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On a class of conformally flat (α, β)-metrics with special curvature properties

Mosayeb Zohrehvand[a]

[a]Department of Mathematical Sciences and Statistic, Malayer University, Malayer, Iran.

E-mail: m.zohrehvand@malayeru.ac.ir

Abstract. This paper is devoted to study of a class of conformally flat (α, β)-metrics that have of the form $F = \alpha \exp(2s)/s$, where $s := \beta/\alpha$. They are called Kropina change of exponential (α, β)-metrics. We prove that if F has relatively isotropic mean Landsberg curvature or almost vanishing Ξ-curvature then it is a Riemannian metric or a locally Minkowski metric. Also, we prove that, if F be a weak Einstein metric, then it is either a Riemannian metric or a locally Minkowski metric.

Keywords: Conformally flat metric, exponential (α, β)-metric, Ξ-curvature, mean Landsberg curvature.

1. **Introduction**

The important and interesting applications of conformal geometry in physical theories have caused that this field has more studying and consideration. For example, in general relativity the light-like geodesics are invariant under the conformal relation between pseudo-Riemannian metrics. Also, the Weyl theorem states that by studying the conformal and projective properties of a Finsler metric, the properties of metric can be determined uniquely[12, 17].

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Two Finsler metrics F and $\tilde{F}$ on a differentiable manifold M are said to be conformally related if $F = e^{\kappa(x)}\tilde{F}$ where $\kappa(x)$ is a scalar function on M and is called the conformal factor. In the cace of $\tilde{F}$ is a locally Minkowski metric, we say that F is a conformally flat metric.

One of the interesting problems in conformal geometry is obtaining the local metric structure of conformally flat Finsler metrics [3]. But, in general case, this is a difficult problem. Thus, the researchers considered special classes of Finsler metrics. Ichijyō and Hashiguchi gave a condition that a Randers metric is conformally flat [10]. Randers metrics are contained in an important class of Finsler metric i.e. (α, β)-metrics which have many applications in physics, biology and etc (see [1]).

The Finsler metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is an (α, β)-metric where $\alpha := \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric, $\beta := b_i(x)y^i$ is a 1-form and $\phi(s)$ is a C^∞ function that satisfied a certain inequality [15]. (α, β)-metrics have been widely studied because they are computable and also, the researches on (α, β)-metrics enrich Finsler geometry and suggested many references for further studies.

The study of conformally flat (α, β)-metrics with special curvature properties is an interesting field. L. Kang considered conformally flat Randers metrics of scalar flag curvature and proved that they are projectively flat and then classified such metrics completely [11]. Conformally flat (α, β)-metrics with isotropic S-curvature are considered in [4] and it is shown that they are Riemannian or locally Minkowski metric and also conformally flat weak Einstein (α, β)-metrics of polynomial type are classified. In [6] it is proved that any non-Riemannian conformally flat weakly Landsberg (α, β)-metric must be a locally Minkowski metric. Conformally flat (α, β)-metrics with constant flag curvature are considered by Chen et al. and they showed that the such metrics are either locally Minkowski or Riemannian metrics [5]. Tayebi and Razgordani studied conformally flat weak Einstein fourth root (α, β)-metrics and proved that they were also either locally Minkowskian or Riemannian [22]. For more references see [2, 16, 19, 21]

The Kropina metric $F = \alpha^2/\beta$ is an (α, β)-metric which firstly was investigated by V.K. Kropina [13]. This metric appears when the general dynamical system represented by a Lagrangian function [3]. Due to this, for any Finsler metric F, one can consider the transformation

$$F(x, y) \to \bar{F}(x, y) := \frac{F^2}{\beta}. \tag{1.1}$$

The transformation (1.1) is called the Kropina change of Finsler metric F, because $\bar{F}$ is reduced to the Kropina metric, when F reduced to a Riemannian metric α.

A class of (α, β)-metrics that deserve more attention are exponential (α, β)-metrics. They are of the form $F = \alpha\exp(s)$, $s := \beta/\alpha$, and have studied by many authors [18, 20, 24, 25]. This class of metrics is interesting, because the exponential metric

$$F = \alpha \exp\left(\int_0^s \frac{q\sqrt{b^2 - t^2}}{1 + qt\sqrt{b^2 - t^2}} dt\right),$$

is an almost regular unicorn metric, where $b := \|\beta\|_\alpha$ and q is a constant. A unicorn metric is a Landsberg metric that is not Berwaldian [23]. Tayebi and Amini considered conformally flat exponential (α, β)-metric with some special curvature properties [20].

This paper is devoted to study of the conformally flat Kropina change of exponential (α, β)-metric i.e.

$$F = \alpha \mathrm{exp}(2s)/s, \ \ s := \frac{\beta}{\alpha}.$$

For a Finsler metric F, we have the basic tensors, fundamental tensor $\mathbf{g}_y$ and Cartan tortion $\mathbf{C}$. By taking horizontal covariant derivative of Cartan torsion along the geodesics we obtain the tensor field $\mathbf{L}$ that is called Landsberg curvature. The trace of $\mathbf{C}$ and $\mathbf{L}$ are called the mean Cartan torsion $\mathbf{I}$ and the mean Landsberg curvature $\mathbf{J}$, respectively. A Finsler metric F is called relatively isotropic mean Landsberg curvature if there exists a scalar function $c = c(x)$ on M such that $\mathbf{J} + cF\mathbf{I} = 0$. In this paper, we consider the conformally flat Kropina change of exponential (α, β)-metric that has relatively isotropic mean Landsberg curvature and prove the following.

Theorem 1.1. *Let $F = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the conformally flat (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} + c(x)F\mathbf{I} = 0,$$

where $c = c(x)$ is a scalar function on M. Then F reduces to a Riemannian metric or a locally Minkowski metric.

For an n-dimensional Finsler manifold (M, F), Ξ-curvature $\mathbf{\Xi} = \Xi_i dx^i$ is a non-Riemannian quantity that is defined by the non-Riemannian quantity S-curvature $\mathbf{S}$ as follows:

$$\Xi_i := \mathbf{S}_{.i;m} y^m - \mathbf{S}_{;i},$$

where ";" and "." denote the horizontal and vertical covariant derivatives with respect to the Berwald connection of F, respectively. Finsler manifold (M, F) is said to be of almost vanishing Ξ-curvature if

$$\Xi_i := -(n+1)F^2 \left(\frac{\theta}{F}\right)_{y^i}, \tag{1.2}$$

where $\theta := t_i(x)y^i$ is a 1-form on M. In this paper, firstly, the Kropina change of exponential (α, β)-metric with almost vanishing Ξ-curvature are considered and prove that it has vanishing Ξ-curvature. Then, we prove the following theorem.

Theorem 1.2. *Let $F = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the conformally flat (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has almost vanishing Ξ-curvature, then F is a Riemannian metric or a locally Minkowski metric.*

A weak Einstein metric is a Finsler metric F on an n-dimensional manifold M, that the Ricci curvature **Ric** satisfies

$$\mathbf{Ric} = (n-1)\Big(\frac{3\theta}{F} + \sigma\Big)F^2, \tag{1.3}$$

for a scalar function $\sigma := \sigma(x)$ and a 1-form $\theta := t_i(x)y^i$ on M. If $\theta = 0$, then F is called Einstein metric, in this case we have $\mathbf{Ric} = (n-1)\sigma F^2$. If $\mathbf{Ric} = 0$, then F is called Ricci flat. In this paper we study the weak Einstein Kropina change of exponential (α, β)-metrics. At first, we prove that every weak Einstein Kropina change of exponential (α, β)-metric is a Ricci-flat metric. Then, we prove the following theorem.

Theorem 1.3. *Let $F = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the conformally flat (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is a weak Einstein metric, then F is either a Riemannian metric or a locally Minkowski metric.*

2. Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional differentiable manifold M and $TM_0 := \bigcup_{x\in M} T_xM - \{0\}$ the slit tangent bundle. The fundamental tensor $(\mathbf{g}_y) = (g_{ij}(x, y))$ of F is a quadratic form on T_xM that is defined

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y).$$

A curve $x = x^i(t)$ on Finsler space (M, F) is called geodesic if satisfies in the following system of ODEs:

$$\frac{d^2x^i}{dt^2} + G^i(x, \frac{dx}{dt}) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients of F and defined by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

The Riemann curvature of $\mathbf{R} := R^i_k dx^k \frac{\partial}{\partial x^i}$ is defined by

$$R^i_k := 2(G^i)_{x^k} - (G^i)_{x^j y^k}y^j + 2G^j(G^i)_{y^j y^k} - (G^i)_{y^j}(G^j)_{y^k}. \tag{2.1}$$

The trace of the Riemann curvature is called the Ricci curvature **Ric** and is defined by

$$\mathbf{Ric} = R^m_m.$$

In Finsler geometry, there are some geometric quantities that are vanishing for Riemannian metrics and are called non-Riemannian quantities. The Cartan torsion $\mathbf{C}$ is a symmetric trilinear form $\mathbf{C} := C_{ijk}dx^i \otimes dx^j \otimes dx^k$ on TM_0 that is defined as follow

$$C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

One can see that F is a Riemannian metric if and only if $\mathbf{C} = 0$. Thus it is a non-Riemannian quantity.

The mean Cartan torsion of F is the tensor field $\mathbf{I} := I_i dx^i$, that is defined by

$$I_i := g^{jk}C_{ijk}.$$

Furtheremore, one can see that

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{jk})}\Big].$$

The horizontal covariant derivative of Cartan torsion along the geodesics define the tensor field $\mathbf{L} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ on slit tangent bundle TM_0, that is called the Landsberg curvature of F. Thus $L_{ijk} := C_{ijk;m}y^m$, where ";" denoted the horizontal covariant derivative with respect to the Berwald connection of F. Also, the Landsberg curvature can be expressed as following

$$L_{ijk} = -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k}. \tag{2.2}$$

A Finsler metric F is called the Landsberg metric if $\mathbf{L} = 0$.

The mean Landsberg curvature $\mathbf{J} := J_i dx^i$ is a non-Riemannian quantity that is obtained by horizontal covariant derivative of the mean Cartan torsion $\mathbf{I}$ along the geodesics of F. Thus

$$J_i := I_{i;m}y^m. \tag{2.3}$$

Also, the mean Landsberg curvature $\mathbf{J}$ can be obtained as following

$$J_i := g^{jk}L_{ijk}.$$

A Finsler metric F is called weak Landsberg metric if $\mathbf{J} = 0$.

A Finsler metric F is called of relatively isotropic mean Landsberg curvature if $\mathbf{J}/\mathbf{I}$, the relative growth rate of the mean Cartan torsion along geodesics of F be isotropic, i.e. there exists a scalar function $c = c(x)$ on M such that

$$\mathbf{J} + cF\mathbf{I} = 0.$$

For an n-dimensional Finsler manifold (M, F), the Busemann-Hausdorff volume form $dV_F := \sigma_F(x)dx^i \dots dx^n$ is defined by

$$\sigma_F(x) := Vol\Big(\frac{\mathbb{B}^n(1)}{Vol\Big\{y^i \in R^n | F\Big(y^i\frac{\partial}{\partial x^i} < 1\Big)\Big\}}\Big).$$

The S-curvature $\mathbf{S}(y)$ can be defined by

$$\mathbf{S}(y) := \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i}\big[\ln \sigma_F(x)\big],$$

where $y \in T_xM - \{0\}$. From the S-curvature one can obtain the non-Riemaniann quantity Ξ-curvature $\mathbf{\Xi} := \Xi_i dx^i$ as follows:

$$\Xi_i := \mathbf{S}_{.i;m} y^m - \mathbf{S}_{;i},$$

where "." and ";" denote the vertical and horizontal covariant derivative with respect to the Berwald connection of F, respectively.

A Finsler metric F is an (α, β)-metric if $F = \alpha\phi(s)$, $\quad s := \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form with $||\beta_x|| < b_0$, $x \in M$ and $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0. \tag{2.4}$$

In this case, the metric $F = \alpha\phi(s)$ is a positive definite Finsler metric [8]. The fundamental tensor $F = \alpha\phi(s)$ is given by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_i + b_j\alpha_i) + \rho_2\alpha_i\alpha_j,$$

where $\alpha_i := \alpha^{-1}a_{ij}y^j$, and

$$\begin{aligned} &\rho := \phi(\phi - s\phi'), && \rho_0 := \phi\phi'' + \phi'\phi', \\ &\rho_1 := -s(\phi\phi'' + \phi'\phi') + \phi\phi', && \rho_2 := s\{s(\phi\phi'' + \phi'\phi') - \phi\phi'\}. \end{aligned}$$

One can see that

$$g^{ij} = \rho^{-1}\{a^{ij} - \tau b^i b^j - \eta Y^i Y^j\}, \tag{2.5}$$

where $b^i := a^{ij}b_j$ and

$$\begin{aligned} &\eta := \frac{\mu}{1 + Y^2\mu}, \quad \mu := \frac{\rho_2}{\rho}, \quad Y^2 := 1 + (\lambda + \epsilon)s + \lambda\epsilon b^2, \\ &Y^i := \frac{y^i}{\alpha} + \lambda b^i, \quad \lambda := \frac{\epsilon - \delta s}{1 + \delta b^2}, \quad \epsilon := \frac{\rho_1}{\rho_2}, \\ &\delta := \frac{\rho_0 - \epsilon^2\rho_2}{\rho}, \quad \tau := \frac{\delta}{1 + \delta b^2}. \end{aligned}$$

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. We denote

$$\begin{aligned} &r^i_{\ j} := a^{im}r_{mj}, && r_{00} := r_{ij}y^iy^j, && r_i := b^m r_{mi}, \\ &r_0 := r_i y^i, && r_{i0} := r_{im}y^m, && s^i_{\ j} := a^{im}s_{mj}, \\ &s_{i0} := s_{im}y^m, && s_i := b^m s_{mi}, && s_0 := s_i y^i, \end{aligned}$$

The geodesic coefficients G^i of an (α,β)-metric $F=\alpha\phi(s)$ are given by

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\Big\{\Psi b^i + \Theta\alpha^{-1}y^i\Big\}, \tag{2.6}$$

where G^i_α is the geodesic coefficients of α and

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']},$$
$$\Psi := \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}.$$

For more details, see [8].

3. Kropina change of exponential (α,β)-metrics

In this paper we focus on the Kropina change of exponential (α,β)-metric. This is of the form

$$F = \alpha\exp(2s)/s,\ \ s := \beta/\alpha.$$

Since $\phi(s) = \exp(2s)/s$ must be positive function, thus $s > 0$. One can see that F is not a regular (α,β)-metric, but we have the following lemma.

Lemma 3.1. *$F = \alpha\exp(2s)/s,\ s := \beta/\alpha$, is a (non-regular) Finsler metric, if and only if* $0 < \|\beta_x\|_\alpha < 1$

Proof. Let $F = \alpha\exp(2s)/s,\ s := \beta/\alpha$, is a Finsler metric, then from (2.4) we have

$$\frac{s^3 + 2b^2s^2 - 2b^2s + b^2 - 2s^4}{s^3} > 0.$$

For $s = b$, we get $0 < b < 1$. Thus $0 < \|\beta_x\|_\alpha < 1$. The convers is easy to prove. □

3.1. Proof of Theorem 1.1.

Now, we are going to prove Theorem 1.1. In [7] the mean Cartan torsion of an (α,β)-metric are computed.

Lemma 3.2. ([7])*For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, the mean Cartan torsion is given by*

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi')h_i, \tag{3.1}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',$$
$$\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{\frac{3}{2}}}\right]',$$
$$h_j := b_j - \alpha^{-1}sy_j.$$

It is well known that, by Deicke's theorem, F is a Riemannian metric if and only if $\mathbf{I} = 0$. Thus from (3.1) we have the following.

Lemma 3.3. *An (α, β)-metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is a Riemannian metric if and only if $\Phi = 0$.*

From (2.3) and (3.1), one can see that the mean Landsberg curvature of an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is given by

$$\begin{aligned} J_j =& \frac{1}{2\alpha^4\Delta}\Bigg\{\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](s_0+r_0)h_j \\ &+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Q s_0)h_j \\ &+\alpha\Big[-\alpha^2 Q' s_0 h_j+\alpha Q(\alpha^2 s_j-y_j s_0)+\alpha^2\Delta s_{j0} \\ &+\alpha^2(r_{j0}-2\alpha Q s_j)-(r_{00}-2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\Bigg\}, \end{aligned} \tag{3.2}$$

where $y_j := a_{ij}y^i$. For more details see [7, 14]. From (3.1) and (3.2), we obtained that

$$\begin{aligned} J_j + c(x)FI_j = -\frac{1}{2\alpha^4\Delta}\Bigg\{&\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](s_0+r_0)h_j \\ &+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Q s_0)h_j+\alpha\Big[-\alpha^2 Q' s_0 h_j \\ &+\alpha Q(\alpha^2 s_j-y_j s_0)+\alpha^2\Delta s_{j0}+\alpha^2(r_{j0}-2\alpha Q s_j) \\ &-(r_{00}-2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}+c(x)\alpha^4\Phi(\phi-s\phi')h_j\Bigg\}. \end{aligned} \tag{3.3}$$

Since we study conformally flat (α, β)-metrics, we need the following Lemma.

Lemma 3.4. ([1]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Then F is locally Minkowski metric if and only if α is flat and β is parallel with respect to α.*

Now, let $F = \alpha\phi(s)$, $s = \beta/\alpha$, is a conformally flat Finsler metric, it means that, there exists a Minkowski metric $\tilde{F}$ such that $\tilde{F} = e^{\kappa(x)}F$, where $\kappa(x)$ is a scalar function on the manifold. Since $F = \alpha\phi(\beta/\alpha)$, we obtain that $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ is an (α, β)-metric, where

$$\tilde{\alpha} = e^{\kappa(x)}\alpha, \qquad \tilde{\beta} = e^{\kappa(x)}\beta. \tag{3.4}$$

From (3.4), we have

$$\tilde{a}_{ij} = e^{2\kappa(x)}a_{ij}, \qquad \tilde{b}_i = e^{\kappa(x)}b_i.$$

The Christoffel symbols Γ^i_{jk} of α and the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of $\tilde{\alpha}$ are related by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \kappa_k + \delta^i_k \kappa_j - \kappa^i a_{jk},$$

where $\kappa_i := \frac{\partial \kappa}{\partial x^i}$ and $\kappa^i := a^{ij}\kappa_j$. Thus, we obtain

$$\tilde{b}_{i\|j} = \frac{\partial \tilde{b}_i}{\partial x^j} - \tilde{b}_s \tilde{\Gamma}^i_{jk} = e^{\kappa}(b_{i|j} - b_j\kappa_i + b_r\kappa^r a_{ij}). \tag{3.5}$$

where $\tilde{b}_{i\|j}$ denote the coefficients of the covariant derivative of $\tilde{\beta}$ with respect to $\tilde{\alpha}$.

Scince $\tilde{F}$ is a Minkowski metric, from Lemma (3.4), we have $\tilde{b}_{i\|j} = 0$. Thus

$$b_{i|j} = b_j\kappa_i - b_r\kappa^r a_{ij}. \tag{3.6}$$

From (3.6) we obtain

$$r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) - b_r\kappa^r a_{ij}, \qquad r_j = -\frac{1}{2}(b^r\kappa_r)b_j + \frac{1}{2}\kappa_j b^2, \tag{3.7}$$

$$r_{i0} = \frac{1}{2}[\kappa_i\beta + (\kappa_r y^r)b_i] - \kappa_r b^r y_i, \qquad s_{ij} = \frac{1}{2}(\kappa_i b_j - \kappa_j b_i), \tag{3.8}$$

$$s_j = \frac{1}{2}(b^r\kappa_r)b_j - \kappa_j b^2, \qquad s_{i0} = \frac{1}{2}[\kappa_i\beta - (\kappa_r y^r)b_i]. \tag{3.9}$$

Further, we have

$$r_{00} = (\kappa_r y^r)\beta - (\kappa_r b^r)\alpha^2, \tag{3.10}$$

$$r_0 = \frac{1}{2}(\kappa_r y^r)b^2 - \frac{1}{2}(\kappa_r b^r)\beta, \tag{3.11}$$

$$s_0 = \frac{1}{2}(\kappa_r b^r)\beta - \frac{1}{2}(\kappa_r y^r)b^2. \tag{3.12}$$

From (3.11) and (3.12), we see that a conformally flat (α,β)-metric satisfying $r_0 + s_0 = 0$ which means that the 1-form β has constant length with respect to α.

In order to simplify the computations, we take an orthonormal basis at any point x with respect to α such that $\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}$ and $\beta = by^1$, where $b := \|\beta_x\|_\alpha$. Then, we take the following coordinate transformation

$$\psi : (s, u^A) \longrightarrow (y^i),$$

in T_xM, that is

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \quad 2 \le A \le n, \tag{3.13}$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^n (u^A)^2}$. In this case, we have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{3.14}$$

Then, by (3.6)-(3.14) one can obtain

$$r_{00} = -b\kappa_1\bar{\alpha}^2 + \frac{bs\bar{\kappa}_0\bar{\alpha}}{\sqrt{b^2-s^2}}, \qquad r_0 = -s_0 = \frac{1}{2}b^2\bar{\kappa}_0, \tag{3.15}$$

$$r_{A0} = \frac{1}{2}\frac{\kappa_A bs\bar{\alpha}}{\sqrt{b^2-s^2}} - (b\kappa_1)u_A, \quad r_{10} = \frac{1}{2}b\bar{\kappa_0}, \tag{3.16}$$

$$s_A = -\frac{1}{2}\kappa_A b^2, \qquad s_1 = 0, \tag{3.17}$$

$$s_{A0} = \frac{1}{2}\frac{\kappa_A bs\bar{\alpha}}{\sqrt{b^2-s^2}}, \qquad s_{10} = -\frac{1}{2}b\bar{\kappa}_0, \tag{3.18}$$

$$h_A = -\frac{\sqrt{b^2-s^2}su_A}{b\bar{\alpha}}, \qquad h_1 = b - \frac{s^2}{b}. \tag{3.19}$$

where $\bar{\kappa}_0 := \kappa_A u^A$.

Proof of Theorem 1.1: Since $\tilde{b}_{i\|j} = 0$, we have that $\tilde{b} = constant$. If $\tilde{b} = 0$, then $F = e^{k(x)}\tilde{\alpha}$ is a Riemannian metric. Now, let F not be a Riemannian metric. Suppose that F is a conformally flat (α, β)-metric with relatively isotropic mean Landsberg curvature. By (3.3) and $r_0 + s_0 = 0$, we obtain

$$\begin{aligned}\frac{\alpha^2}{b^2-s^2}\big\{\Psi_1 + s\frac{\Phi}{\Delta}\big\}(r_{00} - 2\alpha Q s_0)h_j + \alpha\big\{ -\alpha^2 Q' s_0 h_j \\ + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) \\ - (r_{00} - 2\alpha Q s_0)y_j\big\}\frac{\Phi}{\Delta} - c(x)\alpha^4\Phi(\phi - s\phi')h_j = 0.\end{aligned} \tag{3.20}$$

Putting $j = 1$ in (3.20), we have

$$\begin{aligned}\frac{\alpha^2}{b^2-s_2}\big\{\Psi_1 + s\frac{\Phi}{\Delta}\big\}(r_{00} - 2\alpha Q s_0)h_1 + \alpha\big\{ -\alpha^2 Q' s_0 h_1 \\ + \alpha Q(\alpha^2 s_1 - y_1 s_0) + \alpha^2\Delta s_{10} + \alpha^2(r_{10} - 2\alpha Q s_1) \\ - (r_{00} - 2\alpha Q s_0)y_1\big\}\frac{\Phi}{\Delta} - c(x)\alpha^4\Phi(\phi - s\phi')h_1 = 0.\end{aligned} \tag{3.21}$$

Substituting (3.14)-(3.19) into (3.21) and then multiplying the resulting equation with $-2\Delta(b^2-s^2)^{3/2}$ we have

$$\begin{aligned}b^2\bar{\alpha}^3\Big\{2\sqrt{b^2-s^2}\Delta\big[bc\Phi(\phi - s\phi') + \Psi_1\kappa_1\big]\bar{\alpha} - \bar{\kappa}_0\big[b^2\Phi Q'(b^2-s^2) \\ + \Phi b^2(sQ+1) + \Delta\Phi b^2 - 2\Psi_1\Delta(b^2 Q + s)\big]\Big\} = 0.\end{aligned} \tag{3.22}$$

From (3.22), we get

$$\Delta\big[bc\Phi(\phi - s\phi') + \Psi_1\kappa_1\big] = 0, \tag{3.23}$$

$$\bar{\kappa}_0\big[b^2\Phi Q'(b^2-s^2) + \Phi b^2(sQ+1) - \Delta\Phi b^2 - 2\Psi_1\Delta(b^2Q+s)\big] = 0. \tag{3.24}$$

One can see that (3.24) simplify as follow

$$\bar{\kappa}_0\big[b^2\Phi\big(Q'(b^2-s^2) + sQ + 1 - \Delta\big) - 2\Psi_1\Delta(b^2Q+s)\big] = 0 \tag{3.25}$$

substituting $\Delta = Q'(b^2-s^2)+sQ+1$, in (3.25), we get

$$\Psi_1\Delta(b^2Q+s)\bar{\kappa}_0 = 0. \tag{3.26}$$

Now let $j = A$ in (3.20), thus we have

$$\begin{aligned}\frac{\alpha^2}{b^2-s^2}\big[\Psi_1+s\frac{\Phi}{\Delta}\big](r_{00}-2\alpha Qs_0)h_A+\alpha\big[-\alpha^2Q's_0h_A\\ +\alpha Q(\alpha^2s_A-u_As_0)+\alpha^2\Delta s_{A0}+\alpha^2(r_{A0}-2\alpha Qs_A)\\ -(r_{00}-2\alpha Qs_0)u_A\big]\frac{\Phi}{\Delta}+c(x)\alpha^4\Phi(\phi-s\phi')h_A=0.\end{aligned} \tag{3.27}$$

Putting (3.14)-(3.19) into (3.27) and using the same method used in the case of $j=1$ and from $\Delta = Q'(b^2-s^2)+sQ+1$, we get

$$\begin{aligned}(s\Delta+s+b^2Q)b^2\Phi\kappa_A\bar{\alpha}^2-\big[(s\Delta+s+b^2Q)b^2\Phi\\ +2s(b^2Q+s)\Psi_1\Delta\big]\bar{\kappa}_0u_A=0,\end{aligned} \tag{3.28}$$

$$s\sqrt{b^2-s^2}[\Phi bc(\phi-s\phi')+\Psi_1\kappa_1]\Delta u_A=0. \tag{3.29}$$

One can easily see that (3.29) is equivalent to (3.23). Also, multiplying (3.28) with u^A implies that

$$s(b^2Q+s)\Psi_1\Delta\bar{\kappa}_0\bar{\alpha}^2=0. \tag{3.30}$$

It is obvious that (3.30) is equivalent to (3.26). Anyway, we showed that a conformally flat (α,β)-metric with relatively isotropic mean Landsberg curvature satisfiy (3.23) and (3.26).

If $b^2Q+s=0$, then we obtain $\phi = k\sqrt{b^2-s^2}$, where k is a constant. This is a contradiction with the assumption that $\phi=\exp(2s)/s$. Thus $b^2Q+s\neq 0$. Then from (3.26) we conclude that $\Psi_1=0$ or $\kappa_A=0$.

If $\Psi_1=0$, then using (3.23) we obtain that $\Phi=0$, and from Lemma 3.3, we see that F is a Riemannian metric.

If $\Psi_1\neq 0$, then $\kappa_A=0$. In this case, we prove that $\kappa_1=0$.
Simplifying (3.23) and multiplying it by Δ^2, we get

$$\Big\{[-s\Phi+(b^2-s^2)\Phi']\Delta-\frac{3}{2}(b^2-s^2)\Phi\Delta'\Big\}\kappa_1-cb\Delta^2\Phi(\phi-s\phi')=0. \tag{3.31}$$

Putting $\phi=\exp(2s)/s$ into (3.31) and using maple program, we can obtain the following

$$\kappa_1s^6(\zeta_{10}s^{10}+\cdots+\zeta_0)+2cbe^{2s}(\xi_{14}s^{14}+\cdots+\xi_0)=0, \tag{3.32}$$

where ζ_i $(0\le i\le 10)$ and ξ_j $(0\le j\le 14)$ are polynomials of b. From (3.32), we get

$$\kappa_1(\zeta_{10}s^{10}+\cdots+\zeta_0)=0, \tag{3.33}$$

$$2cb(\xi_{14}s^{14}+\cdots+\xi_0)=0. \tag{3.34}$$

From (3.33), (3.34) and that $\zeta_0 = 2(2-n)b^6$ and $\xi_0 = 4(1+n)b^6$, we conclude that

$$\kappa_1 = 0, \quad c = 0.$$

Thus $\kappa_1 = \kappa_A = 0$. It follows that $\kappa = constant$, which means that F is a locally Minkowski metric. This completes the proof. □

4. **Proof of Theorem 1.2**

In this section, firstly we study the Kropina change of exponential (α, β)-metric with almost vanishing Ξ-curvature. From (2.6), the spray coefficients of a (α, β)-metric $F = \alpha\phi(s),\ s = \beta/\alpha$, are rewritten as follow

$$G^i = G^i_\alpha + H^i, \tag{4.1}$$

where

$$\begin{aligned} H^i &:= \frac{1}{\alpha}\{A\Theta y^i + \alpha A\Psi b^i + \alpha^2 Q s^i{}_0\}, \\ A &:= r_{00} - 2\alpha Q s_0. \end{aligned}$$

We denote $H := \frac{\partial H^m}{\partial y^m}$. By a simple computation, one can see that

$$\begin{aligned} H =& \frac{1}{\alpha}\{(n+1)A\Theta + \alpha Q' s_0 + A\Psi'(b^2 - s^2)\} \\ &+ 2\Psi\big[r_0 - sQs_0 - Q'(b^2 - s^2)s_0\big]. \end{aligned}$$

Let $\Pi := \frac{\partial G^m}{\partial y^m}$. By definition Ξ-curvature of a Finsler metric can be expressed as follows

$$\Xi_i = \Pi_{y^i x^m} y^m - \Pi_{x^i} - 2\Pi_{y^i y^m} G^m. \tag{4.2}$$

Substituting (4.1) in (4.2), we get

$$\Xi_i = H_{.i|m} y^m - H_{|i} - 2H_{.i.m} H^m, \tag{4.3}$$

where "|" denotes the horizontal covariant derivative with respect to α. By compute each terms of the right hand side of (4.3), one can obtaine a formula for Ξ-curvature of an (α, β)-metric. For details of computaition see [26].

Now, we can prove the following lemma.

Lemma 4.1. *Let $F = \alpha \exp(2s)/s, s := \beta/\alpha$ be an (α, β)-metric on an n-dimensional manifold M $(n \geq 3)$ with almost vanishing Ξ-curvature. Then F has vanishing Ξ-curvature.*

Proof. Let $F = \alpha\phi(s),\ s := \frac{\beta}{\alpha}$ has almost vanishing Ξ-curvature, thus there exists a 1-form $\theta := t_i(x)y^i$ on M, such that

$$\Xi_i = -(n+1)F^2\Big(\frac{\theta}{F}\Big)_{y^i} = -\frac{(n+1)}{\alpha}\big[\alpha^2\phi t_i - \phi\theta y_i - \phi'\theta(\alpha b_i - s y_i)\big]. \tag{4.4}$$

Putting $\phi(s) = \alpha\exp(2s)/s$, $s = \beta/\alpha$ and (4.3) in (4.4) and using Maple program, we obtain

$$A_{i28}\alpha^{28}+\cdots+A_{i1}\alpha+A_{i0} = \lambda\big[\alpha\beta(\alpha^2t_i-\theta y_i)-(2\beta-\alpha)\theta(\alpha^2b_i-\beta y_i)\big]e^{2s}, \quad (4.5)$$

where $A_{i28},\ldots,A_{i0}$ are polynomials of s and

$$\lambda = 2\alpha\beta\big[\beta(\alpha-\beta)(\beta^2-2b^2\alpha^2)+b^2\alpha^4-\beta^4\big]^5(\alpha-\beta)^4.$$

If $\lambda = 0$, then

$$\beta(\alpha-\beta)(\beta^2-2b^2\alpha^2)+b^2\alpha^4-\beta^4 = 0. \quad (4.6)$$

From (4.6), we obtain

$$\beta^2(2\alpha^2b^2+\alpha\beta-2\beta^2) = \alpha^3(2\beta-\alpha)b^2,$$

this means that α^2 contains β as a factor, which is impossible, because $n \geq 3$, (see [9]).

Thus, from (4.5) we have

$$\alpha\beta(\alpha^2t_i-\theta y_i)-(2\beta-\alpha)\theta(\alpha^2b_i-\beta y_i) = 0. \quad (4.7)$$

Contracting (4.7) with b^i, yields

$$(\beta t_ib^i+\theta b^2)\alpha^2-2\theta\beta^2 = 0, \quad (4.8)$$

$$2\theta\beta(b^2\alpha^2-\beta^2) = 0. \quad (4.9)$$

Note that $b^2\alpha^2-\beta^2 \neq 0$, thus from (4.9), it follows that $\theta = 0$. Therefore $\mathbf{\Xi} = 0$. □

In order to prove Theorem 1.2, we need the following Lemma.

Lemma 4.2. ([26]) *Let F and $\tilde{F}$ be two conformally related Finsler metrics on a manifold M with conformal factor $\kappa = \kappa(x)$. Then Ξ and $\tilde{\Xi}$-curvatures satisfy*

$$\tilde{\Xi}_i = \Xi_i + B_{.i;m}y^m - B_{;i} + 2Q^r(\mathbf{S}_{.i.r}+B_{.i.r}),$$

where $\kappa_i := \frac{\partial\kappa}{\partial x^i}$, $\kappa^r := g^{ir}\kappa_i$, $B := F^2\kappa^rI_r$, $Q^r := \frac{F^2}{2}\kappa^r$ and ";" denotes the horizontal covariant derivative with respect to the Berwald connection of F.

Proof of Theorem 1.2: Let $F = \alpha\exp(2s)/s, s := \beta/\alpha$ be a Minkowski metric and $\tilde{F} = e^{\kappa(x)}F$ is of almost vanishing Ξ-curvature. It follows that α has zero sectional curvature and $b_{i|j} = 0$, i.e. $b = constant$. If $b = 0$, then $\tilde{F} = e^{\kappa(x)}\alpha$ is a Riemannian metric. Thus we suppose that $b \neq 0$ and show that $\tilde{F}$ is a Minkowski metric. Since F is Minkowski metric by Lemmas 4.1 and 4.2 we have

$$B_{.i;m}y^m - B_{;i} + 2Q^rB_{.i.r} = 0. \quad (4.10)$$

Now we can compute each term of equation (4.10) (see [26]).We denote

$$\kappa^r := g^{ir}\kappa_i,\ \kappa^r_\alpha := a^{ri}\kappa_i,\ f := b^i\kappa_i,\ \kappa_0 := \kappa_i y^i,\ \kappa_{ij} := \frac{\partial\kappa}{\partial x^i\partial x^j},$$
$$f_i := b^r\kappa_{ri},\ f_0 := f_i y^i,\ \kappa_{0i} := \kappa_{ri}y^r,\ \kappa_{00} := \kappa_{ij}y^iy^j,\ \chi := \frac{\kappa_0}{\alpha} + \lambda f,$$
$$\chi_0 := \frac{\kappa_{00}}{\alpha} + \lambda f_0,\ \chi_{.i} := \frac{\partial\chi}{\partial y^i},\ \chi_{0i} := \frac{\kappa_{0i}}{\alpha} - \frac{\kappa_{00}y_i}{\alpha^3} + \lambda' f_0 s_{.i},$$
$$Y^r_{.i} := \frac{\partial Y^r}{\partial y^i},\ \zeta^r := \kappa^r_\alpha - \tau f b^r - \eta\chi Y^r,\ s_{.i} := \frac{1}{\alpha^2}\big(b_i\alpha - s y_i\big),$$
$$y_i := a_{ir}y^r,\ b^i := a^{ir}b_r,\ \Phi_1 := -\frac{\Phi}{2\Delta\phi}(\phi - s\phi').$$

One can see that

$$\kappa^r = \frac{1}{\rho}\zeta^r,\ I_r = \Phi_1\frac{h_r}{\alpha} = \Phi_1 s_{.r},\ I_r\kappa^r = \Phi_1 s_{.r}\kappa^r,\ \kappa^r_{.i} = -\frac{\rho'}{\rho^2}\zeta^r s_{.r} + \frac{1}{\rho}\zeta^r_{.i},$$
$$\kappa^r_{.i.k} = -\Big(\frac{\rho'}{\rho^2}\Big)'\zeta^r s_{.i}s_{.k} - \frac{\rho'}{\rho^2}\Big(\zeta^r_{.k}s_{.i} + \zeta^r s_{.i.k} + \zeta^r_{.i}s_{.k}\Big) + \frac{1}{\rho}\zeta^r_{.i.k},$$
$$I_{r.i} = \Phi_1' s_{.r}s_{.i} + \Phi_1 s_{.i.r},$$
$$I_{r.i.k} = \Phi_1'' s_{.r}s_{.i}s_{.k} + \Phi_1'(s_{.r.i}s_{.k} + s_{.k.r}s_{.i} + s_{.i.k}s_{.r}) + \Phi_1 s_{.r.i.k},$$

Note that F is a Minkowski metric, thus the horizontal covariant derivative with respect to F reduces to common derivative with respect to the position variable x. Thus

$$B_{.i;m}y^m = \frac{\partial B_{.i}}{\partial x^m}y^m = E_{1i} + F^2(E_{2i} + E_{3i}), \tag{4.11}$$

where

$$E_{1i} := (F^2)_{.i}\frac{\partial\kappa^r}{\partial x^m}y^m I_r = 2g_{ij}y^j\frac{\Phi_1}{\rho\alpha^2}\big\{(f_0\alpha - s\kappa_{00}) - (\tau f_0 + \eta\lambda\chi_0)(b^2 - s^2)\alpha\big\}.$$

$$\begin{aligned}E_{2i} := {} & \frac{\partial\kappa^r_{.i}}{\partial x^m}y^m I_r = -\frac{\rho'\Phi_1}{\rho^2\alpha}\big\{f_0(1-\tau b^2) - \eta\chi_0 Y^r b_r\big\}s_{.i} + \frac{s\rho'\Phi_1}{\rho^2\alpha^2}\big\{\kappa_{00} - \tau f_0 s\alpha \\ & - \eta\chi_0 Y^r y_r\big\}s_{.i} - \frac{\Phi_1}{\rho\alpha}\big\{(\tau' f_0 b^2 + \eta'\chi_0 Y^r b_r)s_{.i} + \eta Y^r b_r\chi_{0i} + \eta\chi_0 Y^r_{.i}b_r\big\} \\ & + \frac{s\Phi_1}{\rho\alpha^2}\big\{(\tau' f_0\alpha s - \eta'\chi_0 Y^r y_r)s_{.i} + \eta Y^r y_r\chi_{0i} + \eta\chi_0 Y^r_{.i}y_r\big\}.\end{aligned}$$

$$\begin{aligned}E_{3i} := {} & \frac{\partial\kappa^r}{\partial x^m}y^m I_{r.i} = \frac{1}{\rho}\Big\{\frac{\Phi_1'}{\alpha}\big(f_0 - \frac{s\kappa_{00}}{\alpha})s_{.i} + \Phi_1(-\frac{f_0y_i + \kappa_{00}b_i}{\alpha^3} - \frac{s\kappa_{0i}}{\alpha^2} \\ & + 3s\frac{\kappa_{00}y_i}{\alpha^4})\Big\} - \frac{\eta\chi_0}{\rho}\Big\{\frac{\Phi_1'}{\alpha^2}(Y^r b_r\alpha - Y^r y_r)s_{.i} - \frac{\Phi_1}{\alpha^4}\big[sY^r a_{ir}\alpha^2 \\ & + (Y^r b_r y_i + Y^r y_r b_i)\alpha - 3Y^r y_r y_i\big]\Big\} - \frac{\tau f_0}{\rho}\Big\{\frac{\Phi_1'}{\alpha}(b^2 - s^2)s_{.i} \\ & + \frac{\Phi_1}{\alpha^4}\big[3s^2y_i - b^2y_i\alpha - 2sb_i\alpha^2)\big]\Big\}\end{aligned}$$

and

$$B_{;i} := F^2\frac{\partial\kappa^r}{\partial x^i}I_r = \frac{F^2\Phi_1}{\rho\alpha}\Big\{f_i - \frac{s\kappa_{0i}}{\alpha} - \tau f_i(b^2-s^2) - \eta\lambda(b^2-s^2)\big(\lambda f_i + \frac{\kappa_{0i}}{\alpha}\big)\Big\}. \tag{4.12}$$

Also, we have

$$2Q^rB_{.i.r} = F^2\{2\kappa^rI_r\kappa_i + 2E_4(F^2)_{.i} + 2\kappa_0E_{5i} + F^2E_{6i}\}, \tag{4.13}$$

where

$$E_4 := (\kappa^kI_k)_{.r}\kappa^r = \frac{\Phi_1}{\rho^2}\Big\{-\frac{\rho'}{\rho}(\zeta^rs_{.r})^2 + \zeta^r_{.k}\zeta^ks_{.r}\Big\} + \frac{1}{\rho^2}\Big\{\Phi_1'(\zeta^rs_{.r})^2 + \Phi_1\zeta^r\zeta^ks_{.r.k}\Big\},$$

$$E_{5i} := (\kappa^rI_r)_{.i} = \frac{\Phi_1}{\rho}\Big\{-\frac{\rho'}{\rho}\zeta^rs_{.r}s_{.i} + \zeta^r_{.i}s_{.r}\Big\} + \frac{1}{\rho}\Big\{\Phi_1'\zeta^rs_{.r}s_{.i} + \Phi_1\zeta^rs_{.r.i}\Big\},$$

and

$$E_{6i} := E_{61i} + E_{62i} + E_{63i} + E_{64i},$$

where

$$E_{61i} := -\kappa^r_{.i.k}I_r\kappa^k = -\frac{\Phi_1}{\rho}\Big\{\big(\frac{\rho'}{\rho^2}\big)'(\zeta^rs_{.r})^2s_{.i} + \frac{\rho'}{\rho^2}(\zeta^k\zeta^r_{.k}s_{.r}s_{.i} - \zeta^rs_{.r}\zeta^ks_{.k.i} + \zeta^ks_{.k}\zeta^r_{.i}s_{.r}) - \frac{1}{\rho}\zeta^r_{.i.k}s_{.r}\zeta^k\Big\},$$

$$E_{62i} := I_{r.k}\kappa^r_{.i}\kappa^k = -\frac{\rho'}{\rho^3}\Big\{\Phi_1'(\zeta^rs_{.r})^2 + \Phi_1\zeta^r\zeta^ks_{.k.r}\Big\}s_{.i} + \frac{1}{\rho}\Big\{\Phi_1'\zeta^ks_{.k}\zeta^r_{.i}s_{.r} + \Phi_1\zeta^k\zeta^r_{.i}s_{.k.r}\Big\},$$

$$E_{63i} := \kappa^r_{.k}I_{r.i}\kappa^k = -\frac{\rho'}{\rho^3}\Big\{\Phi_1'(\zeta^rs_{.r})^2s_{.i} + \Phi_1\zeta^ks_{.k}\zeta^rs_{.r.i}\Big\} + \frac{1}{\rho^2}\Big\{\Phi_1'\zeta^r_{.k}\zeta^ks_{.r}s_{.i} + \zeta^k\zeta^r_{.k}s_{.r.i}\Big\},$$

$$E_{64i} := I_{r.i.k}\kappa^r\kappa^k = \frac{1}{\rho^2}\Big\{\Phi_1''(\zeta^rs_{.r})^2s_{.i} + \Phi_1'(\zeta^r\zeta^ks_{.r.k}s_{.i} + 2\zeta^rs_{.r}\zeta^ks_{.k.i}) + \Phi_1\zeta^r\zeta^ks_{.r.k.i}\Big\}.$$

It is easy to see that

$$g_{ij}y^j = (\rho + s\rho_1 + \rho_2)y_i + (s\rho_0 + \rho_1)\alpha b_i,$$

$$s_{.i.r} = -\frac{b_ry_i + y_rb_i}{\alpha^3} + \frac{sa_{ir}}{\alpha^2} + \frac{3sy_ry_i}{\alpha^4},$$

$$
\begin{aligned}
s_{.i.r.k} =& -\frac{1}{\alpha^3}\{a_{ik}b_r + a_{kr}b_i + a_{ri}b_k\} + \frac{3s}{\alpha^4}\{a_{ri}y_k + a_{ik}y_r + a_{kr}y_i\}\\
&+ \frac{3}{\alpha^5}\{b_r y_i y_k + b_i y_k y_r + b_k y_r y_i\} - \frac{15s}{\alpha^6} y_i y_r y_k,\\
\chi_{.i} =& \frac{\kappa_i}{\alpha} - \frac{\kappa_0 y_i}{\alpha^3} + \lambda' f s_{.i},\\
\chi_{.i.k} =& -\frac{1}{\alpha^3}\{\kappa_i y_k + \kappa_k y_i + \kappa_0 a_{ik}\},\\
Y^r b_r =& s + \lambda b^2, \qquad Y^r y_r = \alpha(1 + \lambda s),\\
\kappa^r b_r =& \frac{1}{\rho}\{f - \tau f b^2 - \eta\chi Y^r b_r\}, \qquad \kappa^r y_r = \frac{1}{\rho}\{\kappa_0 - \tau f \alpha s - \eta \chi Y^r y_r\},\\
Y^r_{.i} =& \frac{\delta^r_i}{\alpha} - \frac{y^r y_i}{\alpha^3} + \lambda' b^r s_{.i},\\
Y^r_{.i.k} =& -\frac{1}{\alpha^3}\{\delta^r_i y_k + \delta^r_k y_i - a_{ik} y^r\} + \frac{3y^r y_i y_k}{\alpha^5} + \lambda'' b^r s_{.i} s_{.k} + \lambda' b^r s_{.i.k},\\
\zeta^r_{.i} =& \{\tau' f b^r + \eta' \chi Y^r\} s_{.i} + \eta \chi_{.i} Y^r + \eta \chi Y^r_{.i},\\
\zeta^r_{.i.k} =& \big\{(\tau'' f b^r + \eta'' \chi Y^r) s_{.k} + \eta'(\chi_{.k} Y^r + \chi Y^r_{.k})\big\} s_{.i} + \{\tau' f b^r + \eta' \chi Y^r\} s_{.i.k}\\
&+ \eta'\{\chi_{.i} Y^r + \chi Y^r_{.i}\} s_{.k} + \{\chi_{.i.k} Y^r + \chi_{.i} Y^r_{.k} + \chi_{.k} Y^r_{.i} + \chi Y^r_{.i.k}\}.
\end{aligned}
$$

Now, we take the same coordinate transformation that introduced in the proof of Theorem 1.1 and one can see that

$$\kappa_0 = \frac{\kappa_1}{\sqrt{b^2 - s^2}} s\bar{\alpha} + \bar{\kappa}_0, \tag{4.14}$$

$$\kappa_{00} = \frac{\kappa_{11}}{b^2 - s^2} s^2 \bar{\alpha}^2 + \frac{2\bar{\kappa}_{10}}{\sqrt{b^2 - s^2}} s\bar{\alpha} + \bar{\kappa}_{00}, \tag{4.15}$$

$$\kappa_{0i} = \frac{\kappa_{1i}}{\sqrt{b^2 - s^2}} s\bar{\alpha} + \bar{\kappa}_{0i}, \tag{4.16}$$

$$f = \kappa_1 b, \tag{4.17}$$

$$f_0 = \frac{f_1}{\sqrt{b^2 - s^2}} s\bar{\alpha} + \bar{f}_0, \tag{4.18}$$

where

$$\bar{\kappa}_0 := \sum_{A=2}^{n} \kappa_A u^A,\ \bar{\kappa}_{10} := \sum_{A=2}^{n} \kappa_{1A} u^A,\ \bar{\kappa}_{00} := \sum_{A,B=2}^{n} \kappa_{AB} u^A u^B,$$

$$\bar{\kappa}_{0i} := \sum_{A=2}^{n} \kappa_{Ai} u^A,\ \bar{f}_0 := \sum_{A=2}^{n} f_A u^A.$$

Substitiuting (4.11)-(4.13) in (4.10) and using Malpe program, yields

$$A_{0i} + A_{2i}\bar{\alpha}^2 + A_{4i}\bar{\alpha}^4 + A_{6i}\bar{\alpha}^6 + \sqrt{b^2 - s^2}(A_{1i}\bar{\alpha} + A_{3i}\bar{\alpha}^3 + A_{5i}\bar{\alpha}^5) = 0 \tag{4.19}$$

where $A_{0i}, \ldots A_{6i}$ are polynomials of s. Thus

$$A_{0i} + A_{2i}\bar{\alpha}^2 + A_{4i}\bar{\alpha}^4 + A_{6i}\bar{\alpha}^6 = 0, \tag{4.20}$$

$$A_{1i} + A_{3i}\bar{\alpha}^2 + A_{5i}\bar{\alpha}^4 = 0. \tag{4.21}$$

From (4.21), for $i = 1$ we obtain

$$A_5\bar{\alpha}^5 + A_4\bar{\alpha}^4 + A_3\bar{\alpha}^3 + A_2\bar{\alpha}^2 + A_1\bar{\alpha} + A_0 = 0, \tag{4.22}$$

where $A_0, \ldots, A_5$ are polynimials of s. From (4.22) have

$$A_4\bar{\alpha}^4 + A_2\bar{\alpha}^2 + A_0 = 0 \tag{4.23}$$

$$A_5\bar{\alpha}^4 + A_3\bar{\alpha}^2 + A_1 = 0 \tag{4.24}$$

By Maple program, (4.24) implies that

$$\mathcal{M}_{37}s^{37} + \mathcal{M}_{36}s^{36} + \mathcal{M}_{35}s^{35} + \cdots + \mathcal{M}_0 = 0, \tag{4.25}$$

where $\mathcal{M}_i,\ 0 \leq i \leq 37$ are funstions of κ and $\bar{\alpha}$, and specially

$$\mathcal{M}_{37} := 49152n\bar{\kappa}_0\kappa_1, \tag{4.26}$$

$$\mathcal{M}_{36} := 64n(3\bar{\kappa}_{10} - 8866\bar{\kappa}_0\kappa_1), \tag{4.27}$$

$$\mathcal{M}_{35} := 32(3 - 65n)\bar{\kappa}_{10} + 512((5871 - 826b^2)n - 36)\bar{\kappa}_0\kappa_1 + 3072nb\bar{\kappa}_0^2, \tag{4.28}$$

$$\mathcal{M}_0 := 3(1 + n)b^{17}(\kappa_1^2\bar{\alpha}^2 + \bar{\kappa}_0^2 b^2). \tag{4.29}$$

From (4.25) we have $\mathcal{M}_i = 0,\ (0 \leq i \leq 37)$. From (4.26) and (4.27) we obtain $\bar{\kappa}_0\kappa_1 = \bar{\kappa}_{10} = 0$. Thus from (4.28) we get $\bar{\kappa}_0 = 0$, i.e.

$$\kappa_A = 0. \tag{4.30}$$

Putting (4.30) in (4.29) we obtain

$$\kappa_1 = 0. \tag{4.31}$$

From (4.30) and (4.31) it follows that $\kappa = constant$. Thus $\tilde{F} = e^{\kappa}F$ is locally Minkowski metric. ☐

5. Proof of Theorem 1.3

In this section, we study conformally flat weak Einstein Finsler metric $F = \alpha\exp(2s)/s,\ s := \beta/\alpha$. The Ricci curvature of a conformally flat (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ is computed in [4]. In fact we have the following.

Lemma 5.1. *Let $F = \alpha\phi(s),\ s := \beta/\alpha$ be a conformally flat (α, β)-metric on a manifold M, that is, there exists a locally Minkowski metric $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ such that $F = \exp(\kappa)\tilde{F}$, where $\kappa = \kappa(x)$ is a scalar function on M. Then, the Ricci curvature of F is determined by*

$$\begin{aligned}\mathbf{Ric} =& c_1\|\nabla\kappa\|_{\tilde{\alpha}}^2\tilde{\alpha}^2 + c_2\kappa_0^2 + c_3\kappa_0 f\tilde{\alpha} + c_4 f^2\tilde{\alpha}^2 \\ &+ c_5 f_1\tilde{\alpha} + c_6\tilde{\alpha}^2 + c_7\kappa_{00}.\end{aligned} \tag{5.1}$$

where

$$
\begin{aligned}
c_6 &:= c_{61}\tilde{a^{ij}}\kappa_{ij} + c_{62}f_2,\\
c_{61} &:= -\frac{\phi}{\phi - s\phi'},\\
c_{62} &:= \frac{\phi\phi''}{(\phi - s\phi')\big[(\phi - s\phi')(b^2 - s^2)\phi''\big]}.
\end{aligned}
$$

and $\|\nabla\kappa\|^2_{\tilde{\alpha}} := \tilde{a}^{ij}\kappa_i\kappa_j, f := \tilde{b}^i\kappa_i,\ f_1 := \tilde{b}^i y^j\kappa_{ij},\ f_2 := \tilde{b}^i\tilde{b}^j\kappa_{ij}$. *Here* c_1, c_2, c_3, c_4, c_5, c_7 *are the functions only in* s *and are independent of the* $\tilde{\alpha}, \kappa_0, \kappa_{00}, f, f_1$, f_2 *and* $\tilde{a}^{ij}\kappa_{ij}$.

Firstly, we show that if $F = \alpha\exp(2s)/s$ be a conformally flat weak Einstein metric then it is a Ricci flat metric.

Lemma 5.2. *Let* $F = \alpha\phi(s),\ s := \beta/\alpha$, *be a conformally flat weak Einstein metric on a manifold* M *with the dimension* $n \geq 3$, *where* $\phi(s) = \exp(2s)/s$. *Then* F *is a Ricci flat metric.*

Proof. Since F is a conformally flat weak Einstein metric, from (1.3) and (5.1) we have

$$
\begin{aligned}
(n-1)\Big(\frac{3\theta}{F} + \sigma\Big)F^2 =& c_1\|\nabla\kappa\|^2_{\tilde{\alpha}}\tilde{\alpha}^2 + c_2\kappa_0^2 + c_3\kappa_0 f\tilde{\alpha} + c_4 f^2\tilde{\alpha}^2\\
&+ c_5 f_1\tilde{\alpha} + c_6\tilde{\alpha}^2 + c_7\kappa_{00},
\end{aligned}
\tag{5.2}
$$

where $\sigma = \sigma(x)$ and $\theta := t_i(x)y^i$. Let

$$A_1 := (s-1),\quad A_2 := (s^3 + 2b^2s^2 - 2b^2s + b^2 - 2s^4).$$

We put $\phi(s) = \exp(2s)/s$ and by use of Maple program have

$$
\begin{aligned}
&c_1 = \frac{\bar{c}_1}{8A_1^3A_2^2},\quad c_2 = \frac{\bar{c}_2}{16A_1^3A_2^2},\quad c_3 = \frac{\bar{c}_3}{8A_1^3A_2^4},\quad c_4 = \frac{\bar{c}_4}{16A_1^3A_2^4},\\
&c_5 = \frac{\bar{c}_5}{4A_1A_2^2},\quad c_{61} = \frac{\bar{c}_{61}}{2A_1},\quad c_{62} = \frac{\bar{c}_{62}}{2A_1A_2},\quad c_7 = \frac{\bar{c}_7}{4A_1A_2^2},
\end{aligned}
\tag{5.3}
$$

where $\bar{c}_1, \ldots, \bar{c}_7$ are polynomials of s.

We take the same coordinate transformation that is used in Theorem 1.1 for $\tilde{\alpha}$ and $\tilde{\beta}$. In this case we have

$$\tilde{\alpha} = \frac{\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha},\qquad \tilde{\beta} = \frac{\tilde{b}s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}.$$

Thus

$$F = \exp(\kappa)\tilde{\alpha}\phi(s) = \frac{\exp(k)\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}\phi(s),$$
$$f = \kappa_1\tilde{b}, \quad f_1 = \frac{\tilde{b}s\kappa_{11}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha} + \tilde{b}\bar{\kappa}_{10},$$
$$f_2 = \tilde{b}^2\kappa_{11}, \qquad \theta = \frac{t_1 s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}^2 + \bar{t}_0, \tag{5.4}$$

where $\bar{t}_0 := \Sigma_{A=2}^n t_A u^A$. Using (5.4), Eq. (5.2) is equivalent with the following equations:

$$\Big[c_1\tilde{b}^2\|\nabla\kappa\|^2_{\bar{\alpha}} + c_2\kappa_1^2 s^2 + c_3\tilde{b}^2\kappa_1^2 s + c_4\kappa_1^2\tilde{b}^4 + c_5\kappa_{11}\tilde{b}^2 s$$
$$+ (c_{61}\delta^{ij}\kappa_{ij} + c_{62}\kappa_{11}\tilde{b}^2)\tilde{b}^2 + c_7\kappa_{11}s^2\Big]\frac{\bar{\alpha}^2}{\tilde{b}^2 - s^2} + c_2\bar{\kappa}_0^2 + c_7\bar{\kappa}_{00}$$
$$= \frac{(n-1)e^{\kappa}(3t_1\tilde{b}s + \sigma e^{\kappa}\tilde{b}^2\phi)}{\tilde{b}^2 - s^2}\bar{\alpha}^2\phi, \tag{5.5}$$
$$(2c_2 s + c_3\tilde{b}^2)\kappa_1\kappa_A + (c_5\tilde{b}^2 + 2c_7 s)\kappa_{1A} = 3(n-1)e^{\kappa}t_A\tilde{b}\phi. \tag{5.6}$$

Substituting (5.3) in (5.5) and (5.6) and multiplying by $16A_1^3A_2^4(\tilde{b}^2 - s^2)$ we get

$$\Big[2\bar{c}_1\tilde{b}^2\|\nabla\kappa\|^2_{\bar{\alpha}}A_2^2 + \bar{c}_2A_2^2\kappa_1^2 s^2 + 2\bar{c}_3\tilde{b}^2\kappa_1^2 s + \bar{c}_4\kappa_1^2\tilde{b}^4 + 4\bar{c}_5A_1^2A_2^2\kappa_{11}\tilde{b}^2 s$$
$$+ (8\bar{c}_{61}A_1^2A_2^4\delta^{ij}\kappa_{ij} + 8\bar{c}_{62}A_1^2A_2^3\kappa_{11}\tilde{b}^2)\tilde{b}^2 + c_7\kappa_{11}s^2\Big]\bar{\alpha}^2 + (\bar{c}_2A_2^2\bar{\kappa}_0^2$$
$$+ 4\bar{c}_7A_1^2A_2^2\bar{\kappa}_{00})(\tilde{b}^2 - s^2) = (n-1)e^{\kappa}(3t_1\tilde{b}s + \sigma e^{\kappa}\tilde{b}^2\phi)\bar{\alpha}^2\phi, \tag{5.7}$$
$$(2\bar{c}_2A_2^2 s + 2\bar{c}_3\tilde{b}^2)\kappa_1\kappa_A + (4\bar{c}_5A_1^2A_2^2\tilde{b}^2 + 8\bar{c}_7A_1^2A_2^2 s)\kappa_{1A} = 3(n-1)e^{\kappa}t_A\tilde{b}\phi. \tag{5.8}$$

Since $\phi(s) = \exp(2s)/s$, one can see that, the left hand side of (5.7) is a polynomial of s, meanwhile the right hand side is a multiple of exponential function $\phi(s) = \exp(2s)$. Thus

$$3t_1\tilde{b}s + \sigma e^{\kappa}\tilde{b}^2\phi = 0$$

Thus $\sigma = t_1 = 0$. By the same way, from (5.8), we get $t_A = 0$. Therefore $\theta = \sigma = 0$ and thus $\mathbf{Ric} = 0$. □

Now, lets to prove Theorem 1.3.

Proof of Theorem 1.3: We suppose that $\tilde{b} \neq 0$. Using $t_i = \sigma = 0$, from the Eq. (5.5), it follows that, there exists a function $\xi := \xi(s)$ such that

$$c_2\kappa_A\kappa_B + c_7\kappa_{AB} = \xi(s)\delta_{AB}. \tag{5.9}$$

For $A \neq B$, Eq. (5.9) reduced to

$$c_2\kappa_A\kappa_B + c_7\kappa_{AB} = 0. \tag{5.10}$$

Multiplying (5.10) by $16A_1^3A_2^4$ and using Maple program, one can see that

$$192(2-n)\kappa_{AB}s^{18} - 16\big[(2-n)(3\kappa_A\kappa_B - 62\kappa_{AB}) + 6\kappa_{AB}n\big]s^{17} + \mathcal{N}_{16}s^{16} + \cdots + \mathcal{N}_0 = 0,$$

where $\mathcal{N}_0, \ldots, \mathcal{N}_{16}$ are finctions of $\kappa_A, \kappa_B, \kappa_{AB}$. It follows that

$$\kappa_{AB} = \kappa_A\kappa_B = 0, \ (A \neq B).$$

Since F is Ricci flat, (5.6) becomes to

$$(2c_2s + c_3\tilde{b}^2)\kappa_1\kappa_A + (c_5\tilde{b}^2 + 2c_7s)\kappa_{1A} = 0. \tag{5.11}$$

Multiplying (5.11) with $4A_1A_2^4$ and by Maple program, we get

$$96(n-2)\kappa_{1A}s^{16} - 8[(n-2)(38\kappa_{1A} + 3\kappa_1\kappa_A) - 6\kappa_{1A}]s^{15} + \mathcal{P}_{14}s^{14} + \cdots + \mathcal{P}_0 = 0. \tag{5.12}$$

From (5.12), we have that

$$\kappa_{1A} = \kappa_1\kappa_A = 0 \tag{5.13}$$

Now, we prove that $\kappa_1 = \kappa_A = 0$. By multiple (5.5) in $16A_1^3A_2^4(\tilde{b}^2 - s^2)$ and use Maple program we get

$$192(n-2)(\bar{\kappa}_{00} - \kappa_{11}\bar{\alpha}^2)s^{20} - 16\big[\big(62(n-2) - 3\big)(\bar{\kappa}_{00} - \kappa_{11}\bar{\alpha}^2) + 3(n-2)(\bar{\kappa}_0^2 - \kappa_1^2\bar{\alpha}^2)\big]s^{19} + \mathcal{M}_{18}s^{18} + \cdots + \mathcal{M}_0 = 0, \tag{5.14}$$

where $\mathcal{M}_i$, $(0 \leq i \leq 18)$ are polynomials of $\kappa_1, \kappa_{11}, \bar{\kappa}_0, \bar{\kappa}_{00}$.

From (5.14), we have

$$\bar{\kappa}_0^2 - \kappa_1^2\bar{\alpha}^2 = 0 \tag{5.15}$$

From (5.13) and (5.15), it follows that $\kappa_1 = \kappa_A = 0$. Thus κ is a constant and therefore F is a locally Minkowski metric.

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