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Special projective algebra of exponential metrics of isotropic S-curvature

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Abstract. Exponential metrics are popular Finsler metrics. Let F be an exponential (α, β)-metric of isotropic S-curvature on manifold M. In this paper, We study a Lie sub-algebra of projective vector fields of a Finsler metric F is introduced and denoted by SP(F). We classify SP(F) of isotropic S-curvature as a certain Lie sub-algebra of the Killing algebra $K(M, \alpha)$.

Keywords: Projective vector field, Exponential Finsler metirc, **S**-curvature.

1. Introduction

The projective Finsler metrics are smooth solutions to the historical Hilbert's fourth problem. The projective vector fields are a way to characterize the projective metrics. The collection of all projective vector fields on a Finsler space is a finite-dimensional Lie algebra with respect to the usual Lie bracket, called the projective algebra denoted by $p(M, F)$. The collection of all projective vector fields on a Finsler space $p(M, F)$ is a finite-dimensional Lie algebra with respect to the usual Lie bracket, called the projective algebra. A specific Lie sub-algebra of projective algebra of Finsler spaces, called the special projective algebra and denoted by $SP(F)$.

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In [8], Rafie-Rad studied on the projective vector fields on the class of Randers metrics and introduced Lie sub-algebra of projective vector fields of a Finsler metric. In [4], B. Rezaei and M.Rafie-Rad studied the projective algebra of some (α,β)-metrics of isotropic S-curvature. In [10], the auther show that if the Matsumoto metric admits a projective vector field, then this is a conformal vector field with to Riemannianmetric α or F has vanishing S-curvature.

In this paper, we characterize the special projective vector field V on manifold M with exponential metric of isotropic S-curvature. We prove the following theorem:

Theorem 1.1. *Let $(M, F = \alpha e^{\beta/\alpha})$ be exponential metric of isotropic S-curvature on a manifold and $b := \|\beta\|_\alpha$ is constant. Then one of the following statements holds:*

(a) *β is parallel with respect to α and the projective algebra $p(M,F)$ of F is coincides with the projective algebra $p(M,\alpha)$ of α.*

(b) *Every special projective vector field V on (M,F) is an Killing vector field on (M,α) and $\pounds_{\hat{V}}\beta = 0$.*

2. **Preliminaries**

Let F be a Finsler metric on an n-dimensional manifold M. It induces a spray G on TM. In local coordinates in TM, it is expressed by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 satisfying $G^i(x,\lambda y) = \lambda^2 G^i(x,y)$ $\lambda > 0$. Assume the following conventions:

$$G^i{}_j = \frac{\partial G^i}{\partial y^i}, \quad G^i{}_{jk} = \frac{\partial G^i{}_j}{\partial y^k}, \quad G^i{}_{jkl} = \frac{\partial G^i{}_{jk}}{\partial y^l}.$$

The local functions $G^i{}_{jk}$ give rise to a torison-free connection in π^*TM called the berwald connection which is this paper, see [5].

Let

$$\alpha(\mathbf{y}) := \sqrt{g_{ij}(x)y^iy^j}, \quad \mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM.$$

α is a family of Euclidean norms on tangent spaces. Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ a 1-form on a manifold M. An (α,β)-metric is a scalar function F on TM defined by $F := \alpha\phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity such that F is a positive definite Finsler metric. A special (α,β)-metric defined by $\phi(s) = e^s$ is called exponential metric.

Denote the Levi-Civita connection of α by ∇ and define $b_{i|j}$ by $(b_{i|j})\theta^j := db_i - b_j\theta_i{}^j$, where $\theta^i := dx^i$, $\theta_i{}^j := \Gamma^j_{ik}dx^k$.

In order to study the geometric properties of (α,β)-metrics, one needs a formula for the spray coefficients of an (α,β)-metrics. Let

$$r_{ij} = (\nabla_j b_i + \nabla_i b_j)/2, \quad s_{ij} = (\nabla_j b_i - \nabla_i b_j)/2, \quad r^i{}_j := a^{ik} r_{kj},$$
$$r_{\circ\circ} := r_{ij} y^i y^j, \quad r_{i\circ} := r_{ij} y^j, \quad s^i{}_j := a^{ik} s_{kj},$$
$$s_j := b^i s_{ij}, \quad s_\circ := s_i y^i, \quad s_{i\circ} := s_{ij} y^j.$$

The spray coefficients G^i of F and G^i_α of α are related as follows:

$$G^i = G^i_\alpha + \alpha Q s^i{}_\circ + \alpha^{-1}\Theta\{r_{\circ\circ} - 2\alpha Q s_\circ\} y^i + \Psi\{r_{\circ\circ} - 2\alpha Q s_\circ\} b^i, \tag{2.1}$$

$$Q = \frac{\phi'}{\phi - s\phi'}, \ \Theta = \frac{\phi\phi' - s(\phi\phi'' - \phi'\phi')}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}},$$
$$\Psi = \frac{\phi''}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}.$$

There is a notion of distortion $\tau = \tau(x,y)$ on TM associated with the Busemann-Hausdorff volume form on manifold, i.e., $dV_F = \sigma_F(x) dx^1 \cdots dx^n$, which is defined by

$$\tau(\mathbf{y}) := \ln\Big[\frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma_F(x)}\Big],$$

,

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathrm{R}^n \ \Big| \ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}. \tag{2.2}$$

For a vector $\mathbf{y} \in T_xM$. Let $c(t), -\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. Define

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau\Big(\dot{c}(t)\Big)\Big]|_{t=0}.$$

We say S-curvature is isotropic if there exists a scalar function $c(x)$ on M such that $S(x,y) = (n+1)c(x)F(x,y)$, and constant S-curvature if $c(x) =$ constant, see [2, 6, 7].

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. By the definition of the S-curvature, we have

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big], \tag{2.3}$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [5]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$. To prove the Theorem 1.1, we need the following theorem which is proved in [3].

Theorem 2.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a (α,β)-metric on a manifold of dimension n and $b := \|\beta\|_\alpha$ is constant. Suppose that*

$$\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s,$$

for any constant $k_1 > 0$, k_2 and k_3. Then F is of isotropic S-curvature, $S = (n+1)cF$, if and only if one of the following holds:

(i) *β satisfies*

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), s_j = 0 \tag{2.4}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}$$

where k is a constant. In this case, $S = (n+1)cF$ with $c = k\varepsilon$.

(ii) *β satisfies*

$$r_{ij} = 0, s_j = 0 \tag{2.5}$$

In this case, $S = 0$, regardless of the choice of a particular ϕ.

3. **Projective vector fields on Finsler spaces**

Every vector field X on M induces naturally a transformation under the following infinitesimal coordinate transformations on TM, $(x^i, y^i) \longrightarrow (\bar{x}^i, \bar{y}^i)$ given by

$$\bar{x}^i = x^i + V^i dt, \qquad \bar{y}^i = y^i + y^k \frac{\partial V^i}{\partial x^k} dt.$$

This leads us to the notion of *the complete lift* $\hat{V}$ (see [9]) of V to a vector field on TM_0 given by

$$\hat{V} = V^i \frac{\partial}{\partial x^i} + y^k \frac{\partial V^i}{\partial x^k} \frac{\partial}{\partial y^i}.$$

Almost any geometric object in Finsler geometry depends on the both points and velocities, hence the Lie derivatives of such geometric objects should rather should be regarded with respect to $\hat{V}$. For computational use, it is known $\pounds_{\hat{V}} y^i = 0$, $\pounds_{\hat{V}} dx^i = 0$ and the differential operators $\pounds_{\hat{V}}$, $\frac{\partial}{\partial x^i}$, exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute as well. The vector field V is called a projective vector field, if there is a function P on TM_0 such that

$$\pounds_{\hat{V}} G^i{}_k = P\delta^i{}_k + P_k y^i,$$

where $P_k = P_{.k}$, see [1]. Thereby, given any appropriate t, the local flow $\{\phi_t\}$ associated to V is projective transformation. If V is a projective vector field, then [1]:

$$\begin{aligned}
\pounds_{\hat{V}} G^i &= P y^i, \\
\pounds_{\hat{V}} G^i{}_{jk} &= \delta^i{}_j P_k + \delta^i{}_k P_j + y^i P_{k.j}, \\
\pounds_{\hat{V}} G^i{}_{jkl} &= \delta^i{}_j P_{k.l} + \delta^i{}_k P_{j.l} + \delta^i{}_l P_{k.j} + y^i P_{k.j.l}, \\
2\pounds_{\hat{V}} \mathbf{E}_{jl} &= (n+1) P_{j.l}.
\end{aligned}$$

On the Riemannian spaces, given any projective vector field V the function $P = P(x, y)$ is linear with respect to y. A projective vector field V is called a *special projective vector field* if $\pounds_{\hat{V}}\mathbf{E} = 0$, equivalently, $P(x, y) = P_i(x)y^i$.

Remark 3.1. *On a weakly-Berwald space, every projective vector field is special.*

4. **Proof of Theorem 1.1**

Let $F = \alpha e^s$, $s := \beta/\alpha$ be exponential Finsler metric of isotropic S-curvature on a manifold M and $b := \|\beta\|_\alpha$ is constant. According to theorem 2.1, F is of isotropic S-curvature, $S = (n + 1)cF$, if and only if β satisfies $r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), s_j = 0$ or $r_{ij} = 0, s_j = 0$. Plugging $r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), s_j = 0$ in (2.1) the geodesic coefficients of F can be calculated by

$$G^i = G^i_\alpha + \frac{\alpha^2}{\alpha-\beta}s^i_\circ + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)y^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}e^s + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)b^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}\alpha. \tag{4.1}$$

Assuming $s^i_\circ = 0$, equation (4.1) can be seen as follows:

$$G^i = G^i_\alpha + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)y^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}e^s + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)b^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}\alpha. \tag{4.2}$$

Let us suppose that V is a projective vector field on (M, F). By assuming, V is a special projective field, that is to exists a function P of the form $P(x, y) = P_k(x)y^k$ on M such that

$$\pounds_{\hat{V}}G^i = Py^i.$$

If $s^i_\circ = 0$, by (4.2) we can write this equation as follows

$$\pounds_{\hat{V}}(G^i_\alpha + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)y^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}e^s + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)b^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}\alpha) = Py^i.$$

Therefore, Equation mentioned above is equivalent to the following equality

$$\begin{aligned} 0 \;=\; & -Py^i + \pounds_{\hat{V}}G^i_\alpha + \frac{\varepsilon(e^s y^i + \alpha b^i)}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}\pounds_{\hat{V}}(b^2\alpha^3-\beta^2\alpha) \\ & + \frac{\varepsilon(e^s y^i + \alpha b^i)(b^2\alpha^3-\beta^2\alpha)}{(2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2)^2}\pounds_{\hat{V}}(2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2) \\ & + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}\pounds_{\hat{V}}(e^s y^i + \alpha b^i). \end{aligned}$$

Let us denote

$$t_{\circ\circ} = \pounds_{\hat{V}}\alpha^2.$$

By simplifying above equation and multiplying both sides of this very equation by $\alpha^3(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2$, we can rewrite (4.3) as follows:

$$K(x, y)\alpha + R(x, y)e^s = 0 \tag{4.3}$$

where

$$
\begin{aligned}
K(x,y) &= \alpha^8(2b^i\varepsilon\pounds_{\hat{V}}b^2 + 2b^2\varepsilon\pounds_{\hat{V}}b^i + 2b^4\varepsilon\pounds_{\hat{V}}b^i) \\
&+\alpha^7(2b^2b^i\varepsilon\pounds_{\hat{V}}\beta - 2\beta\varepsilon b^i\pounds_{\hat{V}}b^2 - 2\beta\varepsilon b^2\pounds_{\hat{V}}b^i) \\
&+\alpha^6(-4Py^i + 4\pounds_{\hat{V}}G^i_\alpha - 8b^2Py^i + 8b^2\pounds_{\hat{V}}G^i_\alpha - 4b^4Py^i \\
&+4b^4\pounds_{\hat{V}}G^i_\alpha - 2\beta^2\varepsilon\pounds_{\hat{V}}b^i - 4b^2\beta^2\varepsilon\pounds_{\hat{V}}b^i + 2b^4b^i\varepsilon t_{\circ\circ} \\
&+2b^2\varepsilon t_{\circ\circ}b^i - 4b^i\beta\varepsilon\pounds_{\hat{V}}\beta) \\
&+\alpha^5(8\beta Py^i - 8\beta\pounds_{\hat{V}}G^i_\alpha + 8\beta b^2Py^i - 8\beta b^2\pounds_{\hat{V}}G^i_\alpha \\
&+2b^i\beta^2\varepsilon\pounds_{\hat{V}}\beta - 3\beta b^2b^i\varepsilon t_{\circ\circ} + 2\beta^3\varepsilon\pounds_{\hat{V}}b^i) \\
&+\alpha^4(4\beta^2Py^i - 4\beta^2\pounds_{\hat{V}}G^i_\alpha + 8b^2\beta^2Py^i \\
&-8b^2\beta^2\pounds_{\hat{V}}G^i_\alpha + 2\beta^4\varepsilon\pounds_{\hat{V}}b^i - 4b^i\beta^2b^2\varepsilon t_{\circ\circ}) \\
&+\alpha^3(-8\beta^3Py^i + 8\beta^3\pounds_{\hat{V}}G^i_\alpha + b^i\beta^3\varepsilon t_{\circ\circ}) \\
&+\alpha^2(-4\beta^4Py^i + 4\beta^4\pounds_{\hat{V}}G^i_\alpha + 2b^i\varepsilon\beta^4t_{\circ\circ}). \\
R(x,y) &= \alpha^8(2y^i\varepsilon\pounds_{\hat{V}}b^2) + \alpha^7(2y^ib^4\varepsilon\pounds_{\hat{V}}\beta + 4b^2y^i\varepsilon\pounds_{\hat{V}}\beta - 2y^i\varepsilon\beta\pounds_{\hat{V}}b^2) \\
&+\alpha^6(-2\beta b^2\varepsilon y^i\pounds_{\hat{V}}\beta + b^4\varepsilon y^it_{\circ\circ} + b^2\varepsilon y^it_{\circ\circ} - 4y^i\beta\varepsilon\pounds_{\hat{V}}\beta) \\
&+\alpha^5(-b^4\beta\varepsilon y^it_{\circ\circ} - 4b^2y^i\beta^2\varepsilon\pounds_{\hat{V}}\beta - 3\beta b^2y^i\varepsilon t_{\circ\circ}) \\
&+\alpha^4(-b^2\beta^2\varepsilon y^it_{\circ\circ} + 2y^i\beta^3\varepsilon\pounds_{\hat{V}}\beta + \beta^2y^i\varepsilon t_{\circ\circ}) \\
&+\alpha^3(2\beta^4y^i\varepsilon\pounds_{\hat{V}}\beta + 2b^2\beta^3\varepsilon y^it_{\circ\circ} + y^i\beta^3\varepsilon t_{\circ\circ}) \\
&+\alpha^1(-\beta^5\varepsilon y^it_{\circ\circ}).
\end{aligned}
$$

By changing all the terms y to $-y$ in (4.3) we obtain $R(x,y) = K(x,y) = 0$. Equation $R(x) = 0$ is equivalent to following polynimal equation:

$$a_8\alpha^8 + a_7\alpha^7 + a_6\alpha^6 + a_5\alpha^5 + a_4\alpha^4 + a_3\alpha^3 + a_1\alpha^1 = 0 \tag{4.4}$$

where

$$
\begin{aligned}
&a_8 = 2y^i\varepsilon\pounds_{\hat{V}}b^2, \\
&a_7 = 2y^ib^4\varepsilon\pounds_{\hat{V}}\beta + 4b^2y^i\varepsilon\pounds_{\hat{V}}\beta - 2y^i\varepsilon\beta\pounds_{\hat{V}}b^2, \\
&a_6 = -2\beta b^2\varepsilon y^i\pounds_{\hat{V}}\beta + b^4\varepsilon y^it_{\circ\circ} + b^2\varepsilon y^it_{\circ\circ} - 4y^i\beta\varepsilon\pounds_{\hat{V}}\beta, \\
&a_5 = -b^4\beta\varepsilon y^it_{\circ\circ} - 4b^2y^i\beta^2\varepsilon\pounds_{\hat{V}}\beta - 3\beta b^2y^i\varepsilon t_{\circ\circ}, \\
&a_4 = -b^2\beta^2\varepsilon y^it_{\circ\circ} + 2y^i\beta^3\varepsilon\pounds_{\hat{V}}\beta + \beta^2y^i\varepsilon t_{\circ\circ}, \\
&a_3 = 2\beta^4y^i\varepsilon\pounds_{\hat{V}}\beta + 2b^2\beta^3\varepsilon y^it_{\circ\circ} + y^i\beta^3\varepsilon t_{\circ\circ}, \\
&a_1 = -\beta^5\varepsilon y^it_{\circ\circ}.
\end{aligned}
$$

From above equation, we can get two fundamental equations

$$a_8\alpha^8 + a_6\alpha^6 + a_4\alpha^4 = 0,$$

$$a_7\alpha^6 + a_5\alpha^4 + a_3\alpha^2 + a_1\alpha^0 = 0. \tag{4.5}$$

From (4.5), we can see that a_1 has the factor α^2 and then

$$t_{\circ\circ} = c^i(x)\alpha^2$$

for some scalar function $c^i(x)$ on M.

By the equation mentioned above we can conclude that the coefficient a_4 must be divided by α^2, hence there is a class of homogenous of degree one functions $g^i = g^i(y)$ on M such that,

$$-b^2\varepsilon y^i t_{\circ\circ} + 2y^i\beta\varepsilon \pounds_{\hat{V}}\beta + y^i\varepsilon t_{\circ\circ} = g^i(y)\alpha^2 \tag{4.6}$$

Replacing this quantity $t_{\circ\circ} = c^i(x)\alpha^2$ into (4.6) and taking into account the non-degeneracy of $\varepsilon, \beta \neq 0$ we conclude that

$$\pounds_{\hat{V}}\beta = 0.$$

Plugging the quantities $t_{\circ\circ} = c^i(x)\alpha^2, \pounds_{\hat{V}}\beta = 0$ in $R(x) = 0$ and sorting again by α, we can get the following equation

$$m_8\alpha^8 + m_7\alpha^7 + m_6\alpha^6 + m_5\alpha^5 + m_3\alpha^3 = 0. \tag{4.7}$$

where

$$\begin{aligned}
m_8 &= 2\varepsilon y^i \pounds_{\hat{V}} b^2 + \varepsilon b^4 y^i c^i(x) + \varepsilon b^2 y^i c^i(x),\\
m_7 &= -2y^i\beta\varepsilon\pounds_{\hat{V}} b^2 - \varepsilon\beta y^i b^4 c^i(x) - 3\varepsilon\beta y^i b^2 c^i(x),\\
m_6 &= \beta^2\varepsilon y^i c^i(x) - \varepsilon\beta^2 y^i b^2 c^i(x),\\
m_5 &= \beta^3\varepsilon c^i(x) y^i + 2b^2\beta^3\varepsilon c^i(x) y^i,\\
m_3 &= -\beta^5\varepsilon y^i c^i(x).
\end{aligned}$$

From above equation, we can get two fundamental equations

$$m_8\alpha^6 + m_6\alpha^4 = 0,$$

$$m_7\alpha^4 + m_5\alpha^2 + m_3\alpha^0 = 0. \tag{4.8}$$

From (4.8), we see that m_3 has the factor α^2 and taking into account the non-degeneracy of $\varepsilon, \beta \neq 0$ we conclude that

$$c^i(x) = 0, \quad \text{for any index i.}$$

Therefore $t_{\circ\circ} = 0$.

If we assume that $s^i_\circ \neq 0$, by (4.3) we can write equation (4.1) as follows

$$\begin{aligned}
\pounds_{\hat{V}}\Big(G^i_\alpha + \frac{\alpha^2}{\alpha-\beta}s^i_\circ + \frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)y^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}e^s\\
+\frac{\varepsilon(b^2\alpha^3-\beta^2\alpha)b^i}{2b^2\alpha^2-2\beta^2-2\beta\alpha+2\alpha^2}\alpha\Big) = Py^i.
\end{aligned}$$

Therefore, Equation mentioned above is equivalent to the following equality

$$\begin{aligned} 0 \quad = \quad & \pounds_{\hat{V}} G^i_\alpha - Py^i + \left(\frac{t_{\circ\circ}}{\alpha-\beta} - \frac{\alpha t_{\circ\circ}}{2(\alpha-\beta)^2} + \frac{\alpha^2 \pounds_{\hat{V}}\beta}{(\alpha-\beta)^2}\right) s^i_\circ \\ & + \frac{\alpha^2}{\alpha-\beta} \pounds_{\hat{V}} s^i_\circ + \frac{\varepsilon(e^s y^i + \alpha b^i)}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \pounds_{\hat{V}} (b^2\alpha^3 - \beta^2\alpha) \\ & + \frac{\varepsilon(e^s y^i + \alpha b^i)(b^2\alpha^3 - \beta^2\alpha)}{(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2} \pounds_{\hat{V}} (2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2) \\ & + \frac{\varepsilon(b^2\alpha^3 - \beta^2\alpha)}{2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2} \pounds_{\hat{V}} (e^s y^i + \alpha b^i). \end{aligned} \tag{4.9}$$

By simplifying above equation and multipling both sides of this very equation by $\alpha^2(\alpha-\beta)^2(2b^2\alpha^2 - 2\beta^2 - 2\beta\alpha + 2\alpha^2)^2$, we can rewrite (4.9) as follows:

$$L(x, y)\alpha + D(x, y)e^s = 0 \tag{4.10}$$

where

$$\begin{aligned} L(x, y) \quad = \quad & \alpha^9(2b^4\varepsilon\pounds_{\hat{V}} b^i + 2b^2\varepsilon\pounds_{\hat{V}} b^i + 2b^i\varepsilon\pounds_{\hat{V}} b^2) \\ & +\alpha^8(4\pounds_{\hat{V}} s^i_\circ + 4b^4\pounds_{\hat{V}} s^i_\circ + 8b^2\pounds_{\hat{V}} s^i_\circ - 4b^4\beta\varepsilon\pounds_{\hat{V}} b^i \\ & +2b^2b^i\varepsilon\pounds_{\hat{V}}\beta - 6b^2\beta\varepsilon\pounds_{\hat{V}} b^i - 6b^i\beta\varepsilon\pounds_{\hat{V}} b^2) \\ & +\alpha^7(-4Py^i + 4\pounds_{\hat{V}} G^i_\alpha - 4b^4\beta\pounds_{\hat{V}} s^i_\circ - 2\beta^2\varepsilon\pounds_{\hat{V}} b^i \\ & +4b^4s^i_\circ\pounds_{\hat{V}}\beta - 16b^2\beta\pounds_{\hat{V}} s^i_\circ + 8b^2s^i_\circ\pounds_{\hat{V}}\beta + 6b^i\beta^2\varepsilon\pounds_{\hat{V}} b^2 \\ & -4b^i\beta\varepsilon\pounds_{\hat{V}}\beta + 2b^4\beta^2\varepsilon\pounds_{\hat{V}} b^i + 2b^2\beta^2\varepsilon\pounds_{\hat{V}} b^i + 2b^2b^i\varepsilon t_{\circ\circ} \\ & +2b^4b^i\varepsilon t_{\circ\circ} + 8b^2\pounds_{\hat{V}} G^i_\alpha - 8b^2Py^i + 4b^4\pounds_{\hat{V}} G^i_\alpha \\ & -4b^4Py^i + 4s^i_\circ\pounds_{\hat{V}}\beta - 12\beta\pounds_{\hat{V}} s^i_\circ - 4b^2b^i\beta\varepsilon\pounds_{\hat{V}}\beta) \\ & +\alpha^6(6\beta^3\varepsilon\pounds_{\hat{V}} b^i + 2b^4s^i_\circ t_{\circ\circ} + 4b^2s^i_\circ t_{\circ\circ} - 8\beta b^4\pounds_{\hat{V}} G^i_\alpha \\ & +8\beta b^4Py^i - 24\beta b^2\pounds_{\hat{V}} G^i_\alpha + 24\beta b^2Py^i - 8\beta s^i_\circ\pounds_{\hat{V}}\beta \\ & +6b^2\beta^3\varepsilon\pounds_{\hat{V}} b^i - 2b^i\beta^3\varepsilon\pounds_{\hat{V}} b^2 + 10b^i\beta^2\varepsilon\pounds_{\hat{V}}\beta \\ & -8b^2\beta s^i_\circ\pounds_{\hat{V}}\beta - 16\beta\pounds_{\hat{V}} G^i_\alpha + 16\beta Py^i + 4\beta^2\pounds_{\hat{V}} s^i_\circ \\ & +2s^i_\circ t_{\circ\circ} - 4\beta b^i t_{\circ\circ} b^4\varepsilon - 7\beta b^2\varepsilon b^i t_{\circ\circ} + 2b^ib^2\beta^2\varepsilon\pounds_{\hat{V}}\beta) \\ & +\alpha^5(-4b^4\beta^2Py^i - 16b^2\beta^2Py^i + 8b^2\beta^3\pounds_{\hat{V}} s^i_\circ + 16b^2\beta^2\pounds_{\hat{V}} G^i_\alpha \\ & -8s^i_\circ t_{\circ\circ}\beta + 4b^4\beta^2\pounds_{\hat{V}} G^i_\alpha - 4\beta^2s^i_\circ\pounds_{\hat{V}}\beta - 4\beta^4\varepsilon\pounds_{\hat{V}} b^i \\ & -4\beta b^4s^i_\circ t_{\circ\circ} - 12\beta b^2s^i_\circ t_{\circ\circ} - 8\beta^2b^2s^i_\circ\pounds_{\hat{V}}\beta - 8b^i\beta^3\varepsilon\pounds_{\hat{V}}\beta \\ & -4b^2\beta^4\varepsilon\pounds_{\hat{V}} b^i + 12\beta^3\pounds_{\hat{V}} s^i_\circ - 16\beta^2Py^i + 16\beta^2\pounds_{\hat{V}} G^i_\alpha \\ & +2b^ib^4\beta^2t_{\circ\circ}\varepsilon + 4b^ib^2\beta^2\varepsilon t_{\circ\circ}) \end{aligned}$$

$$
\begin{aligned}
&+\alpha^4(-2\beta^5\varepsilon £_{\hat{V}}b^i+8b^2\beta^3£_{\hat{V}}G^i_\alpha-8b^2\beta^3Py^i+8\beta^3s^i_\circ £_{\hat{V}}\beta\\
&+6\beta^2t_{\circ\circ}s^i_\circ+5b^ib^2\beta^3\varepsilon t_{\circ\circ}-8\beta^3Py^i-4\beta^4£_{\hat{V}}s^i_\circ+8\beta^3£_{\hat{V}}G^i_\alpha\\
&+\beta^3b^i\varepsilon t_{\circ\circ}+4\beta^2b^2s^i_\circ t_{\circ\circ}+2\beta^4b^i\varepsilon £_{\hat{V}}\beta)\\
&+\alpha^3(2\beta^6\varepsilon £_{\hat{V}}b^i+8b^2\beta^4Py^i-8b^2\beta^4£_{\hat{V}}G^i_\alpha+16\beta^4Py^i\\
&+8s^i_\circ t_{\circ\circ}b^2\beta^3-4b^it_{\circ\circ}b^2\beta^4\varepsilon)\\
&+\alpha^2(-6\beta^4s^i_\circ t_{\circ\circ}-3\beta^5b^it_{\circ\circ}\varepsilon)\\
&+\alpha^1(-4\beta^6Py^i+4\beta^6£_{\hat{V}}G^i_\alpha-4\beta^5s^i_\circ t_{\circ\circ}+2\beta^6b^it_{\circ\circ}\varepsilon).
\end{aligned}
$$

$$
\begin{aligned}
D(x,y) \;=\;& \alpha^9(2y^i\varepsilon £_{\hat{V}}b^2)\\
&+\alpha^8(-6\beta y^i\varepsilon £_{\hat{V}}b^2+2\varepsilon b^4y^i£_{\hat{V}}\beta+4b^2y^i\varepsilon £_{\hat{V}}\beta)\\
&+\alpha^7(-10b^2\beta\varepsilon y^i£_{\hat{V}}\beta-4b^4\beta\varepsilon y^i£_{\hat{V}}\beta+b^4y^i\varepsilon t_{\circ\circ}+b^2y^i\varepsilon t_{\circ\circ}\\
&-7\beta b^2\varepsilon b^it_{\circ\circ}+2b^ib^2\beta^2\varepsilon £_{\hat{V}}\beta-4\beta\varepsilon y^i£_{\hat{V}}\beta+6\beta^2\varepsilon y^i£_{\hat{V}}b^2)\\
&+\alpha^6(4b^2\beta^2y^i\varepsilon £_{\hat{V}}\beta-5b^2\beta\varepsilon y^it_{\circ\circ}-3b^4\beta\varepsilon y^it_{\circ\circ}+2b^4\beta^2\varepsilon y^i£_{\hat{V}}\beta\\
&-2\beta^3\varepsilon y^i£_{\hat{V}}b^2+8\beta^2\varepsilon y^i£_{\hat{V}}\beta)\\
&+\alpha^5(-2\beta^3\varepsilon y^i£_{\hat{V}}\beta+\beta^2\varepsilon y^it_{\circ\circ}+3b^4\beta^2y^i\varepsilon t_{\circ\circ}+6b^2\beta^3\varepsilon y^i£_{\hat{V}}\beta\\
&+6y^ib^2\beta^3\varepsilon t_{\circ\circ})\\
&+\alpha^4(b^2\beta^3y^i\varepsilon t_{\circ\circ}-b^4\beta^3\varepsilon y^it_{\circ\circ}-4b^2\beta^4y^i\varepsilon £_{\hat{V}}\beta-\varepsilon\beta^3y^it_{\circ\circ}\\
&-2\beta^4y^i\varepsilon £_{\hat{V}}\beta)\\
&+\alpha^3(-5\beta^4b^2\varepsilon y^it_{\circ\circ}-\beta^4\varepsilon y^it_{\circ\circ}-2y^i\beta^5\varepsilon £_{\hat{V}}\beta)\\
&+\alpha^2(2\beta^5b^2\varepsilon y^it_{\circ\circ}+2\beta^6\varepsilon y^i£_{\hat{V}}\beta)\\
&+\alpha^1(2\beta^6\varepsilon y^it_{\circ\circ})\\
&+\alpha^0(-\beta^7\varepsilon y^it_{\circ\circ}).
\end{aligned}
$$

By changing all the terms y to $-y$ in (4.10) we obtain $L(x,y)=D(x,y)=0$. From equation $D(x)=0$, we can get two fundamental equations

$$
\begin{aligned}
&a_9\alpha^8+a_7\alpha^6+a_5\alpha^4+a_3\alpha^2+a_1\alpha^0=0,\\
&a_8\alpha^8+a_6\alpha^6+a_4\alpha^4+a_2\alpha^2+a_0\alpha^0=0. \qquad (4.11)
\end{aligned}
$$

where

$$
\begin{aligned}
a_9 &= 2y^i\varepsilon £_{\hat{V}}b^2,\\
a_8 &= -6\beta y^i\varepsilon £_{\hat{V}}b^2+2\varepsilon b^4y^i£_{\hat{V}}\beta+4b^2y^i\varepsilon £_{\hat{V}}\beta,\\
a_7 &= -10b^2\beta\varepsilon y^i£_{\hat{V}}\beta-4b^4\beta\varepsilon y^i£_{\hat{V}}\beta+b^4y^i\varepsilon t_{\circ\circ}+b^2y^i\varepsilon t_{\circ\circ},\\
&\quad 7\beta b^2\varepsilon b^it_{\circ\circ}+2b^ib^2\beta^2\varepsilon £_{\hat{V}}\beta-4\beta\varepsilon y^i£_{\hat{V}}\beta+6\beta^2\varepsilon y^i£_{\hat{V}}b^2,\\
a_6 &= 4b^2\beta^2y^i\varepsilon £_{\hat{V}}\beta-5b^2\beta\varepsilon y^it_{\circ\circ}-3b^4\beta\varepsilon y^it_{\circ\circ}+2b^4\beta^2\varepsilon y^i£_{\hat{V}}\beta,\\
&\quad 2\beta^3\varepsilon y^i£_{\hat{V}}b^2+8\beta^2\varepsilon y^i£_{\hat{V}}\beta,
\end{aligned}
$$

$$a_5 = -2\beta^3 \varepsilon y^i \pounds_{\hat{V}}\beta + \beta^2 \varepsilon y^i t_{\circ\circ} + 3b^4\beta^2 y^i \varepsilon t_{\circ\circ} + 6b^2\beta^3 \varepsilon y^i \pounds_{\hat{V}}\beta$$
$$+6y^i b^2 \beta^3 \varepsilon t_{\circ\circ},$$
$$a_4 = b^2\beta^3 y^i \varepsilon t_{\circ\circ} - b^4\beta^3 \varepsilon y^i t_{\circ\circ} - 4b^2\beta^4 y^i \varepsilon \pounds_{\hat{V}}\beta - \varepsilon\beta^3 y^i t_{\circ\circ} - 2\beta^4 y^i \varepsilon \pounds_{\hat{V}}\beta,$$
$$a_3 = -5\beta^4 b^2 \varepsilon y^i t_{\circ\circ} - \beta^4 \varepsilon y^i t_{\circ\circ} - 2y^i \beta^5 \varepsilon \pounds_{\hat{V}}\beta,$$
$$a_2 = 2\beta^5 b^2 \varepsilon y^i t_{\circ\circ} + 2\beta^6 \varepsilon y^i \pounds_{\hat{V}}\beta,$$
$$a_1 = 2\beta^6 \varepsilon y^i t_{\circ\circ},$$
$$a_0 = -\beta^7 \varepsilon y^i t_{\circ\circ}.$$

From (4.11), we see that a_0 has the factor α^2 and then $t_{\circ\circ} = c^i(x)\alpha^2$ for some scalar function $c^i(x)$ on M.
Replacing this quantity $t_{\circ\circ} = c^i(x)\alpha^2$ into (4.9) and sorting sorting again by α, we have equation

$$\overline{L}(x,y)\alpha + \overline{D}(x,y)e^s = 0 \tag{4.12}$$

By similar computations we can conclude $\overline{L}(x,y) = \overline{D}(x,y) = 0$. Equation $\overline{D}(x,y) = 0$ is as

$$\overline{m_9}\alpha^9 + \overline{m_8}\alpha^8 + \overline{m_7}\alpha^7 + \overline{m_6}\alpha^6 + \overline{m_5}\alpha^5 + \overline{m_4}\alpha^4 + \overline{m_3}\alpha^3 + \overline{m_2}\alpha^2 = 0. \tag{4.13}$$

where

$$\overline{m_3} = -2\beta^5 \varepsilon y^i \pounds_{\hat{V}}\beta + 2\beta^6 \varepsilon y^i c^i(x),$$
$$\overline{m_2} = +2\beta^6 \varepsilon y^i \pounds_{\hat{V}}\beta - \beta^7 \varepsilon y^i c^i(x).$$

From (4.13), we have two fundamental equation

$$\overline{m_9}\alpha^6 + \overline{m_7}\alpha^4 + \overline{m_5}\alpha^2 + \overline{m_3}\alpha^0 = 0,$$

$$\overline{m_8}\alpha^6 + \overline{m_6}\alpha^4 + \overline{m_4}\alpha^2 + \overline{m_2}\alpha^0 = 0.$$

By the equations mentioned above we conclude that $\overline{m_2}, \overline{m_3}$ must be divided by α^2, therefore there are two scalar function $q^i(x), g^i(x)$ on M where

$$-2\varepsilon y^i \pounds_{\hat{V}}\beta + 2\beta\varepsilon y^i c^i(x) = q^i(x)\alpha^2, \tag{4.14}$$

$$2\varepsilon y^i \pounds_{\hat{V}}\beta - \beta\varepsilon y^i c^i(x) = g^i(x)\alpha^2. \tag{4.15}$$

Let us compute the terms given by (4.14) and (4.15),

$$\beta\varepsilon y^i c^i(x) = (q^i(x) + g^i(x))\alpha^2. \tag{4.16}$$

Taking into account the non-degeneracy of $\varepsilon, \beta \neq 0$ yields

$$c^i(x) = 0,$$

therefore

$$t_{\circ\circ} = 0.$$

Plugging $c^i(x) = 0$ in (4.14) follows that

$$\pounds_{\hat{V}}\beta = 0.$$

Now, let us assume β satisfies

$$r_{\circ\circ}=0, \quad s_\circ=0.$$

In this case, $\mathbf{S}=0$. Substituting $r_{\circ\circ}=0$ and $s_\circ=0$ in (2.1), the spray coefficients of F can be calculated by $G^i=G^i_\alpha+\alpha Q s^i_\circ$, i.e.

$$G^i=G^i_\alpha+\frac{\alpha^2}{\alpha-\beta}s^i_\circ. \tag{4.17}$$

Suppose that $s^i_\circ=0$, so we observe

$$G^i=G^i_\alpha.$$

In this case one can see that the projective algebra $p(M,F)$ of F is coincides with the projective algebra $p(M,\alpha)$ of α and this proves (a).

If $s^i_\circ\neq 0$ and V be a projective vector field on (M,F). From remark 3.1, V is a special projective vector field on M, so

$$\pounds_{\hat{V}}G^i=Py^i.$$

where $P(x,y)=P_k(x)y^k$. From (4.17)

$$\pounds_{\hat{V}}G^i=\pounds_{\hat{V}}(G^i_\alpha+\frac{\alpha^2}{\alpha-\beta}s^i_\circ)=\pounds_{\hat{V}}G^i_\alpha+\pounds_{\hat{V}}(\frac{\alpha^2}{\alpha-\beta}s^i_\circ)=Py^i.$$

Therefore

$$\pounds_{\hat{V}}G^i=\pounds_{\hat{V}}G^i_\alpha+\frac{t_{\circ\circ}}{\alpha-\beta}s^i_\circ-\frac{1}{2}\frac{\alpha t_{\circ\circ}}{(\alpha-\beta)^2}s^i_\circ+\frac{\alpha^2\pounds_{\hat{V}}\beta}{(\alpha-\beta)^2}s^i_\circ+\frac{\alpha^2}{\alpha-\beta}\pounds_{\hat{V}}s^i_\circ. \tag{4.18}$$

By replacing y^i in (4.18) with $-y^i$ we have:

$$\pounds_{\hat{V}}G^i=\pounds_{\hat{V}}G^i_\alpha-\frac{t_{\circ\circ}}{\alpha+\beta}s^i_\circ+\frac{1}{2}\frac{\alpha t_{\circ\circ}}{(\alpha+\beta)^2}s^i_\circ+\frac{\alpha^2\pounds_{\hat{V}}\beta}{(\alpha+\beta)^2}s^i_\circ-\frac{\alpha^2}{\alpha+\beta}\pounds_{\hat{V}}s^i_\circ \tag{4.19}$$

Let us compute the terms given by (4.18), (4.19)

$$\alpha t_{\circ\circ}s^i_\circ(\alpha^2-3\beta^2)+4\alpha^3\beta s^i_\circ\pounds_{\hat{V}}\beta+2\alpha^3\pounds_{\hat{V}}s^i_\circ(\alpha^2-\beta^2)=0. \tag{4.20}$$

Eq. (4.20) is equivalent to following polynimal equation:

$$a_1+\alpha^2a_3+\alpha^4a_5=0. \tag{4.21}$$

where

$$\begin{aligned}
a_1&=-3\beta^2s^i_\circ t_{\circ\circ},\\
a_3&=s^i_\circ t_{\circ\circ}-2\beta^2\pounds_{\hat{V}}s^i_\circ+4\beta s^i_\circ\pounds_{\hat{V}}\beta,\\
a_5&=2\pounds_{\hat{V}}s^i_\circ.
\end{aligned}$$

we see that a_1 has the factor α^2 and then

$$t_{\circ\circ}=c^i(x)\alpha^2$$

for some scalar function $c^i(x)$ on M. Plugging it in (4.21), changes it into the following equation

$$\alpha^2 a_5 + a_3 + a_1 = 0. \tag{4.22}$$

where

$$\begin{aligned} a_5 &= 2\pounds_{\hat{V}} s^i_\circ, \\ a_3 &= s^i_\circ c^i(x)\alpha^2 - 2\beta^2 \pounds_{\hat{V}} s^i_\circ + 4\beta s^i_\circ \pounds_{\hat{V}}\beta, \\ a_1 &= -3\beta^2 s^i_\circ c^i(x). \end{aligned}$$

From which it follows that α^2 must divide $a_1 + a_3$, hence there is a class of functions $\mu^i = \mu^i(x)$ on M such that,

$$-3\beta^2 s^i_\circ c^i(x) + s^i_\circ c^i(x)\alpha^2 - 2\beta^2 \pounds_{\hat{V}} s^i_\circ + 4\beta s^i_\circ \pounds_{\hat{V}}\beta = \mu^i(x)\alpha^2 \tag{4.23}$$

Convecting the two sides of (4.23) with y_i and taking the facts that $y_i = a_{ij}y^j$, $y_i s^i_\circ = 0$ and $\pounds_{\hat{V}} y_i = 0$, Eq.(4.23) reads as $\mu^i(x) y_i \alpha^2 = 0$.

After a derivation with respect to y^k , we have

$$2\mu^i(x)a_{ik} = 0, \quad \mu^i = 0.$$

Plugging $\mu^i = 0$ in (4.23) and then (4.22) follows that

$$\pounds_{\hat{V}} s^i_\circ = 0$$

and thus,

$$-3\beta^2 s^i_\circ c^i(x) + s^i_\circ c^i(x)\alpha^2 + 4\beta s^i_\circ \pounds_{\hat{V}}\beta = 0 \tag{4.24}$$

From $s^i_\circ \neq 0$ we get:

$$-3\beta^2 c^i(x) + c^i(x)\alpha^2 + 4\beta \pounds_{\hat{V}}\beta = 0.$$

Taking into account the non-degeneracy of $\alpha^2, \beta \neq 0$ yields $c^i(x) = 0$, $\pounds_{\hat{V}}\beta = 0$ and completes the proof.

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