

New Special Finsler Spaces

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Abstract. The pullback approach to global Finsler geometry is adopted. Some new types of special Finsler spaces are introduced and investigated, namely, Ricci, generalized Ricci, projectively recurrent and m-projectively recurrent Finsler spaces. The properties of these special Finsler spaces are studied and the relations between them are singled out.

Keywords: recurrent, Ricci recurrent, concircularly recurrent, generalized Ricci, projectively recurrent, m-projectively recurrent.

1. Introduction

Many types of recurrence in Riemannian geometry have been studied by many authors [3, 4, 7, 8, 9, 10, 11, 12]. On the other hand, some types of recurrence in Finsler geometry have been also studied [5, 6, 13, 14].

In a recent paper [14], we have introduced and investigated *intrinsically* three classes of recurrence in Finsler geometry: simple recurrence, Ricci recurrence and concircular recurrence. Each of these classes consists of four types of recurrence. We also investigated the interrelationships between the different types of recurrence.

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The present paper is a continuation of [14], where we introduce and investigate some new types of special Finsler spaces, namely, Ricci, generalized Ricci, projectively recurrent and m-projectively recurrent Finsler spaces. Some Finsler tensors are defined and their properties are studied. These tensors are used to define the projectively recurrent and m-projectively recurrent Finsler spaces. The relations between the above mentioned spaces are investigated.

2. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1, 15, 16, 17, 18, 20]. We shall use the notations of [15].

In what follows, we denote by $\pi : TM \rightarrow M$ the tangent bundle to M , $\mathfrak{F}(TM)$ the algebra of C^∞ functions on TM , $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$ -module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π -vector fields and will be denoted by barred letters \bar{X} . The tensor fields on $\pi^{-1}(TM)$ will be called π -tensor fields. The fundamental π -vector field is the π -vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all u in the slit tangent bundle $\mathcal{T}M := TM / \{0\}$

We have the following short exact sequence of vector bundles [2]

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

with the well known definitions of the bundle morphisms ρ and γ . The vector space $V_u(TM) = \{X \in T_u(TM) : d\pi(X) = 0\}$ is the vertical space to M at u .

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. The vector space $H_u(TM) = \{X \in T_u(TM) : D_X \bar{\eta} = 0\}$ is called the horizontal space to M at u . The connection D is said to be regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall u \in TM.$$

If M is endowed with a regular connection, then the vector bundle maps $\rho|_{H(TM)}$ and $K|_{V(TM)}$ are isomorphisms. In this case, the map $\beta := (\rho|_{H(TM)})^{-1}$ is called the horizontal map of D .

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of D , denoted by Q and T respectively, are defined by

$$Q(\bar{X}, \bar{Y}) = \mathbf{T}(\beta\bar{X}, \beta\bar{Y}), \quad T(\bar{X}, \bar{Y}) = \mathbf{T}(\gamma\bar{X}, \beta\bar{Y}) \quad \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)),$$

where \mathbf{T} is the (classical) torsion tensor field associated with D .

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of D , denoted by R , P and S respectively, are defined by

$$\begin{aligned} R(\bar{X}, \bar{Y})\bar{Z} &= \mathbf{K}(\beta\bar{X}, \beta\bar{Y})\bar{Z}, \\ P(\bar{X}, \bar{Y})\bar{Z} &= \mathbf{K}(\beta\bar{X}, \gamma\bar{Y})\bar{Z}, \\ S(\bar{X}, \bar{Y})\bar{Z} &= \mathbf{K}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z}, \end{aligned}$$

where \mathbf{K} is the (classical) curvature tensor field associated with D .

The contracted curvature tensors of D , denoted by \widehat{R} , \widehat{P} and \widehat{S} (known also as the (v)h-, (v)hv- and (v)v-torsion tensors respectively), are defined by

$$\begin{aligned}\widehat{R}(\overline{X}, \overline{Y}) &= R(\overline{X}, \overline{Y})\overline{\eta}, \\ \widehat{P}(\overline{X}, \overline{Y}) &= P(\overline{X}, \overline{Y})\overline{\eta}, \\ \widehat{S}(\overline{X}, \overline{Y}) &= S(\overline{X}, \overline{Y})\overline{\eta}.\end{aligned}$$

Theorem 2.1. [18] *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L . There exists a unique regular connection ∇ on $\pi^{-1}(TM)$, called Cartan connection, such that*

- (a): ∇ is metric: $\nabla g = 0$,
- (b): The (h)h-torsion of ∇ vanishes: $Q = 0$,
- (c): The (h)hv-torsion T of ∇ satisfies: $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

3. Ricci (generalized Ricci) Finsler spece

In this section, we introduce and study some new special Finsler spaces, called Ricci and generalized Ricci Finsler spaces. Some classes of generalized Ricci Finsler spaces are distinguished. These new spaces have been defined in Riemannian geometry [3, 4, 7, 8, 9, 10, 11, 12]. We extend them to the Finslerian case. The only linear connection we deal with in the sequel is the Cartan connection ∇ .

For an n -dimensional Finsler manifold (M, L) , we set the following notations:

- $\overset{h}{\nabla}$: the h -covariant derivatives associated with Cartan connection,
- Ric : the horizontal Ricci tensor of Cartan connection,
- Ric_o : the horizontal Ricci tensor of of type (1,1) defined by

$$g(\text{Ric}_o \overline{X}, \overline{Y}) = \text{Ric}(\overline{X}, \overline{Y}),$$
- r : the horizontal scalar curvature of Cartan connection,
- $C := R - \frac{r}{n(n-1)} G$: the concircular curvature tensor;

$$G(\overline{X}, \overline{Y})\overline{Z} := g(\overline{X}, \overline{Z})\overline{Y} - g(\overline{Y}, \overline{Z})\overline{X}.$$

Definition 3.1. *A Finsler manifold is said to be horizontally integrable if its horizontal distribution is completely integrable or, equivalently, if $\widehat{R} = 0$.*

Definition 3.2. *Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with non-zero Ricci tensor Ric. Then, (M, L) is said to be:*

- (a): Ricci Finsler manifold if $\text{Ric}_o^2 := \text{Ric}_o \circ \text{Ric}_o = \frac{r}{n-1} \text{Ric}_o$,
- (b): generalized Ricci Finsler manifold if $\text{Ric}_o^2 = \alpha \text{Ric}_o$,

where α is a non-zero scalar function on TM called the associated scalar.

The following result gives some important properties of generalized Ricci Finsler manifolds.

Theorem 3.3. *Let (M, L) be a generalized Ricci Finsler manifold of dimension $n \geq 3$ with associated scalar α . The following assertions hold:*

- (a): *If the Ricci tensor is symmetric, then the scalar curvature r can not vanish.*
- (b): *The Ricci tensor in the direction $Ric_o(\overline{W})$; \overline{W} being a non zero π -vector field, is the associated scalar α .*
- (c): *The tensor Ric_o has two eigenvalues 0 and α .*
- (d): *If (M, L) is horizontally integrable Ricci recurrent, then the associated scalar $\alpha = \frac{r}{2}$.*

Proof.

(a) Let (M, L) be a generalized Ricci Finsler manifold with associated scalar α and g the associated Finsler metric. Then, by Definition 3.2

$$Ric(Ric_o\overline{X}, \overline{Y}) = \alpha Ric(\overline{X}, \overline{Y}). \quad (3.1)$$

Setting $\overline{X} = \overline{Y} = \overline{E}_i$, where $\{\overline{E}_i; i = 1, \dots, n\}$ is an orthonormal basis. Hence,

$$\sum_i Ric(Ric_o\overline{E}_i, \overline{E}_i) = \alpha r.$$

We show that $r \neq 0$. Assuming the contrary, then

$$\sum_i Ric(Ric_o\overline{E}_i, \overline{E}_i) = 0.$$

As the Ricci tensor Ric is symmetric (since (M, L) is horizontally integrable [14]) and g is positive definite, the above relation yields $Ric_o = 0$, which is a contradiction.

(b) Setting $\overline{X} = \overline{W} \neq 0$ and $Y = Ric_o\overline{W}$ in (3.1), we get

$$\alpha = \frac{Ric(Ric_o\overline{W}, Ric_o\overline{W})}{g(Ric_o\overline{W}, Ric_o\overline{W})},$$

which means that α is the Ricci tensor in the direction $Ric_o(\overline{W})$.

(c) Let \overline{V} be an eigenvector associated with the eigenvalue λ of Ric_o , then

$$Ric_o\overline{V} = \lambda\overline{V}.$$

From which, noting that (M, L) is generalized Ricci with associated scalar α , we have

$$(\lambda^2 - \alpha\lambda)\overline{V} = 0.$$

Consequently, $\lambda = 0$ or $\lambda = \alpha$.

(d) As (M, L) is Ricci recurrent with scalar form A , then

$$(\nabla_{\beta\overline{X}} Ric)(\overline{Y}, \overline{Z}) = A(\overline{X}) Ric(\overline{Y}, \overline{Z}) \quad (3.2)$$

and since (M, L) is horizontally integrable, then, we have [14]

$$\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \{(\nabla_{\beta \bar{X}} R)(\bar{Y}, \bar{Z}, \bar{W})\} = 0. \quad (3.3)$$

Contracting (3.2) with respect to \bar{Y} and \bar{Z} , we get

$$(\overset{h}{\nabla} r)(\bar{X}) = rA(\bar{X}). \quad (3.4)$$

From which,

$$(\overset{h}{\nabla} r)(Ric_o \bar{X}) = rA(Ric_o \bar{X}). \quad (3.5)$$

On the hand, contracting (3.2) with respect to \bar{X} and \bar{Y} and using (3.3), we obtain

$$\frac{1}{2}(\overset{h}{\nabla} r)(\bar{Z}) = A(Ric_o \bar{Z}). \quad (3.6)$$

Setting $\bar{Z} = Ric_o \bar{X}$, noting that (M, L) is generalized Ricci with associated scalar α , (3.6) becomes

$$\frac{1}{2}(\overset{h}{\nabla} r)(Ric_o \bar{X}) = \alpha A(Ric_o \bar{X}) \quad (3.7)$$

Now, (3.5) and (3.7) imply that

$$(r - 2\alpha)A(Ric_o \bar{X}) = 0 \quad (3.8)$$

We finally show that $A(Ric_o \bar{X}) \neq 0$. Assume the contrary: $A(Ric_o \bar{X}) = 0$. From which, taking into account (3.4) and (3.6), we get $rA = 0$. Hence, $r = 0$ or $A = 0$. Both yield a contradiction. Then, (3.8) implies that $\alpha = \frac{r}{2}$. \square

Theorem 3.4. *Let (M, L) be a horizontally integrable Ricci recurrent generalized Ricci Finsler manifold of dimension $n \geq 3$ with associated scalar α . If (M, L) is Ricci Finsler, then it is three dimensional.*

Proof. As (M, L) is horizontally integrable Ricci recurrent generalized Ricci with associated scalar α . Then, from Theorem 3.3(d), we have

$$\alpha = \frac{r}{2}. \quad (3.9)$$

On the other hand, if (M, L) is Ricci Finsler, then the associated scalar α has the form

$$\alpha = \frac{r}{n-1} \quad (3.10)$$

As (M, L) is horizontally integrable, the Ricci tensor is symmetric. Consequently, by Theorem 3.3(a), the proof follows immediately from (3.9) and (3.10). \square

Theorem 3.5. *Every Finsler manifold of dimension $n \geq 3$ satisfying $Ric = \frac{r}{n} g$ is a generalized Ricci Finsler manifold with associated scalar $\alpha = \frac{r}{n}$.*

Proof. The proof is clear and we omit it. \square

$\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}$ denotes cyclic sum over $\bar{X}, \bar{Y}, \bar{Z}$.

Definition 3.6. Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with non-zero h -curvature tensor R . We will say that (M, L) is a semi-isotropic Finsler manifold if the h -curvature R has the form:

$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = A(\bar{X}, \bar{Z})A(\bar{Y}, \bar{W}) - A(\bar{X}, \bar{W})A(\bar{Y}, \bar{Z}),$$

where A is a non-zero symmetric tensor of type $(0,2)$, called the associated tensor.

Theorem 3.7. Every horizontally integrable semi-isotropic Finsler manifold, with associated tensor as the Ricci tensor, is generalized Ricci with associated scalar $\alpha = r - 1$.

Proof. As the Ricci tensor of a horizontally integrable Finsler manifold is symmetric [14], then, we have

$$Ric(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = Ric(\bar{X}, \bar{Z})Ric(\bar{Y}, \bar{W}) - Ric(\bar{X}, \bar{W})Ric(\bar{Y}, \bar{Z}).$$

Contracting both sides of the above equation with respect to \bar{Y} and \bar{W} , we obtain

$$Ric(\bar{X}, \bar{Z}) = rRic(\bar{X}, \bar{Z}) - g(Ric_o\bar{X}, Ric_o\bar{Z}).$$

From which, noting that the Ricci tensor Ric is symmetric

$$Ric(Ric_o\bar{X}, \bar{Z}) = (r - 1)Ric(\bar{X}, \bar{Z}).$$

Hence, (M, L) is generalized Ricci with associated scalar $\alpha = r - 1$. \square

Remark 3.8. Theorem 3.5 and Theorem 3.7 give two classes of generalized Ricci Finsler manifolds.

4. Projective (m-projective) recurrence

In this section, we investigate some new types of recurrent Finsler spaces, namely the projectively recurrent and m-projectively recurrent Finsler spaces. Some Finsler tensors are defined and their properties are studied. These tensors are used to define the projectively (m-projectively) recurrent Finsler space.

For a Finsler manifold of dimension $n \geq 3$ with non-zero Ricci tensor Ric , we define the following tensors:

$$\mathbb{P}(\bar{X}, \bar{Y})\bar{Z} := R(\bar{X}, \bar{Y})\bar{Z} - \frac{1}{(n-1)}\{Ric(\bar{X}, \bar{Z})\bar{Y} - Ric(\bar{Y}, \bar{Z})\bar{X}\}, \quad (4.1)$$

$$\begin{aligned} \mathbb{H}(\bar{X}, \bar{Y})\bar{Z} &:= R(\bar{X}, \bar{Y})\bar{Z} - \frac{1}{2(n-1)}\{Ric(\bar{X}, \bar{Z})\bar{Y} - Ric(\bar{Y}, \bar{Z})\bar{X} \\ &\quad + g(\bar{X}, \bar{Z})Ric_o\bar{Y} - g(\bar{Y}, \bar{Z})Ric_o\bar{X}\}. \end{aligned} \quad (4.2)$$

The tensor \mathbb{P} (resp. \mathbb{H}) is called the projective (resp. m-projective) curvature tensor.

Definition 4.1. Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with non-zero Ricci tensor. Then, (M, L) is said to be:

(a): projectively recurrent if $\overset{h}{\nabla} \mathbb{P} = A \otimes \mathbb{P}$,

(b): m -projectively recurrent if $\overset{h}{\nabla} \mathbb{H} = A \otimes \mathbb{H}$,

where A is a non-zero π -form on TM called the associated form.

In particular, if $\overset{h}{\nabla} \mathbb{P} = 0$ (resp. $\overset{h}{\nabla} \mathbb{H} = 0$), then (M, L) is called projectively (resp. m -projectively) symmetric.

The following result gives some properties for the m -projective curvature tensor.

Proposition 4.2. *Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with non-zero Ricci tensor. Then, the m -projective curvature tensor \mathbb{H} has the following properties:*

(a): $\mathbb{H}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = -\mathbb{H}(\bar{Y}, \bar{X}, \bar{Z}, \bar{W})$,

(b): $\mathbb{H}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = -\mathbb{H}(\bar{X}, \bar{Y}, \bar{W}, \bar{Z})$,

(c):

$$\begin{aligned} \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \{ \mathbb{H}(\bar{X}, \bar{Y}) \bar{Z} \} = \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \left\{ T(\hat{R}(\bar{X}, \bar{Y}), \bar{Z}) - \frac{1}{2(n-1)} [\text{Ric}(\bar{X}, \bar{Z}) \bar{Y} \right. \\ \left. - \text{Ric}(\bar{Y}, \bar{Z}) \bar{X} + g(\bar{X}, \bar{Z}) \text{Ric}_o \bar{Y} - g(\bar{Y}, \bar{Z}) \text{Ric}_o \bar{X}] \right\}, \end{aligned}$$

(d):

$$\begin{aligned} \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \left\{ (\nabla_{\beta \bar{X}} \mathbb{H})(\bar{Y}, \bar{Z}, \bar{W}) \right\} = -\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \left\{ P(\bar{X}, \hat{R}(\bar{Y}, \bar{Z})) \bar{W} \right. \\ \left. + \frac{1}{2(n-1)} [(\nabla_{\beta \bar{X}} \text{Ric})(\bar{Y}, \bar{W}) \bar{Z} - (\nabla_{\beta \bar{X}} \text{Ric})(\bar{Z}, \bar{W}) \bar{Y} \right. \\ \left. + g(\bar{Y}, \bar{W}) (\nabla_{\beta \bar{X}} \text{Ric}_o)(\bar{Z}) - g(\bar{Z}, \bar{W}) (\nabla_{\beta \bar{X}} \text{Ric}_o)(\bar{Y}) \right\}, \end{aligned}$$

(e): $(\nabla_{\gamma \bar{\eta}} \mathbb{H})(\bar{X}, \bar{Y}, \bar{Z}) = 0$,

(f):

$$\mathbb{H}(\bar{X}, \bar{Y}) \bar{\zeta} = \frac{1}{2(n-1)} \left\{ g(\bar{Y}, \bar{\zeta}) \text{Ric}_o \bar{X} - g(\bar{X}, \bar{\zeta}) \text{Ric}_o \bar{Y} \right\},$$

where $\bar{\zeta}$ is a concurrent π -vector field [19].

Proof. The proof follows from Theorem 3.6 of [20] and Proposition 2.4 of [19] together with the definition of m -projective curvature tensor. \square

Theorem 4.3. *For a Finsler manifold with non-zero Ricci tensor Ric satisfying $Ric = \frac{r}{n} g$, the three notions of being concircularly recurrent, projectively recurrent and m -projectively recurrent are equivalent.*

Proof. The proof follows from the fact that the concircular curvature tensor C , the projective curvature tensor \mathbb{P} and the m -projective curvature tensor \mathbb{H} coincide under the given assumption $Ric = r/n g$. \square

We know that every recurrent Finsler manifold is Ricci recurrent (Theorem 3.2(a) of [14]). The converse of this theorem is not true. For the converse to be true we need an additional assumption as shown in the next result.

Theorem 4.4. *A Ricci recurrent m-projectively recurrent Finsler manifold with the same recurrence form is recurrent.*

Proof. As (M, L) is Ricci recurrent with recurrence form A , then, we have [14]

$$(\nabla_{\beta\bar{W}}Ric)(\bar{X}, \bar{Y}) = A(\bar{W})Ric(\bar{X}, \bar{Y}). \quad (4.3)$$

Applying the h-covariant derivative on both sides of (4.2), noting that $\nabla g = 0$, we get

$$\begin{aligned} (\nabla_{\beta\bar{W}}\mathbb{H})(\bar{X}, \bar{Y}, \bar{Z}) &= (\nabla_{\beta\bar{W}}R)(\bar{X}, \bar{Y}, \bar{Z}) - \frac{1}{2(n-1)} \left\{ (\nabla_{\beta\bar{W}}Ric)(\bar{X}, \bar{Z})\bar{Y} \right. \\ &\quad - (\nabla_{\beta\bar{W}}Ric)(\bar{Y}, \bar{Z})\bar{X} + g(\bar{X}, \bar{Z})(\nabla_{\beta\bar{W}}Ric_o)(\bar{Y}) \\ &\quad \left. - g(\bar{Y}, \bar{Z})(\nabla_{\beta\bar{W}}Ric_o)(\bar{X}) \right\}, \end{aligned}$$

In view of (4.3), this equation becomes

$$\begin{aligned} (\nabla_{\beta\bar{W}}\mathbb{H})(\bar{X}, \bar{Y}, \bar{Z}) &= (\nabla_{\beta\bar{W}}R)(\bar{X}, \bar{Y}, \bar{Z}) - \frac{A(\bar{W})}{2(n-1)} \left\{ Ric(\bar{X}, \bar{Z})\bar{Y} \right. \\ &\quad \left. - Ric(\bar{Y}, \bar{Z})\bar{X} + g(\bar{X}, \bar{Z})Ric_o\bar{Y} - g(\bar{Y}, \bar{Z})Ric_o\bar{X} \right\}. \quad (4.4) \end{aligned}$$

Now, let (M, L) be an m-projectively recurrent manifold with the same recurrence form A . Then, from Definition 4.1 and (4.4), we obtain

$$(\nabla_{\beta\bar{W}}R)(\bar{X}, \bar{Y}, \bar{Z}) = A(\bar{W})R(\bar{X}, \bar{Y})\bar{Z}.$$

Hence, (M, L) is recurrent with the same recurrence form A . □

Theorem 4.5. *Each recurrent Finsler manifold is m-projectively recurrent.*

Proof. Since (M, L) is recurrent with recurrence form A , then (M, L) is Ricci recurrent with the same recurrence form A . Using Definition 2.1(a) of [14], taking into account (4.4) and (4.2), we conclude that

$$(\nabla_{\beta\bar{W}}\mathbb{H})(\bar{X}, \bar{Y}, \bar{Z}) = A(\bar{W})\mathbb{H}(\bar{X}, \bar{Y})\bar{Z}.$$

Hence, (M, L) is m-projectively recurrent with the same recurrence form A . □

Theorem 4.6. *Let (M, L) be a Ricci recurrent Finsler manifold. Then, (M, L) is m-projectively recurrent if and only if it is projectively recurrent with the same recurrence form.*

Proof. From (4.1) and (4.2), we have

$$\begin{aligned} \mathbb{H}(\bar{X}, \bar{Y})\bar{Z} &:= \mathbb{P}(\bar{X}, \bar{Y})\bar{Z} + \frac{1}{2(n-1)} \left\{ Ric(\bar{X}, \bar{Z})\bar{Y} - Ric(\bar{Y}, \bar{Z})\bar{X} \right. \\ &\quad \left. + g(\bar{Y}, \bar{Z})Ric_o\bar{X} - g(\bar{X}, \bar{Z})Ric_o\bar{Y} \right\}, \quad (4.5) \end{aligned}$$

From which, taking the h-covariant derivative of both sides, we obtain

$$\begin{aligned} (\nabla_{\beta\bar{W}}\mathbb{H})(\bar{X}, \bar{Y}, \bar{Z}) &= (\nabla_{\beta\bar{W}}\mathbb{P})(\bar{X}, \bar{Y}, \bar{Z}) + \frac{1}{2(n-1)} \left\{ (\nabla_{\beta\bar{W}}Ric)(\bar{X}, \bar{Z})\bar{Y} \right. \\ &\quad - (\nabla_{\beta\bar{W}}Ric)(\bar{Y}, \bar{Z})\bar{X} + g(\bar{Y}, \bar{Z})(\nabla_{\beta\bar{W}}Ric_o)(\bar{X}) \\ &\quad \left. - g(\bar{X}, \bar{Z})(\nabla_{\beta\bar{W}}Ric_o)(\bar{Y}) \right\}, \end{aligned}$$

Since, (M, L) is Ricci recurrent with recurrence form A , then by (4.3), the above equation takes the form

$$\begin{aligned} (\nabla_{\beta\bar{W}}\mathbb{H})(\bar{X}, \bar{Y}, \bar{Z}) &= (\nabla_{\beta\bar{W}}\mathbb{P})(\bar{X}, \bar{Y}, \bar{Z}) + \frac{A(\bar{W})}{2(n-1)} \left\{ Ric(\bar{X}, \bar{Z})\bar{Y} \right. \\ &\quad \left. - Ric(\bar{Y}, \bar{Z})\bar{X} + g(\bar{Y}, \bar{Z})Ric_o\bar{X} - g(\bar{X}, \bar{Z})Ric_o\bar{Y} \right\}, \quad (4.6) \end{aligned}$$

Now, let (M, L) be m-projectively recurrent with the same recurrence form A . Then, from Definition 4.1, taking into account (4.5), the above equation reduces to

$$(\nabla_{\beta\bar{W}}\mathbb{P})(\bar{X}, \bar{Y}, \bar{Z}) = A(\bar{W})\mathbb{P}(\bar{X}, \bar{Y})\bar{Z}.$$

Hence, (M, L) is projectively recurrent with the same recurrence form A .

Conversely, let (M, L) be projectively recurrent with the same recurrence form A . Then, from Definition 4.1, taking into account (4.6) and (4.5), we conclude that (M, L) is m-projectively recurrent with the same recurrence form A . \square

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