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One PDE arising from concircular transformation on Finsler spaces

[a] Samaneh Saberali

[a]School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Niavaran Bldg., Niavaran Square, P.O. Box: 193955746, Tehran, Iran.
E-mail: samanehsaberali@ipm.ir

Abstract. In this paper, we study conformal transformations in Finsler manifolds. Let $(M, \mathbf{g})$ be a Finsler manifold. Suppose that F admits a conformal transformation that is concircular. We characterize a Finsler manifold admitting a conformal transformation such that the difference of the two Ricci tensors is a constant multiple of the metric. Furthermore, we find some results on Finsler manifolds with constant flag curvature admiting a special conformal transformation.

Keywords: Finsler metric, geodesic circle, Concircular transformation, Ricci tensor.

1. **Introduction**

A geodesic circle in an Euclidean space is a straight line or a circle with finite positive radius, which can be generalized naturally to Riemannian or Finsler geometry. Firstly, in 1940, Yano introduced concircular transformations on Riemannian manifolds [29]. Exactly, a geodesic circle in a Riemannian manifold, as well as in a Finsler manifold, is a curve with constant first Frenet curvature and zero second one. In other words, a geodesic circle is a torsion free curve with constant curvature. A concircular transformation on a Riemannian manifold is a conformal transformation which preserves geodesic circles ([13], [29]). Many researchers have developed the theory of concircular transformations to different contents ([14, 15, 26]). In 1970, Vogel showed that every concircular

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transformation on a Riemannian manifold is conformal [27]. This notion has been extended to Finsler geometry by Agrawal and Izumi [1]. Also, a similar result is proved by Bidabad-Shen in 2012 [6]. That is, every transformation which preserves geodesic circles reduces to a conformal transformation. So, by the modified definition, a diffeomorphism φ, between two Finsler manifolds (M, F) and $(\bar{M}, \bar{F})$, is said to be concircular if it maps geodesic circles to geodesic circles. Also, two Finsler metrics defined on a manifold are said to be concircular if they have the same geodesic circles.

In [16], Kuhnel-Rademacher studied about the conformal transformation of semi-Riemannian manifolds. They showed that semi-Riemannian manifolds admitting a global conformal transformation such that the difference of the two Ricci tensors is a constant multiple of the metric.

For a Finsler metric $F = F(x, y)$ on a manifold M, the fundamental metric tensor g_{ij}(while g^{ij} is its inverse), the Cartan torsion C^i_{jk} and the mean Cartan torsion I_i (respectively) will be defined as follow:

$$g_{ij}:= \dot{\partial}_i\dot{\partial}_j\Big(\frac{F^2}{2}\Big), \quad 2C_{ijk}:= \dot{\partial}_k g_{ij}, \quad I_i:= g^{jk}C_{ijk} = C^r_{ir}, \quad \Big(\dot{\partial}_i = \frac{\partial}{\partial y^i}\Big). \quad (1.1)$$

Clearly, a Finsler metric will be a Riemannian metric if its Cartan torsion or mean Cartan torsion is null ([12]). In this paper, we consider concircular transformations on a Finsler manifold, where the difference of whose Ricci tensors are a constant multiple of the Finsler metric $\bar{F}$. We obtain Theorems 4.2 and 4.3.

2. Preliminary

Let M be an $n-$dimensional manifold of class C^∞. We denote by $\pi : TM \to M$ the bundle of tangent vectors and by $\pi_0 : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A Finsler structure on M is a function $F : TM \to [0, \infty)$, with the following properties:

I) F is differentiable (C^∞) on TM_0;
II) $F(x, y)$ is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall\lambda > 0$, where we denote an element of TM by (x, y).
III) The Hessian matrix of $\frac{F^2}{2}$ is positive definite on TM_0; $g_{ij}:= \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$.
A Finsler manifold (M, g) is a pair of a differential manifold M and a tensor field $g = (g_{ij})$ on TM defined by a Finsler structure F. The spray of a Finsler structure F is a vector field on TM:

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}. \quad (2.1)$$

where G^i are called the spray (or geodesic) coefficients

$$G^i = \frac{1}{4}g^{il}\{F^2_{x^m y^l}y^m - F^2_{x^l}\} \quad (2.2)$$

and $(g^{ij}) := (g_{ij})^{-1}$.

We denote here by G^i_j the coefficients of nonlinear connection on TM, where $G^i_j = \frac{\partial G^i}{\partial y^j}$. By means of this nonlinear connection the tangent space can be split into the horizontal and vertical subspaces with the corresponding basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, which are related to the typical bases of TM, $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$, by

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j},$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{kl}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^i}).$$

The components of the hh-curvature of Chern connection are expressed here by

$$R^i_{jkl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} + \frac{\delta \Gamma^i_{jk}}{\delta x^l} - \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk}.$$

The geodesics of F are characterized by the second order differential equation:

$$\frac{d^2 c^i}{dt^2} + 2G^i(c(t), \dot{c}(t)) = 0.$$

The Riemann curvature $R_y : T_pM \to T_pM$ is a linear transformation on tangent spaces, which is defined by

$$R_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i} \tag{2.3}$$

$$R^i_k := 2\frac{\partial G^i}{\partial x^i} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.4}$$

For a two-dimensional plane $P \subset T_pM$ and $y \in T_pM \setminus \{0\}$ such that $P = span\{y, u\}$, the pair $\{P, y\}$ is called a flag in T_pM. The flag curvature $K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

We say that F is of scalar curvature if for any $y \in T_pM \setminus \{0\}$ the flag curvature $K(P, y) = \lambda(y)$ is independent of P containing y. This is equivalent to the following condition in a local coordinate system (x^i, y^i) in TM:

$$R^i_k = \lambda F^2 \{\delta^i_k - F^{-1} F_{y^k} y^i\}.$$

If λ is a constant, then F is said to be of constant curvature.
The trace of the Riemann curvature

$$Ric(y) := (n-1)R(y) = R^m_m(y) \tag{2.5}$$

is called the Ricci curvature and

$$R(y) := \frac{1}{n-1} Ric(y)$$

is called the Ricci scalar.

Due to different Ricci tensors defined for a Finsler metric, there are various methods to present the concept of Einstein-Finsler space. One of the prominent Ricci tensors in Finsler geometry is defined by H. Akbar-Zadeh, cf., [2] as follows

$$(Ric)_{ij} := \frac{1}{2}\frac{\partial^2 Ric}{\partial y^i \partial y^j}. \tag{2.6}$$

The tensor $(Ric)_{ij}$ is symmetric and homogeneous of degree zero in y. A Finsler manifold (M, g) is said to be Einstein if

$$Ric(x, y) = (n-1)k(x)F^2$$

or equivalently

$$(Ric)_{ij}(x, y) = (n-1)k(x)g_{ij},$$

where function k is a scalar function on manifold M.

Let $\bar{F}$ and F be two Finsler metrics on an n-dimensional manifold M. There is a relation between the geodesic coefficients $\bar{G}^i$ and G^i as follows:

$$\bar{G}^i = G^i + \frac{\bar{F}_{;k} y^k}{2\bar{F}} y^i + \frac{\bar{F}}{2}\bar{g}^{il}\{\bar{F}_{;k.l}y^k - \bar{F}_{;l}\}. \tag{2.7}$$

If $\bar{F} = e^{c(x)}F$ then we have

$$\bar{G}^i = G^i + (c_k y^k)y^i - \frac{F^2}{2}c^i,$$

where $c^i = g^{il}c_l$.

Let $c : [0, L] \to (M, g)$ be a smooth curve, parameterized by arc length,∇ the Levi-Civita-connection. We denote its derivatives by $\dot{c}$, $\ddot{c} = \nabla_{\dot{c}}\dot{c}$, $\dddot{c} = \nabla_{\dot{c}}\nabla_{\dot{c}}\dot{c}$ and so on. If these derivatives are linearly independent we can define the Frenet frame $e_1, ..., e_k$ and the geodesic Frenet curvatures $k_1, ..., k_{k-1}$ in the usual way [17]:

$$\dot{e}_1 = k_1 e_2$$
$$\dot{e}_i = -k_{i-1}e_{i-1} + k_i e_{i+1},\ i = 2, \cdots, k-1.$$

where $\dot{e}_i$ is the covariant derivative of e_i along c and $k_i := g_{\dot{c}}(\dot{e}_i, e_{i+1})$ is called i-th Frenet curvature. This system of equations is called Frenet equations. c is called geodesic if $k_1 = 0$, and it is called a geodesic circle if k_1 is constant and $k_2 = 0$ (a geodesic is also a geodesic circle). If $k_1 \neq 0$ for such a geodesic circle then we have the beginning e_1, e_2 of the Frenet frame. If $k_2 = 0$ then it follows that e_2 lies in the (e_1, e_2)-plane. In the following examples we write k instead of k_1. Examples of geodesic circles are small circles on the sphere. It is not required that a geodesic circle is a closed curve. It might be something like a spiral even if the length is infinite.

Proposition 2.1. *(Yano [29]) c is a geodesic circle in Riemannian space if and only if* $\dddot{c}$ *is a scalar multiple of* $\dot{c}$. *In this case necessarily* $\dddot{c} = -\langle \ddot{c}, \ddot{c} \rangle . \dot{c}$.

If $c = c(t)$ is a parametrized curve by an arbitrary parameter, regarding $c = c(s(t))$ we obtain the following relations:

$$\begin{aligned} c' &= \|c'\| \cdot \dot{c} \\ c'' &= \|c'\|^2 \cdot \ddot{c} + \frac{\langle c', c'' \rangle}{\|c'\|} \cdot \dot{c} \\ c''' &= \|c'\|^3 \cdot \dddot{c} + 3\langle c', c'' \rangle \ddot{c} + \left(\frac{\langle c', c'' \rangle}{\|c'\|} \right)' \cdot \dot{c}. \end{aligned} \tag{2.8}$$

Corollary 2.2. *c is a geodesic circle if and only if* $c''' - 3\frac{\langle c', c'' \rangle}{\|c'\|^2} c''$ *is a multiple of* $\dot{c}$ *(or of c).*

3. **Circles in Finsler Manifolds**

Let (M, g) be a Finsler manifold of class C^∞ and use a Chern connection ∇. A smooth curve $c : I \subset R \to M$ parameterized by arc length s is called a circle if there exist a unitary vector field $Y = Y(s)$ along c and a positive constant k such that

$$\nabla_{\acute{c}} X = kY, \tag{3.1}$$

$$\nabla_{\acute{c}} Y = -kX, \tag{3.2}$$

where $X := \acute{c} = \frac{dc}{ds}$ is the unitary tangent vector field at each point $c(s)$. The number $\frac{1}{k}$ is called the radius of the circle.

Comparing this definition of circle with definition of a geodesic circle in Finsler geometry, we find out that if in the definition of a geodesic circle we exclude the trivial case, $k_1 = 0$, that is, if we remove geodesics, then we obtain the definition of a circle in a Finsler manifold.

In [24], Shen-Yang found a necessary and suficient condition for a conformal transformation to be concircular. On a Finsler manifold, a conformal vector field with the conformal factor u is concircular if and only if u satisfies,

$$u_{i|j} = \lambda g_{ij}, \quad u^r C^k_{ri} = 0, \tag{3.3}$$

where

$$u_i := u_{x^i}, \quad u^i := g^{ir} u_r, \tag{3.4}$$

and $\lambda = \lambda(x)$ is a scalar function on M and the symbol "|" means the horizontal covariant derivative of Cartan (or Chern) connection.

4. **Transnormal Functions**

Let (M, g) be a Finsler manifold and $u: M \to \mathbb{R}$ smooth function, if there exists a continuous function $\mathfrak{b}: u(M) \to \mathbb{R}$ such that

$$g(\mathbf{grad}\, u, \mathbf{grad}\, u) = \mathfrak{b} o u, \tag{4.1}$$

then u is called a Finsler transnormal function.

A critical point is a point o such that $u'(o) = 0$, we define a regular point as a point of M which is not critical. The regular and critical values are images of regular and critical points, respectively, under u. Given a Finsler manifold (M, g) and any two points $p, q \in M$, the Finsler distance from p to q is defined as

$$d(p, q) := inf_\gamma \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt, \tag{4.2}$$

where the infimum is taken over all piece-wise smooth curves $\gamma : [a, b] \to M$ joining p to q. One special example of Finsler transnormal functions is the Finsler distance function.

Proposition 4.1. [3] *Let (M, g) be a connected complete Finsler manifold of dimension $n \geq 2$ and $u : M \to \mathbb{R}$ a transnormal function on it. Then,*

a) *If u has one critical point, M is conformal to an n-dimensional Euclidean space.*

b) *If u has two critical points, M is conformal to an n-dimensional unit sphere in an Euclidean space.*

Theorem 4.2. *Let (M, F) be a Finsler metric and admitting a conformal transformation $\bar{F} = u^{-1}F$, $\bar{F}$ concircular to F satisfying*

$$Ric_{\bar{g}} - Ric_g = (n-1)cF^2 \tag{4.3}$$

for constant c. If λ is a linear function of u with constant coefficients, Then we have the following:

1. *The conformal transformation is homothetic.*
2. *M is conformal to a unit sphere in an Euclidean space.*

Proof of Theorem 4.2. It is known that if two sprays $\bar{G}^i$ and G^i satisfy equation $\bar{G}^i = G^i + H^i$, then their Reimann curvature tensors satisfy

$$\bar{R}^i_k = R^i_k + 2H^i_{;k} - y^m H^i_{;m.k} + 2H^m H^i_{.m.k} - H^i_{.m} H^m_{.k} \tag{4.4}$$

where the symbol "$;''$ denotes the horizontal covariant derivative of Berwald connection of G^i [24].

Now, since $\bar{F} = u^{-1}F$, the sprays $\bar{G^i}$ and G^i (related to $\bar{F}$ and F, respectively) satisfy the following equalities:

$$\bar{G^i} = G^i - \frac{1}{u}u_0 y^i + \frac{1}{2u}F^2 u^i, \tag{4.5}$$

$$\bar{G^i_j} = G^i_j - \frac{1}{u}(u_j y^i + u_0\delta^i_j - y_j u^i + F^2 C^i_{jr}u^r) \tag{4.6}$$

taking H^i

$$H^i = -\frac{1}{u}u_0 y^i + \frac{1}{2u}F^2 u^i \tag{4.7}$$

and plugging (4.7) into (4.4), we obtain

$$\begin{aligned}\bar{R}^i_k = \;& R^i_k + \frac{uu_{0;0} - (u_m u^m)F^2}{u^2}\delta^i_k + \frac{1}{u}F^2 u^i_{;k} + \frac{u^m u_m}{u^2}y^i y^k - \frac{1}{u}(y^i u_{k;0} + y_k u^i_{;0}) \\ & -\frac{u^m u^r}{u^2}F^2(y^i C_{kmr} + y_k C^i_{mr}) + \frac{1}{u^2}F^2(uu^r_{;0} - 3u_0 u^r)C^i_{kr} + \frac{1}{u}F^2 u^r C^i_{kr;0} \\ & + \frac{u^r u^m}{u^2}F^4(C^i_{pr}C^p_{km} - C^i_{mr.k}).\end{aligned} \tag{4.8}$$

Then by (4.8), the Ricci curvatures $\bar{Ric} := \bar{R}^m_m$ and $Ric := R^m_m$ are related by

$$\begin{aligned}\bar{Ric} &= Ric + \frac{n-2}{u}u_{0;0} + \frac{1}{u^2}\Big[uu^m_{;m} - (n-1)u^m u_m + u^r(uI_{r;0} - 3u_0 I_r) \\ &= +uI^r u_{r;0}\Big]F^2 - \frac{1}{u^2}u^r u^m(C^i_{jm}C^j_{ir} - 2I^i C_{imr} + I_{m.r})F^4.\end{aligned} \tag{4.9}$$

Now suppose F and $\bar{F}$ are concircular. Then by Lemma 2 in [24], we have

$$u_{i|j} = \lambda g_{ij},\; u^r C^k_{ri} = 0, (u_i := u_{x^i},\; u^i := g^{ir}u_r), \tag{4.10}$$

where $\lambda = \lambda(x)$ is a scalar function on M and the symbol "|" means the horizontal covariant derivative of Cartan (or Chern) connection of F. Plugging (4.10) into (4.8) and (4.9), we respectively have

$$\bar{R}^i_k = R^i_k + u^{-2}(2\lambda u - u_m u^m)(F^2\delta^i_k - y^i y_k), \tag{4.11}$$

$$Ric_{\bar{g}} = Ric_g + (n-1)u^{-2}(2\lambda u - u_m u^m)F^2 \tag{4.12}$$

Substituting (4.3) into (4.12) yields

$$u^m u_m - 2\lambda u + cu^2 = 0 \tag{4.13}$$

If u is linear function of u with constant coefficients, then we say that u is a special solution of (4.13). Hence, any solution of equation (4.13) can be written in the form

$$u^m u_m - 2(Ku + B)u + cu^2 = 0 \tag{4.14}$$

where K and B are constants. The equation (4.14) along any geodesic with arc-length t reduces to the ordinary differential equation

$$(\frac{du}{dt})^2 - 2(Ku + B)u + cu^2 = 0 \tag{4.15}$$

Now for the special case $K = c_1^2 > 0$ and $B = 0$, we have

$$(u')^2 - c_1^2 u^2 + cu^2 = 0 \tag{4.16}$$

By a suitable choice of the arc-length t, a solution (4.16) is given by

$$\begin{aligned} (u')^2 &= u^2(c_1^2 - c) \\ u' &= \pm u c_2 \end{aligned}$$

then we get

$$u = c_3 e^{\pm c_4} \tag{4.17}$$

Therefore the conformal transformation is homothety.

Also in equation (4.16) differentiating once more we have

$$\begin{aligned} 2u'u'' + 2(c - c_1^2)uu' = 0 \\ u'' + (c - c_1^2)u = 0 \end{aligned} \tag{4.18}$$

we get

$$u(t) = A\cos(c - c_1^2)t, \tag{4.19}$$

$$u'(t) = -A(c - c_1^2)\sin(c - c_1^2)t. \tag{4.20}$$

Equation (4.18) has two critical points corresponding to $t = 0$ and $t = \frac{\pi}{c - c_1^2}$

Then, M is conformal to an n-dimensional unit sphere in an Euclidean space. □

Theorem 4.3. *Let (M, g) be a Finsler manifold and admitting a concircular transformation $\bar{F} = u^{-1}F$. Assume that K and $\bar{K}$ are scalar flag curvature of F and $\bar{F}$ such that K and $\bar{K}$ are constant with condition $\lambda = A_1 u + B_1$, Then M is conformal to a unit sphere in an Euclidean space.*

Proof. Suppose F and $\bar{F}$ are concircular. we have the equation

$$\bar{Ric} - Ric = (n-1)F^2(\bar{K}u^{-2} - K) \tag{4.21}$$

Plugging (4.12) into (4.21) implies

$$(n-1)u^{-2}F^2(2\lambda u - u_m u^m) = (n-1)F^2(\frac{\bar{K}}{u^2} - K)$$

then we have

$$\begin{aligned} u^{-2}(2\lambda u - u_m u^m) - \bar{K}u^{-2} + K &= 0 \\ Ku^2 + 2\lambda u - u_m u^m - \bar{K} &= 0. \end{aligned}$$

In the case of $\lambda = A_1 u + B_1$, where A_1, B_1 are constants. we have,

$$Ku^2 + 2(A_1 u + B_1)u - u_m u^m - \bar{K} = 0. \tag{4.22}$$

The equation (4.22) along any geodesic with arc-length t reduces to the ordinary differential equation

$$Ku^2 + 2(A_1 u + B_1)u - (\frac{du}{dt})^2 - \bar{K} = 0.$$

Now, for the special case $B_1 = 0$, we get

$$Ku^2 + 2A_1u^2 - (\frac{du}{dt})^2 - \bar{K} = 0$$

in the equation above, differentiating we can get the following

$$u'' - (K + 2A_1)u = 0.$$

We can obtain

$$u(t) = C_1 \cos(K + A_1)t$$

where C_1 is a constant. We obtain two critical points, $t = 0$ and $t = \frac{\pi}{K-K_1}$. According to proposition 4.1, M is conformal to a unit sphere in an Euclidean space. □

5. **Example**

Assume a two dimensional Euclidean coordinate system on the surface M. Let $D = \{(x, y) \in R^2 | x^2 + y^2 < T\}$ the open disk, where T is big enough such that D covers the surface. The only force perturbing surface is the wind W(x,y). The associated metric to this problem is a special case of Finsler metric which is called Randers metric and is given by

$$F(y) = \sqrt{\frac{h^2(y, W) + \lambda_1 h(y, y)}{\lambda_1^2}} - \frac{h(y, W)}{\lambda_1},$$

where h is the canonical Euclidean metric and $\lambda_1 = 1 - h(W, W)$. We consider the function $u : D \to R$ defined by $u(x, y) = x^2 + y^2$. We have

$$\mathbf{g}(grad\, u, grad\, u) = 2u,$$

where $\mathbf{g}$ is the metric with components $(g_{ij}) = (\frac{1}{2}[\frac{\partial^2 F^2}{\partial y^i \partial y^j}])$ in the following equation

$$u^m u_m - 2\lambda u + cu^2 = 0 \tag{5.1}$$

$$2u - 2\lambda u + cu^2 = 0 \tag{5.2}$$

We consider $\lambda = K_1 u + B$, we can get

$$2u - 2(K_1 u + B)u + cu^2 = 0 \tag{5.3}$$

In the special case $B = 0$, we have $u = 0$ then the conformal transformation is an isometry.

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