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On the spectral geometry of 4-dimensional Lorentzian Lie group

Davood Seifipour[a*]

[a]Department of Mathematics, Abadan Branch, Islamic Azad University, Abadan, Iran

E-mail: davood.seifipour@gmail.com

Abstract. The main focus of this paper is concern to the study on the point-wise Osserman structure on 4-dimensional Lorentzian Lie group. In this paper we study on the spectrum of the Jacobi operator and spectrum of the skew-symmetric curvature operator on the non-abelian 4-dimensional Lie group G, whenever G equipped with an orthonormal left invariant pseudo-Riemannian metric g of signature $(-,+,+,+)$, i.e, Lorentzian metric, where e_1 is a unit time-like vector. The Lie algebra structure in dimension four has key role in our investigation, also in this case we study on the classification of 1-Stein and mixed IP spaces. At the end we show that G does not admit any space form and Einstein structures.

Keywords: Codazzi manifold, statistical manifold.

1. introduction

Lie groups are an important class in the family of Riemannian (pseudo-Riemannian) manifolds whenever they equipped with the left invariant Riemannian (pseudo-Riemannian) metrics. A Lie group is a smooth manifold which also carries a group structure whose product and inverse operations are smooth as maps of manifolds. One of the main reasons for the study of geometric structures on a Lie group is concern to it's Lie algebra. Since many properties of a lie group are reflected in it's lie algebra, [12]. Furthermore, as

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a bridge they establish useful relations between many different areas of mathematics in particular between algebra, topology, analysis, and geometry. They were also initially introduced as a tool to solve or simplify ordinary and partial differential equations. We recall that a Riemannian metric g on a Lie group G is called left invariant if $g_b(u,v) = g_{ab}((dl_a)_b u, (dl_a)_b v)$, for all $a, b \in G$ and all $u, v \in T_b G$, also a linear affine connection ∇ on Lie group G is left invariant if $\nabla_X Y$ is a left invariant vector field for every left invariant vector fields X, Y on G.

Spectral geometry is an area of the differential geometry that studies the spectrum of operators, and it has intersection with analysis and partial differential equations and differential geometry. Originally spectral geometry investigates on the dependence and properties of eigenvalues and eigenfunctions of the Laplacian, Jacobi operator and other operators. Moreover, in recent years have been received new and deep applications of spectral geometry in computer science, shape recognition, machine learning, heat propagation and vibration [11], [13]. Furthermore we emphasize that Osserman spaces, k-Osserman spaces, 1 and 2-Stein spaces are samples of the study of eigenvalues in the spectral geometry.

In recent years, there are a lot of papers and results about Osserman spaces in both the Riemannian and the Lorentzian setting such as [1], [2], [3], [5], [6], [7], but as mentioned in [6], [7], spectral geometry is a big branch and it has a lot of unsolved problems, also there are a few papers and results in the spectral geometry of Lie groups, so in this paper we focus on the spectral geometry of 4-dimensional Lie group G in the Lorentzian setting.

In the theory of relativity, we are mainly concerned about four dimensional Lorentzian manifolds for obvious reasons. Lorentzian manifolds are the best tool for describing the general theory of relativity. The Jacobi operator $J_X(Y) = R(Y,X)X$ is a very important tool for understanding the relation between the curvature and the geometry of pseudo-Riemannian manifold (M, g). This paper concentrates on the spectrum of the Jacobi operator and spectrum of the skew symmetric curvature operator and their geometric consequences on the non abelian 4-dimensional Lie group G equipped with an orthonormal left invariant Lorentzian metric g of signature $(-,+,+,+)$, where e_1 is a unit time-like vector in the throughout of this paper. The main focus of this study is a characterization of space-like Osserman, 1-Stein, and mixed IP spaces on the Lie group G. At the end we prove that 4-dimensional non Abelian Lie group G equipped with an orthonormal left invariant Lorentzian metric g is not a space form.

In differential geometry and mathematical physics an Einstein manifold is a Riemannian or pseudo-Riemannian differentiable manifold whose Ricci tensor is proportional to the metric, i.e, if M is the underlying n-dimensional manifold and g is the metric tensor, the Einstein manifold means $Ric = \kappa g$ for some constant κ, where Ric denotes the Ricci tensor of g. Einstein manifold with $\kappa = 0$ are called Ricci-flat manifolds. The study of left invariant Einstein Riemannian (pseudo-Riemannian) metrics on a Lie group G of a given dimension is an important and interesting question. Moreover, it is still incomplete, especially in the case of left invariant indefinite metrics. In this paper we will see that the geometry of four dimensional Lie groups with an orthonormal left invariant indefinite metrics such as Lorentzian is different from the Riemannian counterpart. At the end we prove that G is not an Einstein space.

2. **Preliminaries**

The Jacobi operator is a self adjoint operator and it plays an important role in the curvature theory. Let $spec\{J_X\}$ be the set of eigenvalues of the Jacobi operator J_X and $S^{\pm}(M,g)$ be the pseudo-sphere bundles of unit space-like $(+)$ and unit time-like $(-)$ tangent vectors. One says that (M,g) is space-like Osserman (resp. time-like Osserman) at $p \in M$, if for every $X, Y \in S_p^+(M,g)$ (resp. $X, Y \in S_p^-(M,g)$), we have the characteristic polynomial of J_X is equal to the characteristic polynomial of J_Y, in fact the eigenvalues of J_X are independent of $X \in S_p^+(M)$ (resp. $X \in S_p^-(M)$). Furthermore, (M,g) is pointwise space-like (resp. pointwise time-like) Osserman, if it is space-like (resp. time-like) Osserman at each $p \in M$, also (M,g) is globally Osserman manifold if, for any point $p \in M$ and any unit tangent vector $X \in T_pM$, the eigenvalues of the Jacobi operator depend neither on X nor on p. In fact the eigenvalues of J_X are constant on $S^{\pm}(M,g)$. We recall that globally Osserman manifolds are clearly pointwise Osserman.

Let (M,g) be a pseudo-Riemannian manifold, $p \in M$, $Z \in S_p(M) = S_p^+(M) \cup S_p^-(M)$, associated to the Jacobi operators, and natural number t, there are functions f_t defined by $f_t(p,Z) = g(Z,Z)^t trace(J_Z^{(t)})$, where $J_Z^{(t)}$ is the t^{th} power of the Jacobi operator J_Z. We say that pseudo-Riemannian manifold (M,g) is k-Stein at $p \in M$, if $f_t(p,Z)$ is independent of $Z \in S_p(M)$ for every $1 \leqslant t \leqslant k$, also (M,g) is k-Stein, if it is k-Stein at each $p \in M$. Moreover, (M,g) is 2-step Jacobi nilpotent at $p \in M$, if $J_X(J_Y(Z)) = 0$, for every $X, Y, Z \in T_pM$. Moreover, (M,g) is Jacobi-Tsankov if $J_X(J_Y(Z)) = J_Y(J_X(Z))$, for every $X, Y, Z \in T_pM$, also if (M,g) is a pseudo-Riemannian manifold of signature (p,q), then M is called a space form if it is complete and it has constant sectional curvature κ.

According to [5], [6], we have the following

Remark 2.1. *It's known that if (M,g) be a pseudo-Riemannian manifold of signature (p,q) and $x \in M$. Then (M,g) is time-like Osserman at $x \in M$ if and only if (M,g) is space-like Osserman at $x \in M$ if and only if (M,g) is m-Stein at $x \in M$, where $m = p + q$.*

Let V be a vector space equipped with a inner product of signature (r, s). A k-plane σ is said to be non-degenerate if the restriction of the metric to the plane is non-degenerate. A non-degenerate k-plane is said to be space-like or time-like if the restriction of the metric to σ is positive or negative definite. A non-degenerate 2-plane which is neither space-like nor time-like is said to be mixed and restriction of the metric has signature $(1, 1)$ in this case. We let $Gr^+_{r,s}(V)$ be Grassmannian consisting of all non-degenerate oriented subspaces of V of signature (r, s) and $\widetilde{Gr}^+_k(V) = \cup_{k=r+s} Gr^+_{r,s}(V)$ be Grassmannian of all non-degenerate oriented k-planes in V.
If $\{X, Y\}$ is an oriented basis for a non-degenerate 2-plane π, then the skew-symmetric curvature operator

$$R_\pi =\mid g(X,X)g(Y,Y) - g(X,Y)^2 \mid^{-\frac{1}{2}} R(X,Y),$$

is a skew-adjoint operator which is independent of the particular oriented basis chosen for π. The natural domains of definition for the skew-symmetric curvature operator R_π are oriented Grassmannians $Gr^+_{0,2}(V)$, $Gr^+_{1,1}(V)$, and $Gr^+_{2,0}(V)$ of oriented space-like, mixed, and time-like 2-planes.

We say that a pseudo-Riemannian manifold (M, g) is space-like, mixed, or time-like Ivanov-Petrova (briefly IP) at $p \in M$ if associated skew symmetric curvature operator has constant eigenvalues on the $Gr^+_{0,2}(T_pM)$, $Gr^+_{1,1}(T_pM)$, or $Gr^+_{2,0}(T_pM)$ respectively, also (M, g) is called Ivanov-Petrova manifold if (M, g) is Ivanov-Petrova at each point $p \in M$.

2.1. **Low dimensional Lie algebras.** Let $\mathfrak{g}_n$ be an n-dimensional Lie algebra over the field of real numbers with generator $e_1, \cdots, e_n$, $n \leq 4$. It is known that it is an abelian Lie algebra. It is knownn that there exists only one non-abelian Lie algebra of dimension two, that is the solvable $\mathfrak{af}(1)$ with Lie bracket $[e_1, e_2] = e_1$. This Lie algebra is denoted by $\mathfrak{g}_{2,1}$. In [9], Mubarakzyanov proved that there exist eight classes of non-abelian Lie algebra of dimension 3. These Lie algebras are denoted by $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,k}$, $k = 1, \cdots, 7$. He also classified the Lie algebras of domensional 4 in Nineteen classes.

3. Geometry of 4-dimensional Lorentzian Lie Group according to the Lie Algebra Structures

In the throughout of this paper we consider a non abelian 4-dimensional Lie group G equipped with an orthonormal left invariant pseudo-Riemannian metric g of signature $(-, +, +, +)$, i.e, Lorentzian metric, where e_1 is a unit time-like vector. The goal of this paper is to examine on the time-like (space-like) Osserman, 1-stein and mixed Ivanov-Petrova (briefly IP) and space form structures. We also show that G is not an Einstein space. The main tool in our study is the Lie algebra structure and Our aim is show that which of 4-dimensional Lie algebra structures on Lie group G can provide pointwise space-like (time-like) Osserman structure on it. The problem of finding all Lie algebras of a given dimension was treated by G. M. Mobarakzyanov and solved for dimensions less equal to five, [9]. In this paper we consider all of non-abelian 4-dimensional Lie algebras up to isomorphism, where their generators are g-orthonormal basis $\{e_i\}_{i=1}^4$, and then study on the geometric structures as mentioned above.

$\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$: The decomposable nilpotent Lie algebra with the only non zero bracket $[e_2, e_3] = e_1$.

From the Koszul formula in Lie groups and direct computations we obtain the non zero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \quad \nabla_{e_2} e_3 = -\nabla_{e_3} e_2 = \frac{1}{2} e_1.$$

We consider $e_3, e_4 \in S_e^+(G, g)$, using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator is as follows

$$J_{e_3}(e_1) = -\frac{1}{4} e_1, \qquad J_{e_3}(e_2) = \frac{3}{4} e_2, \qquad J_{e_3}(e_3) = J_{e_3}(e_4) = 0,$$

$$J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

Thus the spectrum of the Jacobi operators are as follows

$$spec(J_{e_3}) = \left\{0, -\frac{1}{4}, \frac{3}{4}\right\}, \qquad spec(J_{e_4}) = \{0\}.$$

Since $spec(J_{e_3}) \neq spec(J_{e_4})$, hence the spectrum of the Jacobi operator depends to the unit space-like vectors, so G is not space-like Osserman. We have also

$$f_1(e, e_3) = g(e_3, e_3) trace(J_{e_3}) = \frac{1}{2}, \qquad f_1(e, e_4) = g(e_4, e_4) trace(J_{e_4}) = 0.$$

Thus G is not 1-Stein.

If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$ then we have $spec(R_\pi) = \{0, \pm\frac{1}{4}\}$ and $spec(R_\sigma) = \{0\}$, so G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \frac{1}{16} e_2$, so G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = \frac{1}{4}$ and $\kappa_\sigma = 0$, this shows that the sectional curvature is not constant, thus G is not a space form. We have also $Ric(e_1, e_1) = \frac{1}{2}$ and $Ric(e_4, e_4) = 0$, this shows that G is not an Einstein space.

$\mathfrak{g}_{4,1}$: The indecomposable nilpotent Lie algebra with the two non zero brackets as follows $[e_2, e_4] = e_1, [e_3, e_4] = e_2$.

From the Koszul formula in Lie groups and direct computations we obtain non zero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1}e_2 = \frac{1}{2}e_4, \qquad \nabla_{e_1}e_4 = -\frac{1}{2}e_2, \qquad \nabla_{e_2}e_1 = \frac{1}{2}e_4,$$
$$\nabla_{e_2}e_3 = -\frac{1}{2}e_4, \quad \nabla_{e_2}e_4 = \frac{1}{2}e_1 + \frac{1}{2}e_3, \quad \nabla_{e_3}e_2 = -\frac{1}{2}e_4,$$

$$\nabla_{e_3}e_4 = \frac{1}{2}e_2, \qquad \nabla_{e_4}e_1 = -\frac{1}{2}e_2, \quad \nabla_{e_4}e_2 = -\frac{1}{2}e_1 + \frac{1}{2}e_3, \qquad \nabla_{e_4}e_3 = -\frac{1}{2}e_2.$$

We consider $e_3, e_4 \in S_e^+(G, g)$, using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_3}(e_1) = 0, \quad J_{e_3}(e_2) = \frac{1}{4}e_2, \quad J_{e_3}(e_3) = 0, \quad J_{e_3}(e_4) = -\frac{3}{4}e_4,$$

$$J_{e_4}(e_1) = -\frac{1}{4}e_1 + \frac{1}{4}e_3, \quad J_{e_4}(e_2) = e_2, \quad J_{e_4}(e_3) = -\frac{1}{4}e_1 - \frac{3}{4}e_3, \quad J_{e_4}(e_4) = 0.$$

Direct computation gives us

$$\{0, \frac{1}{4}, -\frac{3}{4}\} = spec(J_{e_3}) \neq spec(J_{e_4}) = \{0, 1, -\frac{1}{2}\}.$$

Thus G is not space-like Osserman. We have also

$$f_1(e, e_3) = g(e_3, e_3)trace(J_{e_3}) = -\frac{1}{2}, \qquad f_1(e, e_4) = g(e_4, e_4)trace(J_{e_4}) = 0.$$

Thus G is not 1-Stein.

If we take oriented 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{\sqrt{2}}e_4\}$ of signature $(1,1)$ then $spec(R_\pi) = \{0\}$ and $spec(R_\sigma) = \{\pm\frac{1}{4(2)^{\frac{1}{4}}}\}$, thus G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \frac{1}{16}e_2$, so G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$, then $\kappa_\pi = \frac{1}{4}$ and $\kappa_\sigma = 0$, this shows that the sectional curvature is not constant, thus G is not a space form. We have also

$$Ric(e_1, e_1) = \frac{1}{2}, \quad Ric(e_2, e_2) = \frac{3}{2}.$$

This implies that G is not an Einstein space.

$\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1$: The unsolvable Lie algebra with the three non zero brackets as follows

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2,$$

From the Koszul formula in Lie groups we obtain the non zero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1}e_2 = \frac{3}{2}e_3, \qquad \nabla_{e_1}e_3 = -\frac{3}{2}e_2, \qquad , \nabla_{e_2}e_1 = \frac{1}{2}e_3,$$

$$\nabla_{e_2}e_3 = \frac{1}{2}e_1, \qquad \nabla_{e_3}e_1 = -\frac{1}{2}e_2, \qquad \nabla_{e_3}e_2 = -\frac{1}{2}e_1.$$

Using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_3}(e_1) = -\frac{1}{4}e_1, \qquad J_{e_3}(e_2) = \frac{7}{4}e_2, \qquad J_{e_3}(e_3) = J_{e_3}(e_4) = 0,$$

$$J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

Since $\{0\} = spec(J_{e_4}) \neq spec(J_{e_3}) = \{0, -\frac{1}{4}, \frac{7}{4}\}$, thus as the same proof in last cases this implies that the the spectrum of the Jacobi operator depends to the unit space-like vectors. Hence G is not space-like Osserman. Moreover, by matrix representation of the Jacobi operators we obtain

$$f_1(e, e_3) = g(e_3, e_3)trace(J_{e_3}) = \frac{3}{2}, \qquad f_1(e, e_4) = g(e_4, e_4)trace(J_{e_4}) = 0.$$

Thus G is not 1-Stein.
Considering non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, we get $spec(R_\pi) = \{0, \pm\frac{1}{4}\}$ and $spec(R_\sigma) = \{0\}$, thus G is not mixed IP.
We have also $J_{e_1}(J_{e_1}(e_2)) = \frac{1}{16}e_2$, so G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = \frac{1}{4}$ and $\kappa_\sigma = 0$, this implies that the sectional curvature is not constant, thus G is not a space form. Furthermore, we have $Ric(e_1, e_1) = \frac{1}{2}$ and $Ric(e_4, e_4) = 0$, this implies that G is not an Einstein space.

$\mathfrak{g}_{3,6} \oplus \mathfrak{g}_1$**:** The unsolvable Lie algebra with the three non zero brackets as follows

$$[e_1, e_2] = e_1, \qquad [e_1, e_3] = 2e_2, \qquad [e_2, e_3] = e_3.$$

From the Koszul formula in Lie groups we get the following

$$\nabla_{e_1}e_1 = e_2, \qquad \nabla_{e_1}e_2 = e_1 - e_3, \qquad \nabla_{e_1}e_3 = e_2, \qquad \nabla_{e_2}e_1 = -e_3,$$

$$\nabla_{e_2}e_3 = -e_1, \qquad \nabla_{e_3}e_1 = -e_2, \qquad \nabla_{e_3}e_2 = -e_1 - e_3, \qquad \nabla_{e_3}e_3 = e_2.$$

Using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_3}(e_1) = 4e_1, \qquad J_{e_3}(e_2) = -2e_2, \qquad J_{e_3}(e_3) = J_{e_3}(e_4) = 0,$$

$$J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

In this case we have $\{0, 4, -2\} = spec(J_{e_3}) \neq spec(J_{e_4}) = \{0\}$, therefore the spectrum of the Jacobi operator depends to the unit space-like vectors, thus G is not space-like Osserman. Furthermore, matrix representation of the Jacobi operators give us

$$f_1(e, e_3) = g(e_3, e_3)trace(J_{e_3}) = 2, \qquad f_1(e, e_4) = g(e_4, e_4)trace(J_{e_4}) = 0.$$

Hence G is not 1-Stein.
Considering non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$ of signature

$(1,1)$, we get $spec(R_\pi)=\{0\}$ and $spec(R_\sigma)=\{0,\pm 4\}$, thus G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2))=4e_2$, so G is not 2-step Jacobi nilpotent.
If we take $\pi=\{e_1,e_3\}$ and $\sigma=\{e_1,e_4\}$, then $\kappa_\pi=-4$ and $\kappa_\sigma=0$, this implies that the sectional curvature is not constant, thus G is not a space form. Also $Ric(e_1,e_1)=-2$ and $Ric(e_4,e_4)=0$, this implies that G is not an Einstein space.
$\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$: The decomposable solvable Lie algebra with the only one non zero bracket $[e_1,e_2]=e_1$.
The Koszul formula in Lie groups gives us the following

$$\nabla_{e_1}e_1=e_2,\qquad \nabla_{e_1}e_2=e_1.$$

Using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_2}(e_1)=-e_1,\qquad J_{e_2}(e_2)=e_2,\qquad J_{e_2}(e_3)=J_{e_2}(e_4)=0,$$

$$J_{e_4}(e_1)=J_{e_4}(e_2)=J_{e_4}(e_3)=J_{e_4}(e_4)=0.$$

In this case we have $\{0,-1\}=spec(J_{e_2})\neq spec(J_{e_4})=\{0\}$, therefore the spectrum of the Jacobi operator depends to the unit space-like vectors, thus G is not space-like Osserman. Moreover, matrix representation of the Jacobi operators give us

$$f_1(e,e_2)=g(e_2,e_2)trace(J_{e_2})=-1,\qquad f_1(e,e_4)=g(e_4,e_4)trace(J_{e_4})=0.$$

Since trace depends to the unit space-like vectors, hence G is not 1-Stein.
Considering non degenerate 2-planes $\pi=\{e_1,e_2\}$ and $\sigma=\{e_1,e_4\}$ of signature $(1,1)$, we get $spec(R_\pi)=\{0,\pm 1\}$ and $spec(R_\sigma)=\{0\}$, thus G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2))=e_2$, so G is not 2-step Jacobi nilpotent.
The following is the first case of 4-dimensional Lie algebra structure $\mathfrak{g}$, such that $(G,\mathfrak{g})$ has 1-Stein property. Furthermore, in this case, there exists the same result in the Riemannian setting.
If we take $\pi=\{e_1,e_2\}$ and $\sigma=\{e_1,e_4\}$, then $\kappa_\pi=1$ and $\kappa_\sigma=0$, this implies that the sectional curvature is not constant, therefore G is not a space form. Furthermore, $Ric(e_1,e_1)=1$ and $Ric(e_4,e_4)=0$, this implies that G is not an Einstein space.

$2\mathfrak{g}_{2,1}$: The decomposable solvable Lie algebra with the two non zero bracket $[e_1,e_2]=e_1$ and $[e_3,e_4]=e_3$.
From the Koszul formula in Lie groups we get the following

$$\nabla_{e_1}e_1=e_2,\qquad \nabla_{e_1}e_2=e_1,\qquad\qquad \nabla_{e_3}e_3=-e_4,\qquad \nabla_{e_3}e_4=e_3.$$

Now we consider $v = re_1 + se_2 + te_3 + ke_4 \in S_e^+(G, g)$, from Riemannian curvature tensor we get the Jacobi operator J_v is as follows

$$J_v(e_1) = -s^2 e_1 - sre_2, \quad J_v(e_2) = rse_1 + r^2 e_2, \quad J_v(e_3) = -k^2 e_3 + kte_4,$$
$$J_v(e_4) = tke_3 - t^2 e_4.$$

Matrix representation of J_v, gives us

$$spec(J_v) = \{0, r^2 - s^2, -(k^2 + t^2)\}.$$

This implies that G is not space-like Osserman.

Moreover, if $v = re_1 + se_2 + te_3 + ke_4 \in S_e^+(G, g)$, then we have

$$f_1(e, v) = g(v, v)trace(J_v) = r^2 - (s^2 + k^2 + t^2) = -1.$$

Similarly, if $v \in S_e^-(G, g)$, we get $f_1(e, v) = -1$. This means that $f_1(e, v)$ is independent of unit vector v. Thus G is 1-Stein.

Considering non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$ of signature $(1, 1)$, we get $spec(R_\pi) = \{0, \pm 1\}$ and $spec(R_\sigma) = \{0\}$, thus G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = e_2$, so G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = 1$ and $\kappa_\sigma = 0$, this implies that G is not a space form. We have also $Ric(e_1, e_1) = Ric(e_2, e_2) = 1$ and $Ric(e_3, e_3) = Ric(e_4, e_4) = -1$, thus according to the metric we get that G is not an Einstein space.

$\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with the two non zero bracket $[e_1, e_3] = e_1$ and $[e_2, e_3] = e_1 + e_2$.

From the Koszul formula in Lie groups we obtain the following

$$\nabla_{e_1} e_1 = e_3, \qquad \nabla_{e_1} e_2 = \frac{1}{2} e_3, \qquad \nabla_{e_1} e_3 = e_1 - \frac{1}{2} e_2, \qquad \nabla_{e_2} e_1 = \frac{1}{2} e_3,$$

$$\nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_3 = \frac{1}{2} e_1 + e_2, \qquad \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \qquad \nabla_{e_3} e_2 = -\frac{1}{2} e_1.$$

Using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_2}(e_1) = -\frac{5}{4} e_1, \qquad J_{e_2}(e_2) = 0, \qquad J_{e_2}(e_3) = -\frac{1}{4} e_3, \qquad J_{e_2}(e_4) = 0,$$

$$J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

In this case we have $\{0, -\frac{5}{4}, -\frac{1}{4}\} = spec(J_{e_2}) \neq spec(J_{e_4}) = \{0\}$, therefore the spectrum of the Jacobi operator depends to the unit space-like vectors, thus G is not space-like Osserman. Furthermore, matrix representation of the Jacobi operators give us

$$f_1(e, e_2) = g(e_2, e_2)trace(J_{e_2}) = -\frac{3}{2}, \qquad f_1(e, e_4) = g(e_4, e_4)trace(J_{e_4}) = 0.$$

Since trace depends to the unit space-like vectors, hence G is not 1-Stein.
If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, then we get

$$spec(R_\pi) = \{0, \pm\frac{5}{4}\}, \quad spec(R_\sigma) = \{0\}.$$

hence G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \frac{25}{16}e_2$, so G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = \frac{5}{4}$ and $\kappa_\sigma = 0$, so G is not a space form. Since $Ric(e_1, e_1) = \frac{5}{2}$ and $Ric(e_4, e_4) = 0$, therefore G is not an Einstein space.
$\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with the two non zero bracket as follows

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2.$$

From the Koszul formula in Lie groups we obtain the following

$$\nabla_{e_1} e_1 = e_3, \qquad \nabla_{e_1} e_3 = e_1, \qquad \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_3 = e_2.$$

Using the Levi-Civita connection and Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_2}(e_1) = -e_1, \qquad J_{e_2}(e_2) = 0, \qquad J_{e_2}(e_3) = -e_3, \qquad J_{e_2}(e_4) = 0,$$

$$J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

In this case we have $\{0, -1\} = spec(J_{e_2}) \neq spec(J_{e_4}) = \{0\}$, therefore the spectrum of the Jacobi operator depends to the unit space-like vectors, thus G is not space-like Osserman. Moreover, matrix representation of the Jacobi operators give us

$$f_1(e, e_2) = g(e_2, e_2)trace(J_{e_2}) = -2, \qquad f_1(e, e_4) = g(e_4, e_4)trace(J_{e_4}) = 0.$$

Since trace depends to the unit space-like vectors, hence G is not 1-Stein.
If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, then we get $spec(R_\pi) = \{0, \pm 1\}$ and $spec(R_\sigma) = \{0\}$, therefore G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = e_2$, so G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = 1$ and $\kappa_\sigma = 0$, so G is not a space form. We have also $Ric(e_1, e_1) = 2$ and $Ric(e_4, e_4) = 0$, so G is not an Einstein space.

$\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with the two non zero bracket as follows

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = \alpha e_2,$$

where $-1 \leq \alpha < 1$ and $\alpha \neq 0$. From the Koszul formula in Lie groups we obtain the non zero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1} e_1 = e_3, \qquad \nabla_{e_1} e_3 = e_1, \qquad \nabla_{e_2} e_2 = -\alpha e_3, \qquad \nabla_{e_2} e_3 = \alpha e_2.$$

Using the Levi-Civita connection and Riemannian curvature tensor we obtain the Jacobi operator as follows

$$J_{e_3}(e_1) = -e_1, \qquad J_{e_3}(e_2) = -\alpha^2 e_2, \qquad J_{e_3}(e_3) = J_{e_3}(e_4) = 0,$$

$$J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

In this case we have $\{0, -1, -\alpha^2\} = spec(J_{e_3}) \neq spec(J_{e_4}) = \{0\}$, therefore the spectrum of the Jacobi operator depends to the unit space-like vectors, thus G is not space-like Osserman. Furthermore, matrix representation of the Jacobi operators give us

$$f_1(e, e_3) = g(e_3, e_3) trace(J_{e_3}) = -1 - \alpha^2, \quad f_1(e, e_4) = g(e_4, e_4) trace(J_{e_4}) = 0.$$

Since trace depends to the unit space-like vectors, hence G is not 1-Stein.
If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, then we get $spec(R_\pi) = \{0, \pm\alpha\}$ and $spec(R_\sigma) = \{0\}$, therefore G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \alpha^2 e_2$, so G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = \alpha$ and $\kappa_\sigma = 0$, hence G is not a space form. We have also $Ric(e_1, e_1) = \alpha + 1$, $Ric(e_2, e_2) = \alpha(1 - \alpha)$ and $Ric(e_4, e_4) = 0$, so according to the metric we deduce that G is not an Einstein space.

$\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with the two non zero bracket as follows

$$[e_1, e_3] = \beta e_1 - e_2, \qquad [e_2, e_3] = e_1 + \beta e_2,$$

where $0 \leqq \beta$. From the Koszul formula in Lie groups and direct computations we obtain the left invariant Levi-Civita connection ∇ is as follows

$$\nabla_{e_1} e_1 = \beta e_3, \qquad \nabla_{e_1} e_2 = e_3, \qquad \nabla_{e_1} e_3 = \beta e_1 - e_2,$$

$$\nabla_{e_2} e_1 = e_3, \qquad \nabla_{e_2} e_2 = -\beta e_3, \qquad \nabla_{e_2} e_3 = e_1 + \beta e_2,$$

Using the Levi-Civita connection and Riemannian curvature tensor we obtain the Jacobi operator as follows

$$J_{e_3}(e_1) = (1 - \beta^2) e_1 + 2\beta e_2, \qquad J_{e_3}(e_2) = -2\beta e_1 + (1 - \beta^2) e_2,$$

$$J_{e_3}(e_3) = J_{e_3}(e_4) = 0, , \qquad J_{e_4}(e_1) = J_{e_4}(e_2) = J_{e_4}(e_3) = J_{e_4}(e_4) = 0.$$

We consider two states, if $\beta \neq 1$, then from matrix representations we have

$$f_1(e, e_3) = g(e_3, e_3)trace(J_{e_3}) = 2 - 2\beta^2, \qquad f_1(e, e_4) = g(e_4, e_4)trace(J_{e_4}) = 0.$$

This shows that G is not 1-Stein. Thus according to Remark (2.1), we imply that G is not space-like Osserman. If $\beta = 1$, then the characteristic polynomial of J_{e_3} is $x^2(x^2 + 4)$ and the characteristic polynomial of J_{e_4} is x^4, therefore G is not space-like Osserman.

Now, we examine the 1-Stein property of G. Let

$$v = re_1 + se_2 + te_3 + ke_4 \in S_e^+(G, g).$$

Then direct computations give us the following

$$J_v(e_1) = -2s^2e_1 + (2t^2 - 2sr)e_2 - 2tse_3,$$
$$J_v(e_2) = (2rs - 2t^2)e_1 + 2r^2e_2 - 2tre_3,$$

$$J_v(e_3) = 2ste_1 - 2rte_2 + 4rse_3, \qquad J_v(e_4) = 0.$$

Thus matrix representation of J_v implies that

$$f_1(e, v) = g(v, v)trace(J_v) = -2s^2 + 2r^2 + 4rs.$$

This shows that trace depends to the coordinate of v, so G is not 1-Stein. Now if $\beta = 0$ then it is easy to see that G is not 1-Stein and according to Remark (2.1) we deduce that in this case G is not space-like Osserman.

If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, then we get

$$spec(R_\pi) = \{0, \pm\sqrt{1 + \beta^2}\}, \qquad spec(R_\sigma) = \{0\}.$$

Therefore G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = (\beta^2 + 1)^2e_2$, so G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = (1 + \beta^2)$ and $\kappa_\sigma = 0$, so G is not a space form. A straightforward computation gives us $Ric(e_2, e_2) = 2$ and $Ric(e_3, e_3) = 0$, so G is not an Einstein space.

$\mathfrak{g}_{4,3}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$.

The Koszul formula in Lie groups gives us the following

$$\nabla_{e_1}e_1 = e_4, \qquad \nabla_{e_1}e_4 = e_1, \qquad \nabla_{e_2}e_3 = \nabla_{e_3}e_2 = -\frac{1}{2}e_4,$$

$$\nabla_{e_2}e_4 = \nabla_{e_4}e_2 = \frac{1}{2}e_3, \qquad \nabla_{e_3}e_4 = -\nabla_{e_4}e_3 = \frac{1}{2}e_2.$$

If we consider space-like vectors e_2 and e_3 then we get $spec(J_{e_2}) = \{0, \frac{1}{4}\}$ and $spec(J_{e_3}) = \{0, \frac{1}{4}, -\frac{3}{4}\}$, so G is not space-like Osserman, we have also $f_1(e, e_2) = \frac{1}{2}$ and $f_1(e, e_3) = -\frac{1}{2}$, thus G is not 1-Stein.

If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1,1)$, then we get $spec(R_\pi) = \{0, \pm\frac{1}{2}\}$ and $spec(R_\sigma) = \{0, \pm 1\}$, therefore G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \frac{1}{4}e_2$, so G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = 0$ and $\kappa_\sigma = 1$, thus G is not a space form. A direct computation gives us $Ric(e_2, e_2) = \frac{1}{2}$ and $Ric(e_1, e_1) = 1$, so G is not an Einstein space.

$\mathfrak{g}_{4,2}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = \beta e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$, $\beta \neq 0$.

The Koszul formula in Lie groups gives us the following

$$\nabla_{e_1} e_1 = \beta e_4, \qquad \nabla_{e_1} e_4 = \beta e_1, \qquad \nabla_{e_2} e_2 = -e_4,$$
$$\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = -\frac{1}{2} e_4, \quad \nabla_{e_3} e_3 = -e_4,$$

$$\nabla_{e_2} e_4 = e_2 + \frac{1}{2} e_3, \quad \nabla_{e_4} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_3} e_4 = \frac{1}{2} e_2 + e_3, \quad \nabla_{e_4} e_3 = -\frac{1}{2} e_2.$$

If we take space-like vectors e_3 and e_4 then we get $f_1(e, e_3) = -(\beta + \frac{3}{4})$ and $f_1(e, e_4) = -(\beta^2 + \frac{10}{4})$, thus G is not 1-Stein and according to the Remark (2.1) we imply that G is not space-like Osserman.
If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$ of signature $(1,1)$, then we get

$$spec(R_\pi) = \{0, \pm\frac{\sqrt{5}}{2}\beta\}, \qquad spec(R_\sigma) = \{0, \pm\beta^2\}.$$

Therefore G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \beta^2(\frac{5}{4}e_2 + e_3)$, so G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_2, e_4\}$ and $\sigma = \{e_3, e_4\}$, then $\kappa_\pi = -\frac{3}{4}$ and $\kappa_\sigma = -\frac{7}{4}$, thus G is not a space form. A direct computation gives us

$$Ric(e_2, e_2) = \beta - \frac{3}{2}, \qquad Ric(e_3, e_3) = \beta - \frac{5}{2}.$$

So, G is not an Einstein space.

$\mathfrak{g}_{4,4}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = e_1$, $[e_2, e_4] = e_1 + e_2$, $[e_3, e_4] = e_2 + e_3$.
From the Koszul formula we get the following

$$\nabla_{e_1} e_1 = e_4, \qquad \nabla_{e_1} e_2 = \frac{1}{2} e_4, \qquad \nabla_{e_1} e_4 = e_1 - \frac{1}{2} e_2,$$
$$\nabla_{e_2} e_1 = \frac{1}{2} e_4, \quad \nabla_{e_2} e_2 = -e_4, \quad \nabla_{e_2} e_3 = -\frac{1}{2} e_4$$

$$\nabla_{e_2}e_4 = \frac{1}{2}e_1 + e_2 + \frac{1}{2}e_3, \qquad \nabla_{e_3}e_2 = -\frac{1}{2}e_4,$$

$$\nabla_{e_3}e_3 = -e_4, \qquad \nabla_{e_3}e_4 = \frac{1}{2}e_2 + e_3, \qquad \nabla_{e_4}e_1 = -\frac{1}{2}e_2,$$

$$\nabla_{e_4}e_2 = -\frac{1}{2}e_1 + \frac{1}{2}e_3, \qquad \nabla_{e_4}e_3 = -\frac{1}{2}e_2.$$

If we take space-like vectors e_2 and e_3 then from the Jacobi operators we get $f_1(e, e_2) = -2$ and $f_1(e, e_3) = -\frac{7}{2}$, thus G is not 1-Stein and according to the Remark (2.1) we imply that G is not space-like Osserman.

If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$ of signature $(1, 1)$, then we get $spec(R_\pi) = \{0, \pm\frac{\sqrt{7}}{4}\}$ and $spec(R_\sigma) = \{0, \pm 1\}$, therefore G is not mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \frac{29}{16}e_2 + \frac{9}{8}e_3$, so G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$, then $\kappa_\pi = \frac{5}{4}$ and $\kappa_\sigma = 1$, therefore G is not a space form. We have also $Ric(e_2, e_2) = \frac{1}{2}$ and $Ric(e_1, e_1) = \frac{7}{2}$, so G is not an Einstein space.
Finally we get the first Lie algebra structure such that G equipped with it has mixed IP structure. In this case also G has 1-Stein property, whenever $\alpha = \beta = \gamma$.

$\mathfrak{g}_{4,5}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = \alpha e_1$, $[e_2, e_4] = \beta e_2$, $[e_3, e_4] = \gamma e_3$, $\alpha\beta\gamma \neq 0$.
From the Koszul formula we get the following

$$\nabla_{e_1}e_1 = \alpha e_4, \qquad \nabla_{e_1}e_4 = \alpha e_1, \qquad \nabla_{e_2}e_2 = -\beta e_4,$$

$$\nabla_{e_2}e_4 = \beta e_2, \qquad \nabla_{e_3}e_3 = -\gamma e_4, \qquad \nabla_{e_3}e_4 = \gamma e_3.$$

According to the Jacobi operators we have

$$tr(J_{e_2}) = -\beta(\alpha + \beta + \gamma), \qquad tr(J_{e_3}) = -\gamma(\alpha + \beta + \gamma),$$
$$tr(J_{e_4}) = -(\alpha^2 + \beta^2 + \gamma^2).$$

If $\beta \neq \gamma$ then $tr(J_{e_2}) \neq tr(J_{e_3})$, so G is not 1-Stein and according to the Remark(2.1) we deduce that G is not space-like Osserman. If $\beta = \gamma \neq \alpha$ then $tr(J_{e_2}) = tr(J_{e_4})$ is not possible, because equation implies that $\beta = \alpha$ and this is contradiction. Therefore in this case G is not 1-Stein and according to the Remark(2.1) we obtain that G is not space-like Osserman.

If $\alpha = \beta = \gamma$ and $v = re_1 + se_2 + te_3 + ke_4$ be a space-like unit tangent vector then we have

$$J_v(e_1) = -\alpha^2(s^2 + t^2 + k^2)e_1 - \alpha^2(sr)e_2 - \alpha^2(tr)e_3 - \alpha^2(kr)e_4,$$

$$J_v(e_2) = \alpha^2(rs)e_1 + \alpha^2(r^2 - t^2 - k^2)e_2 + \alpha^2(ts)e_3 + \alpha^2(ks)e_4,$$

$$J_v(e_3) = \alpha^2(rt)e_1 + \alpha^2(st)e_2 + \alpha^2(r^2 - s^2 - k^2)e_3 + \alpha^2(kt)e_4,$$

$$J_v(e_4) = \alpha^2(rk)e_1 + \alpha^2(sk)e_2 + \alpha^2(tk)e_3 + \alpha^2(r^2 - s^2 - t^2)e_4.$$

Thus

$$trace(J_v) = -3\alpha^2(-r^2 + s^2 + t^2 + k^2) = -3\alpha^2.$$

Therefore trace is independent of v, so G is 1-Stein. If we take two space-like vectors $v_1 = e_2$ and $v_2 = \frac{1}{\sqrt{2}}e_2 + \frac{1}{\sqrt{2}}e_3$ then according to the Jacobi operator J_v, we get $spec(J_{v_1}) = \{0, -\alpha^2\}$ and $spec(J_{v_2}) = \{0, -\alpha^2, -\alpha\}$, so in this case we imply that G is not space-like Osserman.

If we consider non degenerate 2-planes $\pi = \{e_1, e_2\}$, $\sigma_1 = \{e_1, e_3\}$ and $\sigma_2 = \{e_1, e_4\}$ of signature $(1,1)$, then $spec(R_\pi) = \{0, \pm\alpha\beta\}$, $spec(R_{\sigma_1}) = \{0, \pm\alpha\gamma\}$ and $spec(R_{\sigma_2}) = \{0, \pm\alpha^2\}$. If α, β, γ are pairwise distinct then $spec(R_\pi) \neq spec(R_{\sigma_1})$, so G is not mixed IP. If $\alpha = \beta \neq \gamma$ then $spec(R_\pi) \neq spec(R_{\sigma_1})$, so in this case G is not mixed IP. Moreover, if $\beta = \gamma \neq \alpha$ then $spec(R_\pi) \neq spec(R_{\sigma_2})$, thus G is not mixed IP.

Now, we assume that $\alpha = \beta = \gamma$ and we take non degenerate 2-plane $\pi = \{v_1, v_2\}$ of signature $(1,1)$, where $v_1 = re_1 + se_2 + te_3 + ke_4$ and $v_2 = xe_1 + ye_2 + ze_3 + we_4$, so we have the following relations

$$-r^2 + s^2 + t^2 + k^2 = -1, \quad -x^2 + y^2 + z^2 + w^2 = 1, \quad -rx + sy + tz + kw = 0. \tag{3.1}$$

Matrix representation of the skew-symmetric curvature operator is as follows

$$R_\pi(e_1) = \alpha^2(sx - ry)e_2 + \alpha^2(tx - rz)e_3 + \alpha^2(kx - rw)e_4,$$

$$R_\pi(e_2) = \alpha^2(sx - ry)e_1 + \alpha^2(sz - ty)e_3 + \alpha^2(sw - ky)e_4,$$

$$R_\pi(e_3) = \alpha^2(tx - rz)e_1 + \alpha^2(ty - sz)e_2 + \alpha^2(tw - kz)e_4,$$

$$R_\pi(e_4) = \alpha^2(kx - rw)e_1 + \alpha^2(ky - sw)e_2 + \alpha^2(kz - tw)e_3.$$

If we take $a = sx - ry$, $b = tx - rz$, $c = kx - rw$, $d = sz - ty$, $e = sw - ky$ and $f = tw - kz$, then the characteristic polynomial of R_π is $X^4 + p\alpha^4 X^2 + q = 0$, where $p = (f^2 + d^2 + e^2 - a^2 - b^2 - c^2)$ and $q = \alpha^8\{2abef - 2acdf + 2becd - f^2a^2 - b^2e^2 - c^2d^2\}$. Direct computation gives us $q = 0$, if we multiply the first and second equation in (3.1) and subtract of square of third equation, we deduce that $p = -1$. Therefore $spec(R_\pi) = \{0, \pm\alpha^2\}$, so the spectrum of R_π is independent of π, thus G is mixed IP. We have also $J_{e_1}(J_{e_1}(e_2)) = \alpha^2\beta^2 e_2$, so G is not 2-step Jacobi nilpotent.

If $\alpha \neq \beta$ then we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, and we have $\kappa_\pi = \alpha\beta$ and $\kappa_\sigma = \alpha^2$, so in this case G is not a space form. If $\alpha = \beta \neq \gamma$ then we take $\pi = \{e_1, e_4\}$ and $\sigma = \{e_1, e_3\}$, and we get $\kappa_\pi = \alpha^2$ and $\kappa_\sigma = \alpha\gamma$, and this

shows that G is not a space form. If $\alpha = \beta = \gamma$ then we have $\pi = \{e_1, e_3\}$ and $\sigma = \{e_2, e_3\}$, and we obtain $\kappa_\pi = \alpha\gamma$ and $\kappa_\sigma = -\beta\gamma$, thus G is not a space form.
We have also $Ric(e_1, e_1) = \alpha(\alpha+\beta+\gamma)$, $Ric(e_2, e_2) = \beta(\alpha-\beta-\gamma)$, $Ric(e_3, e_3) = \gamma(\alpha - \beta - \gamma)$, and $Ric(e_4, e_4) = \alpha^2 - \beta^2 - \gamma^2$. If $\beta \neq \gamma$ then comparison between $Ric(e_2, e_2)$ and $Ric(e_3, e_3)$ give us that G is not an Einstein space. If $\beta = \gamma \neq \alpha$ then in this case G is not an Einstein space, because if there exist a real number λ such that $Ric(e_i, e_j) = \lambda g(e_i, e_j)$ for all i, j, then this relation is hold for $i = j = 2$ and $i = j = 4$, thus $\alpha\beta - 2\beta^2 = \alpha^2 - 2\beta^2$, this give us $\alpha = \beta$, that is a contradiction. If $\alpha = \beta = \gamma$ then $Ric(e_1, e_1) = 3\alpha^2$ and $Ric(e_2, e_2) = -\alpha^2$, so in this case G is not an Einstein space.
In the following we obtain the second Lie algebra structure that can provide mixed IP structure of G.

$\mathfrak{g}_{4,6}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = \alpha e_1$, $[e_2, e_4] = \beta e_2 - e_3$, $[e_3, e_4] = e_2 + \beta e_3$, $\alpha > 0$.
The Koszul formula gives us the following

$$\nabla_{e_1} e_1 = \alpha e_4, \qquad \nabla_{e_1} e_4 = \alpha e_1, \qquad \nabla_{e_2} e_2 = -\beta e_4, \qquad \nabla_{e_2} e_4 = \beta e_2,$$

$$\nabla_{e_3} e_3 = -\beta e_4, \qquad \nabla_{e_3} e_4 = \beta e_3, \qquad \nabla_{e_4} e_2 = e_3, \qquad \nabla_{e_4} e_3 = -e_2.$$

According to the Jacobi operators J_{e_2} and J_{e_4} we have $trace(J_{e_2}) = -\beta(\alpha+2\beta)$ and $trace(J_{e_4}) = -(\alpha^2 + 2\beta^2)$. If $\beta = 0$ then $trace(J_{e_2}) = 0$ and $trace(J_{e_4}) = -\alpha^2$, thus G is not 1-Stein, so according to the Remark (2.1) G is not space-like Osserman.
If $\beta \neq 0$ and $\beta \neq \alpha$ then $trace(J_{e_2}) \neq trace(J_{e_4})$, so G is not 1-Stein and Remark (2.1) gives us G is not space-like Osserman.
If $\beta \neq 0$ and $\beta = \alpha$ and $v = re_1 + se_2 + te_3 + ke_4$ be a space-like unit tangent vector then matrix representation of J_v is exactly as the same as matrix representation of J_v in $\mathfrak{g}_{4,5}$, so we have G is 1-Stein but it is not space-like Osserman.
If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$, $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, then $spec(R_\pi) = \{0, \pm\alpha\beta\}$ and $spec(R_\sigma) = \{0, \pm\alpha^2\}$. If $\alpha \neq \beta$ then $spec(R_\pi) \neq spec(R_\sigma)$, so in this case G is not mixed IP.
Now we consider non degenerate 2-plane $\pi = \{v_1, v_2\}$ of signature $(1, 1)$, where $v_1 = re_1 + se_2 + te_3 + ke_4$ and $v_2 = xe_1 + ye_2 + ze_3 + we_4$. If $\alpha = \beta$ then the matrix representation of the skew-symmetric curvature operator R_π is exactly as the same as in last case, i.e, $\mathfrak{g}_{4,5}$. So we have $spec(R_\pi) = \{0, \pm\alpha^2\}$, hence the spectrum is independent of non degenerate 2-plane π of signature $(1, 1)$, thus in this case G is mixed IP. Furthermore, $J_{e_1}(J_{e_1}(e_4)) = \alpha^4 e_4$, thus G is not 2-step Jacobi nilpotent.
If $\alpha \neq \beta$ then we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, and we have $\kappa_\pi = \alpha\beta$

and $\kappa_\sigma = \alpha^2$, so in this case G is not a space form. If $\alpha = \beta$ then we take $\pi = \{e_1, e_4\}$ and $\sigma = \{e_2, e_3\}$, and we have $\kappa_\pi = \alpha^2$ and $\kappa_\sigma = -\beta^2$, so in this case G is not a space form. Moreover, direct computation gives us

$$Ric(e_1, e_1) = \alpha^2 + 2\alpha\beta, \qquad Ric(e_2, e_2) = Ric(e_3, e_3) = \alpha\beta - 2\beta^2,$$
$$Ric(e_4, e_4) = \alpha^2 - 2\beta^2.$$

If $\alpha \neq \beta$ then G is not an Einstein space, because if G is an Einstein space then comparison between $Ric(e_2, e_2)$ and $Ric(e_4, e_4)$ give us $\alpha = \beta$, that is a contradiction. If $\alpha = \beta$ then $Ric(e_1, e_1) = 3\alpha^2$ and $Ric(e_2, e_2) = -\alpha^2$, so in this case G is not an Einstein space.

$\mathfrak{g}_{4,7}$**:** The indecomposable solvable Lie algebra with $[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$.
From the Koszul formula we obtain the following relations

$$\nabla_{e_1} e_1 = 2e_4, \qquad \nabla_{e_1} e_2 = \frac{1}{2} e_3, \qquad \nabla_{e_1} e_3 = -\frac{1}{2} e_2, \qquad \nabla_{e_1} e_4 = 2e_1,$$

$$\nabla_{e_2} e_1 = \frac{1}{2} e_3, \qquad \nabla_{e_2} e_2 = -e_4, \qquad \nabla_{e_2} e_3 = \frac{1}{2} e_1 - \frac{1}{2} e_4, \qquad \nabla_{e_2} e_4 = e_2 + \frac{1}{2} e_3,$$

$$\nabla_{e_3} e_1 = -\frac{1}{2} e_2, \qquad \nabla_{e_3} e_2 = -\frac{1}{2} e_1 - \frac{1}{2} e_4, \qquad \nabla_{e_3} e_3 = -e_4, \qquad \nabla_{e_3} e_4 = \frac{1}{2} e_2 + e_3,$$

$$\nabla_{e_4} e_2 = \frac{1}{2} e_3, \qquad \nabla_{e_4} e_3 = -\frac{1}{2} e_2.$$

From matrix representation of the Jacobi operators we have $trace(J_{e_2}) = -3$ and $trace(J_{e_3}) = -4$, so G is not 1-Stein, thus from Remark (2.1) we imply that G is not space-like Osserman.

If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$, $\sigma = \{e_1, e_4\}$ of signature $(1, 1)$, then $spec(R_\pi) = \{\pm(\frac{23+17\sqrt{2}}{8})^{\frac{1}{2}}\}$ and $spec(R_\sigma) = \{\pm 4\}$, thus G is not mixed IP. Furthermore, $J_{e_1}(J_{e_1}(e_4)) = 16e_4$, thus G is not 2-step Jacobi nilpotent.

If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = \frac{9}{4}$ and $\kappa_\sigma = 4$, therefore G is not a space form. Furthermore, since $Ric(e_1, e_1) = \frac{17}{2}$ and $Ric(e_2, e_2) = \frac{3}{2}$, so G is not an Einstein space.

$\mathfrak{g}_{4,8}$**:** The indecomposable solvable Lie algebra with $[e_2, e_3] = e_1$, $[e_1, e_4] = (1 + \beta)e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = \beta e_3$, $-1 \leq \beta \leq 1$.

The Koszul formula gives us the non zero components of the Levi-Civita connection are as follows

$$\nabla_{e_1} e_1 = (1 + \beta)e_4, \quad \nabla_{e_1} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2, \quad \nabla_{e_1} e_4 = (1 + \beta)e_1,$$

$$\nabla_{e_2}e_1 = \frac{1}{2}e_3, \qquad \nabla_{e_2}e_2 = -e_4, \qquad \nabla_{e_2}e_3 = \frac{1}{2}e_1, \qquad \nabla_{e_2}e_4 = e_2,$$

$$\nabla_{e_3}e_1 = -\frac{1}{2}e_2, \qquad \nabla_{e_3}e_2 = -\frac{1}{2}e_1, \qquad \nabla_{e_3}e_3 = -\beta e_4, \qquad \nabla_{e_3}e_4 = \beta e_3,$$

From matrix representation of the Jacobi operators we have

$$trace(J_{e_2}) = -2\beta - \frac{3}{2}, \quad trace(J_{e_4}) = -2(\beta^2+\beta+1).$$

So, G is not 1-Stein, thus from Remark (2.1) we imply that G is not space-like Osserman.

Now we take non degenerate 2-planes $\pi = \{e_1, e_2\}$, $\sigma = \{e_1, e_3\}$ of signature $(1,1)$, there are two states, if $\beta \neq 0$ then $spec(R_\pi) = \{\pm(\beta + \frac{5}{4})\}$ and $spec(R_\sigma) = \{0, \pm\sqrt{\frac{1}{4}+\beta+\beta^2}\}$, so G is not mixed IP. Moreover, if $\beta = 0$ then $spec(R_\pi) = \{0, \pm\frac{5}{4}\}$ and $spec(R_\sigma) = \{0, \pm\frac{1}{2}\}$, thus in this case G is not mixed IP. Furthermore, $J_{e_1}(J_{e_1}(e_3)) = (\beta^2 + \beta + \frac{1}{4})^2 e_3$, thus G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then $\kappa_\pi = (\beta + \frac{5}{4})$ and $\kappa_\sigma = (1+\beta)^2$, therefore G is not a space form. We have also

$$Ric(e_1,e_1) = 2\beta^2 + 4\beta + \frac{5}{2}, \quad Ric(e_2,e_2) = Ric(e_3,e_3) = 1, \quad Ric(e_4,e_4) = 2\beta.$$

If G is an Einstein space then there exist a real number λ such that $Ric(e_i, e_j) = \lambda g(e_i, e_j)$ for all i, j, especially $i = j = 1$ and $i = j = 2$ and according to the metric we get $2\beta^2 + 4\beta + \frac{5}{2} = -1$, But this polynomial has not any root, so G is not an Einstein space.

$\mathfrak{g}_{4,9}$: The indecomposable solvable Lie algebra with $[e_2, e_3] = e_1$, $[e_1, e_4] = 2\alpha e_1$, $[e_2, e_4] = \alpha e_2 - e_3$, $[e_3, e_4] = e_2 + \alpha e_3$, $\alpha \geq 0$.
The non zero components of the Levi-Civita connection are as follows

$$\nabla_{e_1}e_1 = 2\alpha e_4, \qquad \nabla_{e_1}e_2 = \frac{1}{2}e_3, \qquad \nabla_{e_1}e_3 = -\frac{1}{2}e_2, \qquad \nabla_{e_1}e_4 = 2\alpha e_1,$$

$$\nabla_{e_2}e_1 = \frac{1}{2}e_3, \qquad \nabla_{e_2}e_2 = -\alpha e_4, \qquad \nabla_{e_2}e_3 = \frac{1}{2}e_1, \qquad \nabla_{e_2}e_4 = \alpha e_2,$$

$$\nabla_{e_3}e_1 = -\frac{1}{2}e_2, \qquad \nabla_{e_3}e_2 = -\frac{1}{2}e_1, \qquad \nabla_{e_3}e_3 = -\alpha e_4, \qquad \nabla_{e_3}e_4 = \alpha e_3,$$

$$\nabla_{e_4}e_2 = e_3, \qquad \nabla_{e_4}e_3 = -e_2.$$

From matrix representation of the Jacobi operators we have $trace(J_{e_2}) = -4\alpha^2 + \frac{1}{2}$ and $trace(J_{e_4}) = -6\alpha^2$, so G is not 1-Stein, thus from Remark (2.1) we imply that G is not space-like Osserman.
Now we consider non degenerate 2-planes $\pi = \{e_1, e_2\}$, $\sigma = \{e_1, e_4\}$ of signature $(1,1)$. There are two states, if $\alpha \neq 0$ then direct computations give us

$spec(R_\pi) = \{0, \pm(2\alpha^2 + \frac{1}{4})\}$ and $spec(R_\sigma) = \{\pm 2\alpha\}$, thus in this case G is not mixed IP. Moreover, if $\alpha = 0$ then $spec(R_\pi) = \{0, \pm\frac{1}{4}\}$ and $spec(R_\sigma) = \{0\}$, so in this case G is not mixed IP. Moreover, $J_{e_1}(J_{e_1}(e_3)) = (2\alpha^2 + \frac{1}{4})^2 e_3$, thus G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_2, e_4\}$, then $\kappa_\pi = (2\alpha^2 + \frac{1}{4}) > 0$ and $\kappa_\sigma = -\alpha^2 < 0$, therefore G is not a space form. Furthermore, if G is an Einstein space then there exist a real number λ such that $Ric(e_i, e_j) = \lambda g(e_i, e_j)$ for all i, j, then for $i = j = 1$ and $i = j = 4$ and according to the metric we obtain $-(8\alpha^2 + \frac{1}{2}) = 2\alpha^2$, so we get $10\alpha^2 + \frac{1}{2} = 0$, that is a contradiction, thus G is not a Einstein space.
$\mathfrak{g}_{4,10}$**:** The indecomposable solvable Lie algebra with $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$, $[e_2, e_4] = e_1$.
The non zero components of the Levi-Civita connection are as follows

$$\nabla_{e_1} e_1 = e_3, \qquad \nabla_{e_1} e_2 = e_4, \qquad \nabla_{e_1} e_3 = e_1, \qquad \nabla_{e_1} e_4 = -e_2,$$

$$\nabla_{e_2} e_1 = e_4, \qquad \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_3 = e_2, \qquad \nabla_{e_2} e_4 = e_1.$$

From matrix representation of the Jacobi operators we have

$$spec(J_{e_2}) = \{0, \pm 1, -2\}, \quad spec(J_{e_3}) = \{0, -1\}.$$

so in this case G is not space-like Osserman. We have also

$$trace(J_{e_2}) = -2, \quad trace(J_{e_4}) = 2.$$

Therefore G is not 1-Stein.
If we take non degenerate 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$ of signature $(1, 1)$, then we have $spec(R_\pi) = \{0, \pm 2\}$ and $spec(R_\sigma) = \{\pm 1\}$, so in this case G is not mixed IP. Moreover, $J_{e_1}(J_{e_1}(e_2)) = 4e_2$, thus G is not 2-step Jacobi nilpotent.
If we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_3\}$, then $\kappa_\pi = 2$ and $\kappa_\sigma = 1$, thus G is not a space form. Furthermore, we have $Ric(e_1, e_1) = 2$ and $Ric(e_3, e_3) = 0$, so G is not an Einstein space.
Also according to [4] we have the following

Remark 3.1. *The minimal dimension for Jacobi-Tsankov manifold (M, g) which is not 2-step Jacobi nilpotent is 14.*

Also according to the Remark (2.1) and Remark (3.1), we get the main result of this paper

Theorem 3.2. *If G is a non Abelian 4-dimensional Lie group with an orthonormal left invariant Lorentzian metric, where e_1 is a time-like vector, then the following assertions hold:*

(1) . *There is not any Lie algebra structure that can provide space-like (time-like) Osserman, also G does not admit any 4-Stein space.*

(2) . *The only Lie algebra structures that can provide* 1*-Stein property are* $2\mathfrak{g}_{2,1}$*,* $\mathfrak{g}_{4,5}$ *whenever* $\alpha = \beta = \gamma$*, and* $\mathfrak{g}_{4,6}$ *whenever* $\alpha = \beta$*.*

(3) . *G does not admit any mixed IP structure, except* $\mathfrak{g}_{4,6}$*, (whenever* $\alpha = \beta$*) and* $\mathfrak{g}_{4,5}$*,(whenever* $\alpha = \beta = \gamma$*).*

(4) . *There is not any Lie algebra structure that can provide* 2*-step Jacobi nilpotent. Furthermore, G does not admit any Jacobi-Tsankov structure.*

(5) . *G is not a space form. Furthermore, G is not an Einstein space.*

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