{\hat{V}} \beta = 0 \quad (4.24)$$

From $s_{\circ}^i \neq 0$ we get:

$$-3\beta^2 c^i(x) + c^i(x) \alpha^2 + 4\beta \mathcal{L}_{\hat{V}} \beta = 0.$$

Taking into account the non-degeneracy of $\alpha^2, \beta \neq 0$ yields $c^i(x) = 0, \mathcal{L}_{\hat{V}} \beta = 0$ and completes the proof.

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