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On special class of R-quadratic Finsler metrics

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Abstract. In this paper a special class of R-quadratic generalized (α, β)-metrics are considered. Some properties of this class of Finsler metrics are investigated. In special case, the Riemann curvature of these metrics is calculated. Moreover, it is proved that, in this class of metrics, there is not any (non-Riemannian) R-quadratic metrics of non-zero scalar curvature.

Keywords: Riemann curvature, General (α, β)-metrics, R-quadratic Finsler metrics.

1. Introduction

Finsler geometry is a suitable extension of Riemannian geometry such that the squared line element is not restricted to be quadratic in the displacements. Considering Finsler geometry was already discussed by Riemann, in his lecture in 1854 [8], Much later, the systematic study of these spaces appeared in the dissertation thesis of Finsler in 1918 [5]. It seems that the class of (α, β)-metrics is a good candidate for more study and do computations in Finsler spaces. They are computable and their patterns offer references. There are

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some classes of Finsler metrics which are not always (α, β)-metrics such as general (α, β)-metrics [14] or spherically symmetric Finsler metrics [16]. Because of the symmetry and computability of this class of Finsler metrics, many interesting results have been acquired in the past years [3]. Moreover, (α, β)-metrics provides several wonderful metrical models for physics and biology [1], and the most important one is the so-called Randers metrics [7]. The concept of (α, β)-metrics was introduced by Matsumoto in 1972 as a generalization of Randers metrics which introduced by Randers. Introducing new Finsler metrics which are not always (α, β)-metrics helps us to evaluate the patterns.

In this research we are going to concentrate on an important class of Finsler metrics called general (α, β)-metrics, which are given as

$$F = \alpha\phi(b^2, s),$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M, respectively. $b^2 = b^i b_i$, $s = \frac{\beta}{\alpha}$ and ϕ is a smooth function. On a smooth manifold, there are several interesting curvatures in Finsler geometry. Riemann curvature is a central concept in Riemannian geometry and was introduced by Riemann in 1854. Berwald generalized it to Finsler metrics. A Finsler metric is said to be R-quadratic if its Riemann curvature is quadratic [4]. R-quadratic metrics were first introduced by Báscó and Matsumoto [2]. They form a rich class in Finsler geometry. There are many interesting works related to this subject [10], [6]. This family of Finsler metrics contains Berwald and R-flat metrics. For a Finsler space (M, F), the Riemann curvature is a family of linear transformations

$$\mathbf{R}_y : T_xM \to T_xM,$$

where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$. The the Finsler metric (M, F) is *R-quadratic* if $\mathbf{R}_y$ is quadratic in $y \in T_xM$. Here a special class of general (α, β)-metric of R-quadratic type is considered. In general, it is difficult to find the Riemann curvature tensor for general (α, β)-metrics. Then we consider the metrics under the following assumption

$$^{\alpha}R^i{}_k = \mu(\alpha^2 \delta^i{}_k - y_k y^i), \quad b_{i|k} = c(x)a_{ik}, \tag{1.1}$$

where $^{\alpha}R^i{}_k$ denotes the Riemann curvature of the Riemannian metric α and μ is the Ricci constant. In other words, we prove

Theorem 1.1. *Let (M, F) be a general (α, β)-metric satisfying (1.1). If F is of R-quadratic type then*

$$R^i_k = R_2 \theta^i_{pkq}(x) y^p y^q, \tag{1.2}$$

where

$$\theta_j{}^i{}_{kl}(x) = (a_{jl}b^2 - b_j b_l)\delta^i{}_k - (a_{jk}b^2 - b_j b_k)\delta^i{}_l + (b_l a_{jk} - a_{jl} b_k)b^i,$$

and $R_2 = R_2(r)$ *is given by*

$$R_2 = -\mu(2\chi - s\chi_s) + c^2[2(2\psi_{b^2} - s\psi_{b^2 s}) - \chi_{ss} + 2\chi(2\chi - s\chi_s) + (b^2 - s^2)(2\chi\chi_{ss} - \chi_s^2).$$

Theorem 1.2. *There is not any (non-Riemannian) R-quadratic general* (α, β)*-metric satisfying* (1.1) *of non-zero scalar curvature.*

2. **Preliminaries**

A Finsler metric on a manifold M is a non-negative function F on TM having the following properties

(a) F is C^∞ on $TM \setminus \{0\}$;
(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
(c) For each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \qquad u, v \in T_xM.$$

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$. To measure the non-Euclidean feature of F_x, define

$$\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$$
$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]\Big|_{t=0},$$

for $u, v, w \in T_xM$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM\setminus\{0\}}$ is called the *Cartan torsion.*

A curve $c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(\dot{c}(t)) = 0,$$

where $G^i(y)$ are local functions on TM given by

$$G^i(y) := \frac{1}{4}g^{il}(y)\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\}, \quad y \in T_xM.$$

G^is called the coefficients of the associated spray G to (M, F). The projection of an integral curve of G is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy

$$\ddot{c}^i + 2G^i(\dot{c}) = 0.$$

F is called a Berwald metric if $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$. For $y \in T_xM_0$, define

$$B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$$
$$B_y(u, v, w) = B_j{}^i{}_{kl}u^jv^kw^l\frac{\partial}{\partial x^i},$$

where

$$B^i_{j\,kl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l},$$

and

$$E_y : T_xM \otimes T_xM \to T_xM$$
$$E_y(u,v) = E_{jk}u^j v^k,$$

where

$$E_{jk} = \frac{1}{2}B_j{}^m{}_{km},$$

$u = u^i\frac{\partial}{\partial x^i}$, $v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$. B and E are called the Berwald curvature and mean Berwald curvature respectively. F is called Berwald and Weakly Berwald (WB) metric if $B = 0$ and $E = 0$, respectively [9].

By means of E-curvature, we can define $\bar{E}$-curvature as follow

$$\bar{E}_y : T_xM \otimes T_xM \otimes T_xM \longrightarrow R$$
$$\bar{E}_y(u,v,w) := \bar{E}_{jkl}(y)u^i v^j w^k = E_{ij|k}u^i v^j w^k.$$

It is remarkable that, $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices.

The S-curvature $S(x,y)$ was introduced as follows [11]

$$S(x,y) = \frac{d}{dt}[\tau(\gamma(t),\gamma'(t)]_{|t=0},$$

where $\tau(x,y)$ is the distortion of the metric F and $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\gamma'(0) = y$ on M. It is considerable that [11]

$$E_{ij} = \frac{1}{2}S_{.i.j},$$

where $.i$ denotes the differential with respect to y^i. The non-Riemannian quantity Ξ-curvature is denoted by $\Xi = \Xi_j dx^j$ and is defined as

$$\Xi_j = S_{.j|m}y^m - S_{|j},$$

where "$|$" and denotes the horizontal covariant derivative with respect to Berwald connection [11]. The Finsler metric F is said to have almost vanishing Ξ-curvature if

$$\Xi_i = -(n+1)F^2\big(\frac{\theta}{F}\big)_{.i}, \tag{2.1}$$

where θ is a 1-form on M and $n = dimM$.

For $y \in T_xM$, the Landsberg curvature $L_y : T_xM \otimes T_xM \otimes T_xM \to R$ is defined by

$$L_y(u,v,w) := -\frac{1}{2}g_y(B_y(u,v,w),y).$$

In local coordinates, $L_y(u,v,w) := L_{ijk}(y)u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2}y_l B^l_{ijk}.$$

$L_y(u,v,w)$ is symmetric in u,v and w. L is called Landsberg curvature. A Finsler metric F is called a Landsberg metric if $L_y = 0$.

In 2012, Yu and Zhu introduced a new class of Finsler metrics called general (α, β)-metrics, which are given as

$$F = \alpha\phi(b^2, s),$$

for some C^∞ function $\phi(b^2, s)$, where α is a Riemannian metric and β is a 1-form.

Also F is called an (α, β)-metric, if F can be expressed as $F = \alpha\phi(s)$ for some C^∞ function $\phi(s)$, Riemannian metric α, and 1-form β. For general (α, β)-metric, $F = \alpha\phi(b^2, s)$, one has [14]

$$\phi - s\phi_s > 0, \quad \phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0, \quad for \quad n \geq 3,$$

or

$$\phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0, \quad for \quad n = 2,$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$. Here ϕ_s denotes the differentiation of ϕ with respect to s. The fundamental tensor g_{ij} is given by [14]

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s\rho_1 \alpha_{y^i} \alpha_{y^j},$$

where $\rho = \phi(\phi - s\phi_s)$, $\rho_0 = \phi\phi_{ss} + \phi_s\phi_s$, $\rho_1 = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}$.
Moreover,

$$det(g_{ij}) = \phi^{n+1}(\phi - s\phi_s)^{n-2}(\phi - s\phi_s + (b^2 - s^2)\phi_{ss})det(a_{ij}).$$

Further, we have[9]

$$I_i = g^{jk}C_{ijk} = \frac{\partial}{\partial y^i}\Big(ln\sqrt{det(g_{jk})}\Big).$$

Then for general (α, β)-metric, $F = \alpha\phi(b^2, s)$, one has

$$I_k = \frac{A}{\alpha}(b_k - \frac{s}{\alpha}y_k), \tag{2.2}$$

where

$$A = \frac{-3s\phi_{ss} + (b^2 - s^2)\phi_{sss}}{\phi - s\phi_s + (b^2 - s^2)\phi_{ss}} - (n+1)\frac{\phi'}{\phi} - (n-2)\frac{s\phi_{ss}}{\phi - s\phi_s}.$$

The notion of Riemann curvature for Riemannian metrics can be extended to Finsler metrics. For $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) = R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$ where The Riemann curvature R^i_k of G are defined by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.3}$$

A Finsler metric F is said to be quadratic if R_y is quadratic in $y \in T_xM$. At each point $x \in M$ let

$$R_j{}^i{}_{kl} := \frac{1}{3}\frac{\partial}{\partial y^j}\left\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\right\}, \tag{2.4}$$

where ${R_j}^i{}_{kl}$ is the Riemann curvature of Berwald connection. Then we have $R^i_k = {R_j}^i{}_{kl}(x,y)y^jy^l$. Therefore R^i_k is quadratic in $y \in T_xM$ if and only if ${R_j}^i{}_{kl}$ are functions of position alone [9].

Theorem 2.1. [9] *Every compact R-quadratic Finsler metric is a Landsberg metric.*

Theorem 2.2. [15] *Riemann curvature of general (α,β)-metric $F = \alpha\varphi(b^2, s)$ satisfying 1.1 is given by*

$$R^i{}_k = \alpha^2 \left\{ R_1 {}^\alpha h^i{}_k + \alpha R_2 s_{.k} b^i + R_3 s_{.k} y^i \right\}, \tag{2.5}$$

where

$$R_1 = \mu(1+s\psi) + c^2[\psi^2 - 2s\psi_{b^2} - \psi_s + 2\chi(1+s\psi+u\psi_s),$$

$$R_2 = -\mu(2\chi - s\chi_s) + c^2[2(2\psi_{b^2} - s\psi_{b^2 s}) - \chi_{ss} + 2\chi(2\chi - s\chi_s) + (b^2-s^2)(2\chi\chi_{ss} - \chi_s^2),$$

$$R_3 = -\mu(2\psi - s\psi_s) + c^2[2(2\psi_{b^2} - s\psi_{b^2 s}) - \psi\psi_s - \psi_{ss} + 2\chi(\psi - s\psi_s + (b^2-s^2)\psi_{ss})$$
$$-\chi_s(1+s\psi+(b^2-s^2)\psi_s)],$$

and

$$\psi = \frac{\varphi_s + 2s\varphi_{b^2}}{2\varphi} - \frac{\chi}{\varphi}[s\varphi + (b^2-s^2)\varphi_s],$$

$$\chi = \frac{\varphi_{ss} - 2(\varphi_{b^2} - s\varphi_{b^2 s})}{2(\varphi - s\varphi_s) + (b^2-s^2)\varphi_{ss}},$$

$\psi_{b^2} = \frac{\partial\psi}{\partial b^2}$, $\psi_s = \frac{\partial\psi}{\partial s}$, $\psi_{ss} = \frac{\partial^2\psi}{\partial s^2}$, $c^2 = k - \mu b^2$ *for some constant* k *and* ${}^\alpha h^i{}_k = \delta^i{}_k - \ell_k\ell^i$.

Note that here we denote by $\ell_k = \alpha_{.k}$ and $l_k = F_{.k}$.

3. On a Special Class of General (α,β)-Metrics of R-quadratic type

In this section, we are going to study some new interesting properties of a special class of general (α,β)-metrics of R-quadratic type. First we show the following lemma.

Lemma 3.1. *Let (M,F) be a general (α,β)-metric satisfying (1.1). If F is of R-quadratic type then*

$$I_m R^m{}_k = 0. \tag{3.1}$$

Proof. First note that we have the following identity [9]

$$R^i_{j\ kl.m} = B_j{}^i{}_{ml|k} - B_j{}^i{}_{mk|l},$$

where "|" denotes the horizontal derivative with respect to Berwald connection. Contracting the above equation by $-\frac{1}{2}y_i$ yields

$$L_{jmk|l} - L_{jml|k} = -\frac{1}{2}y_i R_j{}^i{}_{kl.m}.$$

But F is of R-quadratic type then one has

$$L_{jmk|l} = L_{jml|k}. \tag{3.2}$$

By contracting the above equation by y^l and g^{jm}, respectively, one gets

$$L_{jmk|0} = 0, \tag{3.3}$$

$$J_{k|l} = J_{l|k}. \tag{3.4}$$

In other words, we have the following identity for a Finsler Metric F (the identity (10-13) in [9])

$$L_{ijk|l}y^l = -C_{ijp}R^p{}_k - \frac{1}{2}g_{pj}R_i{}^p{}_{kl} - \frac{1}{2}g_{ip}R_j{}^p{}_{kl}.$$

By contracting the above equation by g^{ij} one has

$$J_{k|0} = -I_p R^p_k - R^m_{m\ kl}y^l.$$

Then based on (3.4) we have

$$I_m R^m{}_k = -R^m_{m\ kl}y^l. \tag{3.5}$$

But F is of R-quadratic type. Then

$$(I_m R^m{}_k)_{.l} = -R^m_{m\ kl}(x). \tag{3.6}$$

On the other hands, based on (2.2) and Theorem (2.2), for general (α, β)-metrics satisfying (1.1), one gets

$$I_m R^m{}_k = \alpha^2 \omega s_{.k},$$

where

$$\omega = A\{R_1 + (b^2 - s^2)R_2\}, \qquad s_{.k} = (\frac{\beta}{\alpha})_{.k} = \frac{b_k}{\alpha} - \frac{s}{\alpha^2}y_k.$$

Then

$$(I_m R^m_k)_{.l} = 2\omega s_{.k}y_l + \alpha^2(\omega s_{.k.l} + \omega_s s_{.k}s_{.l}), \tag{3.7}$$

where

$$\alpha^2 s_{.k.l} = -b_k\ell_l - b_l\ell_k + 3s\ell_k\ell_l - sa_{kl},$$
$$\alpha^2 s_{.k}s_{.l} = -sb_k\ell_l - sb_l\ell_k + s^2\ell_k\ell_l + b_kb_l,$$

Then by 3.6 one finds

$$-R_m{}^m{}_{kl}(x) = (\omega - s\omega_s)(b_k\ell_l) - (\omega + s\omega_s)(b_l\ell_k - s\ell_k\ell_l) - (s\omega a_{kl} - \omega_s b_k b_l), \tag{3.8}$$

which one easily concludes

$$\omega - s\omega_s = 0, \quad and \quad \omega + s\omega_s = 0,$$

and then $\omega = 0$. It means that $I_m R^m{}_k = 0$.

3.1. **Proof of Theorem 1.1.** Now, we could prove the Theorem 1.1. According to the above lemma, we find that

$$\omega = A\big\{R_1 + (b^2 - s^2)R_2\big\} = 0.$$

But F is non-Riemannian, then $A \neq 0$ and we have

$$R_1 + (b^2 - s^2)R_2 = 0. \tag{3.9}$$

On the other hands, for every Finsler metric we have

$$l_m R^m{}_k = F_{.m} R^m{}_k = 0,$$

and

$$F_{.m} = (\phi - s\phi_s)\ell_m + \phi_s b_m.$$

Then

$$0 = F_{.m} R^m{}_k = (\phi - s\phi_s)\ell_m R^m{}_k + \phi_s b_m R^m{}_k.$$

But

$$\ell_m R^m{}_k = \alpha^3(sR_2 + R_3)s_{.k},$$
$$b_m R^m{}_k = \alpha^3(R_1 + b^2 R_2 + sR_3)s_{.k}.$$

Therefore

$$\phi(R_3 + sR_2) + \phi_s(R_1 + (b^2 - s^2)R_2) = 0,$$

which by (3.9) we have

$$R_3 = -sR_2. \tag{3.10}$$

Putting (3.9) and (3.10) in Theorem 2.2 one gets

$$R^i{}_k = -\alpha^2 R_2\big\{(b^2 - s^2)^\alpha h^i{}_k - \alpha(b^i - s\ell^i)s_{.k}\big\}.$$

And

$$Ric = (n-2)(b^2 - s^2)\alpha^2 R_2.$$

Then

$$\begin{aligned} R^i{}_k &= -R_2\big\{(a_{pq}b^2 - b_p b_q)\delta^i{}_k - (a_{kp}b^2 - b_p b_k)\delta^i{}_q + (b_q a_{kp} - a_{pq}b_k)b^i\big\} \quad (3.11)\\ &= -R_2\theta_p{}^i{}_{kq}(x)y^p y^q, \end{aligned}$$

where

$$\theta_j{}^i{}_{kl}(x) = (a_{jl}b^2 - b_j b_l)\delta^i{}_k - (a_{kj}b^2 - b_j b_k)\delta^i{}_l + (b_l a_{kj} - a_{jl}b_k)b^i. \tag{3.12}$$

It is clear that

$$\theta_j{}^i{}_{kl} = -\theta_j{}^i{}_{lk}, \quad \theta_j{}^i{}_{kl} + \theta_k{}^i{}_{lj} + \theta_l{}^i{}_{jk} = 0.$$

Now, we show that $(R_2)_s = 0$. According to (2.4), one could get

$$R^i{}_{kl} = \frac{1}{3}(R_2)_{.s}[\theta_0{}^i{}_{k0}s_{.l} - \theta_0{}^i{}_{l0}s_{.k}] + R_2\theta_0{}^i{}_{lk},$$

where $\theta_0{}^i{}_{l0} = \theta_p{}^i{}_{lq}y^p y^q$. But noting (3.12)

$$\theta_0{}^i{}_{kl}s_{.l} = [sb_k b_l + \frac{s}{\alpha}b^2 y_k y_l - \alpha\beta s_{.k}s_{.l}] - [\frac{b^2}{\alpha}b_l y_k + \frac{s^2}{\alpha}b_k y_l].$$

Then

$$\theta_0{}^i{}_{k0}s_{.l} - \theta_0{}^i{}_{l0}s_{.k} = 0$$

and one has

$$R_j{}^i{}_{kl}(x) = R_2\theta_j{}^i{}_{lk}(x).$$

It means that $R_2 = R_2(r)$. □

3.2. **Proof of Theorem 1.2.** Here we give a proof that, in this special class of Finsler metrics, every non-Riemannian metric of scalar curvature and R-quadratic type is of zero flag curvature. In other words, we prove Theorem 1.2. Now, assume that the R-quadratic general (α, β)-metric F is of scalar flag curvature, then

$$R^i{}_k = \lambda(x, y)F^2 h^i{}_k = \lambda(x, y)(F^2\delta^i{}_k - y^i y_k).$$

According to the above equation and lemma 3.1, one finds

$$I_m R^m{}_k = \lambda(x, y)F^2 I_k = 0,$$

But F is not Riemannian then $I \neq 0$ and consequently $\lambda = 0$.

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