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Flag curvature of invariant 3-power metrics on homogeneous spaces

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Abstract. In this paper, we consider invariant 3-power metric $F = (\alpha + \beta)^3/\alpha^2$ such that induced by invariant Riemannian metrics $\tilde{a}$ and invariant vector fields $\tilde{X}$ on homogeneous spaces. We give an explicit formula for the flag curvature of invariant 3-power metrics.

Keywords: 3-power (α, β) -metric, Flag curvature, Homogeneous space, Invariant 3-power metrics.

1. Introduction

Finsler geometry is a natural generalization of Riemannian geometry. A Riemannain metric is quadratic in the fiber coordinates y while a Finsler metric is not necessary be quadratic in y [15]. The geometry of invariant Finsler metrics on homogeneous manifolds is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers (see [1, 4, 5, 6, 8, 11, 12]).

The flag curvature is a generalization of the sectional curvature of Riemannian geometry. Alternatively, flag curvatures can be treated as Jacobi endomorphisms. The flag curvature has also led to a pinching (sphere) theorem for Finsler metrics. Installing a flag on a Finsler manifold (M, F) implies choosing:

(1) a basepoint $x \in M$ at which the flag will be planted,

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(2) a flagpole given by a nonzero $y \in T_xM$, and
(3) an edge $V \in T_xM$ transverse to the flagpole.

Note that the flagpole $y \neq 0$ singles out an inner product

$$g_y := g_{ij}(x,y)dx^i \otimes dx^j.$$

This g_y allows us to measure the angle between V and y. It also enables us to calculate the area of the parallelogram formed by V and $l := y/F(x,y)$.

The flag curvature is defined as

$$K(x,y,V) := \frac{V^i(y^j R_{jikl} y^l)V^k}{g_y(y,y)g_y(V,V) - g_y^2(y,V)},$$

where the index i on R^i_{jkl} has been lowered by g_y. When the Finsler function F comes from a Riemannian metric, g_y is simply the Riemannian metric, R_{jikl} is the usual Riemann tensor, and $K(x,y,V)$ reduces to the familiar sectional curvature of the 2-plane spanned by $\{y, V\}$.

An (α,β)-metric is a Finsler metric of the form $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. Some important (α,β)-metrics are Randers metric, infinite metric, Matsumoto metric, Kropina metric, etc [10]. For more details about special (α,β)-metrics see [3, 7, 9, 14, 15].

The class of p-power (α,β)-metrics on a manifold M is in the following form

$$F = \alpha\big(1 + \frac{\beta}{\alpha}\big)^p, \tag{1.1}$$

where $p \neq 0$ is a real constant, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. In (1.1), if $p = 1$, then

$$F = \alpha + \beta,$$

satisfying $b := ||\beta||_\alpha < 1$ is called a Randers metric. If $p = 2$, then

$$F = \frac{(\alpha+\beta)^2}{\alpha},$$

satisfying $b := ||\beta||_\alpha < 1$ is called a square metric. Square metrics have been shown to have some special geometric properties. If $p = 1/2$, then

$$F = \sqrt{\alpha(\alpha+\beta)},$$

satisfying $b := ||\beta||_\alpha < 1$ is called a square-root metric. For properties of square-root metrics see [15].

In this paper, we consider $p = 3$, then

$$F = \frac{(\alpha+\beta)^3}{\alpha^2}, \tag{1.2}$$

satisfying $b := ||\beta||_\alpha < 1/2$ is called a 3-power metrics. We give an explicit formula for the flag curvature of invariant 3-power metrics.

2. **Preliminaries**

Let M be a n- dimensional C^∞ manifold and $TM = \cup_{x\in M} T_xM$ the tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties [2]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The $n \times n$ Hessian matrix

$$[g_{ij}] = \frac{1}{2}\Big[\frac{\partial^2 F^2}{\partial y^i \partial y^j}\Big]$$

is positive definite at every point $(x, y) \in TM_0$.

The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

We recall that, by the homogeneity of F we have

$$g_y(u, v) = g_{ij}(x, y)u^i v^j, \quad F = \sqrt{g_{ij}(x, y)u^i v^j}.$$

Definition 2.1. [14] Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x, y) = b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold M. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{2.1}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \le b < b_0. \tag{2.3}$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.2) is called an $(\alpha, \beta)-$metric.

We note that, a Finsler space having the Finsler function:

$$F = \frac{(\alpha+\beta)^3}{\alpha^2},$$

is called a 3-power space.

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T_x^*M such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T_x^*M induces a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{2.4}$$

Also we have

$$\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha.$$

Therefore we can write 3-power metrics as follows:

$$F(x,y) = \frac{(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X_x, y))^3}{\tilde{a}(y,y)}, \tag{2.5}$$

where for any $x \in M$, the following holds

$$\sqrt{\tilde{a}(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < \frac{1}{2}.$$

Suppose $W = W^i \frac{\partial}{\partial x^i}$ be a non-vanishing vector field on an open subset D of M. We can introduce a Riemannian metric g_W and a linear connection ∇^W on the tangent bundle over D as following:

$$g_W(X,Y) = X^i Y^j g_{ij}(x, W), \quad \forall\ X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j},$$

$$\nabla^W_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij}(x, V) \frac{\partial}{\partial x^k}.$$

Now since the Chern connection is torsion free and g-compatible we have:

$$\nabla^W_X Y - \nabla^W_Y X = [X, Y],$$

$$X_{g_W}(Y, Z) = g_W(\nabla^W_X Y, Z) + g_W(Y, \nabla^W_X Z) + 2C_W(\nabla^W_X V, Y, Z),$$

where C denotes the Cartan tensor.

The curvature tensor $R^W(X,Y)Z$ for vector fields X, Y, Z on D is defined by

$$R^W(X,Y)Z = \nabla^W_X \nabla^W_Y Z - \nabla^W_Y \nabla^W_X Z - \nabla^W_{[X,Y]} Z.$$

For a Finsler manifold (M, F) and a flag $(X; P)$ consisting of a nonzero tangent vector $X \in T_xM$ and a plane $P \subset T_xM$ spanned by the tangent vector X and Y , the flag curvature defined as

$$K(X; P) := \frac{g_X(R^X(Y, X)X, Y)}{g_X(X,X) g_X(Y,Y) - g_X(X,Y)}. \tag{2.6}$$

We note that in [11], Parhizkar and Latifi gives a formula for the flag curvature of a left invariant (α, β)-metrics.

3. Flag curvature of invariant 3-power metrics on homogeneous spaces

Let G be a compact Lie group, H a closed subgroup, and $\lll -, - \ggg$ a bi-invariant Riemannian metric on G. Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively. The tangent space of the homogeneous space G/H is given by the orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $\lll -, - \ggg$. Each invariant metric $\mathfrak{g}$ on G/H is determined by its restriction to $\mathfrak{m}$. The arising Ad_H-invariant inner product from g on $\mathfrak{m}$ can extend to an Ad_H-invariant inner product on $\mathfrak{g}$ by taking $\lll -, - \ggg$ for the components in $\mathfrak{h}$. In this way the invariant metric g on G/H determines a unique left invariant metric on G that we also denote by g. The values of $\lll -, - \ggg$ and g at the identity are inner products on $\mathfrak{g}$, we denote them by $\lll -, - \ggg$ and $\ll -, - \gg$ respectively. The inner product $\ll -, - \gg$ determines a positive definite endomorphism ϕ of $\mathfrak{g}$ such that

$$\ll X, Y \gg = \lll \phi X, Y \ggg, \quad \forall X, Y \in \mathfrak{g}.$$

Püttmann has shown that the curvature tensor of the invariant metric $\ll -, - \gg$ on the compact homogeneous space G/H is given by [13]:

$$\begin{aligned} \ll R(X,Y)Z, W \gg = & \frac{1}{2}\Big(\lll B_-(X,Y), [Z,W] \ggg + \lll [X,Y], B_-(Z,W) \ggg \Big) \\ & + \frac{1}{4}\Big(\ll [X,W], [Y,Z]_{\mathfrak{m}} \gg - \ll [X,Z], [Y,W]_{\mathfrak{m}} \gg \\ & - 2 \ll [X,Y], [Z,W]_{\mathfrak{m}} \gg \Big) \\ & + \Big(\lll B_+(X,W), \phi^{-1} B_+(Y,Z) \ggg \\ & - \lll B_+(X,Z), \phi^{-1} B_+(Y,W) \ggg \Big), \end{aligned} \tag{3.1}$$

where

$$B_+(X,Y) = \frac{1}{2}([X, \phi Y] + [Y, \phi X]),$$
$$B_-(X,Y) = \frac{1}{2}([\phi X, Y] + [X, \phi Y]).$$

In this section, we are going to study the flag curvature of invariant 3-power metrics on homogeneous spaces.

Theorem 3.1. *Assume that G be a compact Lie group, H a closed subgroup, $\lll -, - \ggg$ a bi-invariant metric on G and $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Further, assume that $\tilde{a}$ be any invariant Riemannian metric on the homogeneous space G/H such that $\tilde{a}(Y,Z) = \lll \phi Y, Z \ggg$ where $\phi : \mathfrak{g} \to \mathfrak{g}$ is a positive definite endomorphism and $Y, Z \in \mathfrak{g}$. Also suppose that $\tilde{X}$ is an* *invariant vector field on G/H where is parallel with respect to $\tilde{a}$ and $\tilde{X}_H = X$. Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ be the 3-power metric arising from $\tilde{a}$ and $\tilde{X}$ and (P, Y) be a flag in $T_n\frac{G}{H}$ such that $\{U, Y\}$ is an orthonormal basis of P with respect to $\tilde{a}$. Then the flag curvature of the flag (P, Y) is given by*

$$K(P,Y) := \frac{Q\tilde{a}(R(U,Y)Y,U) + N\tilde{a}(X,U)\tilde{a}(R(U,Y)Y,X)}{(1+s)^3\Big(Q + (6+24r+36r^2+24r^3+6r^4)\tilde{a}^2(X,U)\Big)}. \tag{3.2}$$

where

$$r := \frac{\tilde{a}(X,Y)}{\sqrt{\tilde{a}(Y,Y)}}$$

and

$$Q = -2r^6 - 9r^5 - 15r^4 - 10r^3 + 3r + 1, \quad N = 15r^4 + 60r^3 + 90r^2 + 60r + 15,$$

$$\begin{aligned}\tilde{a}(R(U,Y)Y,U) =& \frac{1}{2} \lll [\phi U, Y] + [U, \phi Y], [Y, U] \ggg \\ &+ \frac{3}{4}\tilde{a}([Y,U],[Y,U]_{\mathfrak{m}}) + \lll [U,\phi U], \phi^{-1}([Y,\phi Y]) \ggg \\ &- \frac{1}{4} \lll [U,\phi Y] + [Y,\phi U], \phi^{-1}([Y,\phi U] + [U,\phi Y]) \ggg,\end{aligned}$$

and

$$\begin{aligned}\tilde{a}(R(U,Y)Y,X) =& \frac{1}{4}\Big(\lll [\phi U, Y] + [U,\phi Y], [Y, X] \ggg \\ &+ \lll [U,Y], [\phi Y, X] + [Y, \phi X] \ggg \Big) \\ &+ \frac{3}{4}\tilde{a}([Y,U],[Y,X]_{\mathfrak{m}}) \\ &+ \frac{1}{2} \lll [U,\phi X] + [X,\phi U], \phi^{-1}[Y,\phi Y] \ggg \\ &- \frac{1}{4} \lll [U,\phi Y] + [Y,\phi U], \phi^{-1}([Y,\phi X] + [X,\phi Y]) \ggg .\end{aligned}$$

Proof. Since $\tilde{X}$ is parallel with respect to $\tilde{a}$, then β is parallel with respect to α. Therefore F is a Berwald metric, i.e. the Chern connection of F coincide with the Riemannian connection of $\tilde{a}$. Therefore, F has the same curvature tensor as that of the Riemannian metric $\tilde{a}$ and we denote it by R.

Now by using the formula

$$g_Y(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(Y + sU + tV)\Big]|_{s=t=0},$$

and some computations for the 3-power metric F defined by the following

$$F(x,y) = \frac{(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X_x, y))^3}{\tilde{a}(y,y)},$$

we get:

$$\begin{aligned} g_Y(U,V) \quad = \quad & (1+3r+3r^2+r^3)^2\tilde{a}(U,V) \\ & +(3r^5+15r^4+30r^3+30r^2+15r+3)\tilde{a}(Y,U) \\ & \times\left(\frac{\tilde{a}(X,V)}{\sqrt{\tilde{a}(Y,Y)}}-\frac{\tilde{a}(X,Y)\tilde{a}(Y,V)}{(\tilde{a}(Y,Y))^{3/2}}\right) \\ & +(15r^4+60r^3+90r^2+60r+15)\left(\frac{\tilde{a}(X,V)}{\sqrt{\tilde{a}(Y,Y)}}-\frac{\tilde{a}(X,Y)\tilde{a}(Y,V)}{(\tilde{a}(Y,Y))^{3/2}}\right) \\ & \times\left(\tilde{a}(X,U)\sqrt{\tilde{a}(Y,Y)}-\frac{\tilde{a}(Y,U)\tilde{a}(X,Y)}{\sqrt{\tilde{a}(Y,Y)}}\right) \\ & +\frac{(3r^5+15r^4+30r^3+30r^2+15r+3)}{\sqrt{\tilde{a}(Y,Y)}} \\ & \times(\tilde{a}(X,U)\tilde{a}(Y,V)-\tilde{a}(U,V)\tilde{a}(X,Y)), \end{aligned} \tag{3.3}$$

where

$$r=\frac{\tilde{a}(X,Y)}{\sqrt{\tilde{a}(Y,Y)}}.$$

From equation (3.3) we have:

$$g_Y(U,U)=(15r^4+60r^3+90r^2+60r+15)\tilde{a}^2(X,U)+(-2r^6-9r^5-15r^4-10r^3+3r+1), \tag{3.4}$$

$$g_Y(Y,Y)=(1+r)^6=(1+3r+3r^2+r^3)^2, \tag{3.5}$$

and

$$g_Y(Y,U)=(3r^5+15r^4+30r^3+30r^2+15r+3)\tilde{a}(X,U). \tag{3.6}$$

So we get

$$\begin{aligned} g_Y(Y,Y).g_Y(U,U)-g_Y^2(Y,U) =& (1+3r+3r^2+r^3)^2 \\ & \times\Big((-2r^6-9r^5-15r^4-10r^3+3r+1) \\ & +(6+24r+36r^2+24r^3+6r^4)\tilde{a}^2(X,U)\Big). \end{aligned} \tag{3.7}$$

Furthermore, we have

$$\begin{aligned} g_Y(R(U,Y)Y,U) =& (-2r^6-9r^5-15r^4-10r^3+3r+1)\tilde{a}(R(U,Y)Y,U) \\ & \big((3r^5+15r^4+30r^3+30r^2+15r+3)\tilde{a}(X,U) \\ & -(15r^4+60r^3+90r^2+60r+15)\tilde{a}(X,U)r\big) \\ & \times\tilde{a}(R(U,Y)Y,Y) \\ & +\big((15r^4+60r^3+90r^2+60r+15)\big)\tilde{a}(X,U)\tilde{a}(R(U,Y)Y,X). \end{aligned} \tag{3.8}$$

Now by using Püttmann's formula and some computations we get:

$$\begin{aligned}\tilde{a}(R(U,Y)Y,U) =&\frac{1}{2} \lll [\phi U,Y]+[U,\phi Y],[Y,U] \ggg \\ &+\frac{3}{4}\tilde{a}([Y,U],[Y,U]_{\mathfrak{m}})+ \lll [U,\phi U],\phi^{-1}([Y,\phi Y]) \ggg \\ &-\frac{1}{4} \lll [U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi U]+[U,\phi Y]) \ggg,\end{aligned} \tag{3.9}$$

$$\tilde{a}(R(U,Y)Y,Y)=0, \tag{3.10}$$

and

$$\begin{aligned}\tilde{a}(R(U,Y)Y,X) =&\frac{1}{4}\Big(\lll [\phi U,Y]+[U,\phi Y],[Y,X] \ggg \\ &+ \lll [U,Y],[\phi Y,X]+[Y,\phi X] \ggg \Big) \\ &+\frac{3}{4}\tilde{a}\big([Y,U],[Y,X]_{\mathfrak{m}}\big) \\ &+\frac{1}{2} \lll [U,\phi X]+[X,\phi U],\phi^{-1}[Y,\phi Y] \ggg \\ &-\frac{1}{4} \lll [U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi X]+[X,\phi Y]) \ggg .\end{aligned} \tag{3.11}$$

Substituting the equations (3.4)-(3.11) in (2.6) give us the proof. □

References

1. P. Bahmandoust and D. Latifi, *Naturally reductive homogeneous* (α,β)*- spaces*,Int. J. Geom. Methods Mod. Phys. **17** (8), (2020), 2050117.
2. D. Bao, S. S. Chern, Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, NEWYORK,(2000).
3. M. Ebrahimi and D. Latifi, *On flag curvature and homogeneous geodesics of left invariant Randers metrics on the semi-direct product* $a \oplus_p r$, Journal of Lie Theory, **29**, (2019), 619-627.
4. D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces*, J. Geom. Phys. **57**, (2007), 1421–1433.
5. D. Latifi, A. Razavi, *On homogeneous Finsler spaces*, Rep. Math. Phys, **57**, (2006) 357-366. Erratum: Rep. Math. Phys. 60, (2007), 347.
6. D. Latifi, *Bi-invariant Randers metrics on Lie groups*, Publ. Math. Debrecen., **76** 1-2, (2010), 219–226.
7. D. Latifi and M. Toomanian, *Invariant naturally reductive Randers metrics on homogeneous spaces*, Math Sci., **6** 63, (2012).
8. D. Latifi, *Bi-invariant* (α,β)*- metrics on Lie groups*, Acta Universitatis Apulensis **65**, (2021), 121-131.
9. D. Latifi, *On generalized symmetric square metrics*, Acta Universitatis Apulensis, **68**, (2021), 63-70.
10. M. Matsumoto, *Theory of Finsler spaces with* (α,β)*-metric*, Rep. Math. Phys. **31**, (1992), 43-83.

11. M. Parhizkar and D. Latifi, *On the flag curvature of invariant* (α, β)*- metrics*, Int. J.Geom. Methods Mod. Phys., **13**, (2016), 1650039, 1-11.
12. M. Parhizkar and D. Latifi, *On invariant Matsumoto metrics*, Vietnam J. Math., **47**, (2019), 355–365.
13. T. Püttmann, *Optimal pinching constants of odd dimensional homogeneous spaces*, Invent. Math., **138**, (1999), 631–684.
14. M. L. Zeinali, *On generalized symmetric Finsler spaces with some special* (α, β) *-metrics*, Journal of Finsler Geometry and its Applications, **1**, No. 1, (2020), 45-53.
15. M. L. Zeinali, *Some results in generalized symmetric square-root spaces*, Journal of Finsler Geometry and its Applications, **3**, No. 2, (2022), 13-19.

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