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On projectively related (α, β)-metrics

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Abstract. In this paper, we find necessary and sufficient conditions under which the infinite series metric and Randers metric on a manifold M of dimension $n \geq 3$ be projectively related.

Keywords: Projective change, (α, β)-metric, Douglas metric, **H**-curvature.

1. **Introduction**

For a Finsler metric $F = F(x, y)$, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}, \quad y \in T_x M.$$

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. Hereby, there is a scalar function $P(x, y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + P y^i,$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively. The problem of projectively related Finsler metrics is quite old in geometry and its origin is formulated in Hilberts Fourth Problem: determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's Fourth Problem. A Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called

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projectively flat if all geodesics are straight in U. In this case, F and the Euclidean metric on U are projectively related [24]. The study of projectively related Finsler metrics was initiated by Berwald and his studies mainly concern the 2-dimensional Finsler spaces [4]. Further substantial contributions on this topic are from Rapcsák [20], Szabó [25] and Bácsó-Matsumoto [2][3][14]. The problem of projectively related Finsler metrics is strongly connected to projectively related sprays, as Shen pointed out in [21].

An (α, β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Randers metrics $F = \alpha + \beta$ are the simplest (α, β)-metrics which were first introduced by physicist Randers from the standpoint of general relativity [19].

In [23], Shen-Yu studied projectively related Randers metrics. They show that two Randers metrics are pointwise projectively related if and only if they have the same Douglas tensors and the corresponding Riemannian metrics are projectively related. In [8], Cui-Shen find necessary and sufficient conditions under which a Berwald metric and a Randers metric are projectively related. Then Zohrehvand-Rezaii found necessary and sufficient conditions under which a Matsumoto metric and a Randers metric are projectively related [28]. Recently, Chen-Cheng and Yu-You independently consider other Finsler metrics projectively related to a Randers metric [5][27].

Let us consider the r-th series (α, β)-metric

$$F = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k,$$

where we assume $\alpha < \beta$. If $r = 1$, then we get the Randers metric $F = \alpha + \beta$. If $r = 2$, then we have $F = \alpha + \beta + \frac{\alpha^2}{\beta}$, which is one of the parallel Berwald metric in the sense of Matsumoto [15]. If $r = \infty$, then we get infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$. We have not at all investigated the geometrical meaning about the infinite series metric by this time. But this metric is remarkable as the difference between a Randers metric $F = \alpha + \beta$ and a Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$.

In this paper, first we prove the following.

Theorem 1.1. *Let $F = \frac{\beta^2}{\beta - \alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two (α, β)-metrics on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms with $b^2 \neq 10$. Then they have the same Douglas tensor if and only if F and $\bar{F}$ are Douglas metrics.*

Then, we find equations to characterize projective change between two special classes of (α, β)-metrics, infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ and Randers metric

$\bar{F} = \bar{\alpha} + \bar{\beta}$ on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms. More precisely, we prove the following.

Theorem 1.2. *Let $F = \frac{\beta^2}{\beta-\alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two (α,β)-metrics on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms with $b^2 \neq 10$. Then F is projectively related to $\bar{F}$ if and only if the following equations hold*

: *(a) α is projectively related to $\bar{\alpha}$;*
: *(b) β is parallel with respect to α;*
: *(c) $\bar{\beta}$ is closed.*

In the sequel, we use quantities with a *bar* to denote the corresponding quantities of the metric $\bar{F}$.

For a vector $y \in T_xM_0$, define the E-curvature by $\mathbf{E_y} = E_{ij}dx^i \otimes dx^j|_p : T_pM \otimes T_pM \to \mathbb{R}$, where $E_{ij}(x,y) := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left[\frac{\partial G^m}{\partial y^m}\right]$. We call $\mathbf{E}_y$ the mean Berwald curvature. The Finsler metrics with vanishing E-curvature are called weakly Berwald metrics. In this paper, we consider a Randers metric which is projectively related with a weakly Berwald infinite series metric and prove the following.

Corollary 1.3. *Let $F = \frac{\beta^2}{\beta-\alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two (α,β)-metrics on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms. Suppose that F is a weakly Berwald metric. Then F is projectively related to $\bar{F}$ if and only if the following equations hold*

: *(a) α is projectively related to $\bar{\alpha}$;*
: *(b) β is closed;*
: *(c) $\bar{\beta}$ is closed.*

In [1], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. More precisely, $H_{ij} := E_{ij|m}y^m$, where "$|$" denotes the horizontal covariant differentiation with respect to the Berwald connection. In the class of Finsler metrics of scalar flag curvature, vanishing this quantity results that the Finsler metric is of constant flag curvature and this fact clarifies its geometric meaning [16][18][26]. In continue, we consider a infinite series metric $F = \frac{\beta^2}{\beta-\alpha}$ which are projectively related with a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ satisfies $\bar{\mathbf{H}} = 0$ and prove the following.

Corollary 1.4. *Let $F = \frac{\beta^2}{\beta-\alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be projectively related (α,β)-metrics on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms with $b^2 \neq 10$. Suppose that $\bar{F}$ satisfies $\bar{\mathbf{H}} = 0$. Then F and $\bar{F}$ reduce to Berwald metrics.*

Any geometric object which is identical between two projectively related metrics is called a projective invariant. There are some well-known projective invariants of Finsler metrics namely, Douglas curvature [3] and generalized Douglas-Weyl curvature [17]. Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}|_x$, where

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big[G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big].$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric. It is remarkable that, the notion of Douglas metrics was proposed by B*á*cs*ó*-Matsumoto as a generalization of Berwald metrics [3]. Finally, we consider the infinite series metrics and Randers metrics with vanishing Douglas curvature and prove the following.

Corollary 1.5. *Let $F = \frac{\beta^2}{\beta - \alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two (α, β)-metrics on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms with $b^2 \neq 10$. Then F is projectively related to $\bar{F}$ if and only if they are Douglas metrics and α is projectively related to $\bar{\alpha}$.*

2. **Preliminary**

General (α, β)-metrics were first studied by Matsumoto in 1972 as a direct generalization of Randers metrics [13]. An (α, β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For an (α, β)-metric, let us define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma_{ik}^j dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. An (α, β)-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Put

$$\begin{aligned} &r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^i y^j, \quad r_j := b^i r_{ij}, \\ &s_{i0} := s_{ij}y^j, \quad s_j := b^i s_{ij}, \\ &r_0 := r_j y^j, \quad s_0 := s_j y^j. \end{aligned}$$

For an (α, β)-metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, if we put

$$Q := \frac{\phi'}{\phi - s\phi'}$$

then

$$Q' = \frac{\phi\phi''}{(\phi - s\phi')^2},$$
$$Q'' = \frac{\phi'\phi'' + \phi\phi'''}{(\phi - s\phi')^2} + \frac{2s\phi\phi''^2}{(\phi - s\phi')^3}.$$

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Delta := 1 + sQ + (b^2 - s^2)Q'. \tag{2.1}$$

Let $G^i = G^i(x, y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x, y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = G^i_\alpha + \alpha Q s^i_0 + (-2Q\alpha s_0 + r_{00})(\Theta\frac{y^i}{\alpha} + \Psi b^i), \tag{2.2}$$

where

$$\Theta := \frac{Q - sQ'}{2\Delta} = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]}$$
$$\Psi := \frac{Q'}{2\Delta} = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

By (2.2), it follows that every trivial (α, β)-metric satisfies $G^i = G^i_\alpha$ and then it reduces to a Berwald metric.

3. **Proof of Theorem 1.1**

For an infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$, the following are hold

$$\phi = \frac{s^2}{s-1}, \quad \phi - s\phi' = \frac{s^2}{(s-1)^2}.$$

Then

$$Q = \frac{s-2}{s},$$
$$\Theta = \frac{s(s-4)}{2[s^2(s-1) + 2(b^2 - s^2)]}$$
$$\Psi = \frac{1}{s^2(s-1) + 2(b^2 - s^2)}. \tag{3.1}$$

For a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, we have

$$\bar{Q} := 1,$$
$$\bar{\Theta} := \frac{1}{2(1+s)},$$
$$\bar{\Psi} := 0. \tag{3.2}$$

Let

$$D_{j\ kl}^{i} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\big). \tag{3.3}$$

It is easy to verify that $\mathcal{D} := D_{j\ kl}^{i} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call $\mathcal{D}$ the Douglas tensor. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics.

For an (α, β)-metric , the Douglas tensor is determined by

$$D_{j\ kl}^{i} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\big), \tag{3.4}$$

where

$$T^i := \alpha Q s_0^i + \Psi(r_{00} - 2\alpha Q s_0) b^i, \tag{3.5}$$

and

$$T_{y^m}^m = Q' s_0 + \Psi' \alpha^{-1}(b^2 - s^2)(r_{00} - 2\alpha Q s_0) + 2\Psi\big[r_0 - Q'(b^2 - s^2)s_0 - Qss_0\big]. \tag{3.6}$$

Now, let F and $\bar{F}$ be two (α, β)-metrics which have the same Douglas tensor, i.e., $D_{jkl}^i = \bar{D}_{jkl}^i$. From (3.3) and (3.4), we have

$$\frac{\partial^3}{\partial y^i \partial y^j \partial y^k}\Big[T^i - \bar{T}^i - \frac{1}{n+1}(T_{y^m}^m - \bar{T}_{y^m}^m) y^i\Big] = 0. \tag{3.7}$$

Then there exists a class of scalar function $H_{jk}^i := H_{jk}^i(x)$ such that

$$T^i - \bar{T}^i - \frac{1}{n+1}(T_{y^m}^m - \bar{T}_{y^m}^m) y^i = H_{00}^i, \tag{3.8}$$

where $H_{00}^i = H_{jk}^i(x) y^i y^j$, T^i and $T_{y^m}^m$ are given by (3.5) and (3.6) respectively. In this paper, we assume that $\lambda := \frac{1}{n+1}$.

Proof of Theorem 1.1: Since the sufficiency is obvious, then we just need to prove the necessity. If F and $\bar{F}$ have the same Douglas tensor, then (3.8) holds. Plugging (3.1) and (3.2) into (3.8) implies that

$$\frac{\sum_{j=1}^{8} A_j^i \alpha^j}{\sum_{j=0}^{6} B_j \alpha^j} - \bar{\alpha}\bar{s}_0^i = H_{00}^i, \tag{3.9}$$

where

$$A_1^i := \beta^7 s_0^i - 3\lambda\beta^5 r_{00} y^i$$
$$A_2^i := 6\lambda\beta^5 s_0 y^i - 8\beta^6 s_0^i + 6\lambda\beta^4 r_{00} y^i$$
$$A_3^i := 21\beta^5 s_0^i + \beta^4 r_{00} b^i + 10\lambda\beta^4 s_0 y^i + 3\lambda b^2\beta^3 r_{00} y^i - 2\lambda\beta^4 r_0 y^i$$

$$A_4^i := 6\lambda\beta^2(5\beta s_0 - b^2\beta s_0 + \beta r_0 - b^2 r_{00}) y^i - 2(9 - 2b^2)\beta^4 s_0^i - \beta^3(2\beta s_0 + 3r_{00}) b^i$$

$$\begin{aligned}
A^i_5 &:= 24\lambda b^2\beta^2 s_0 y^i - 20b^2\beta^2 s^i_0 + 10\beta^3 s_0 b^i\\
A^i_6 &:= -28\lambda b^2\beta s_0 y^i - 4\lambda b^2\beta r_0 y^i + 24b^2\beta^2 s^i_0 - 12\beta^2 s_0 b^i + 2b^2\beta r_{00} b^i\\
A^i_7 &:= 4b^2\beta(b^2 s^i_0 - s_0 b^i)\\
A^i_8 &:= -8b^2(b^2 s^i_0 - s_0 b^i) \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
B_0 &:= \beta^7\\
B_1 &:= -6\beta^6\\
B_2 &:= 9\beta^5\\
B_3 &:= 4b^2\beta^4\\
B_4 &:= -12b^2\beta^3\\
B_5 &:= 0\\
B_6 &:= 4b^4\beta. \qquad (3.11)
\end{aligned}$$

Then (3.9) is equal to following

$$\sum_{j=1}^{8} A^i_j\alpha^j = (H^i_{00} + \bar{\alpha}\bar{s}^i_0)\sum_{j=0}^{6} B_j\alpha^j. \qquad (3.12)$$

Replacing y^i by $-y^i$ in (3.12) yields

$$\sum_{j=0}^{3} A^i_{(2j+1)}\alpha^{2j+1} - \sum_{j=1}^{4} A^i_{(2j)}\alpha^{2j} = (H^i_{00} - \bar{\alpha}\bar{s}^i_0)\sum_{j=0}^{1} B_{(2j+1)}\alpha^{2j+1}$$
$$-(H^i_{00} - \bar{\alpha}\bar{s}^i_0)\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}. \qquad (3.13)$$

$[(3.12) + (3.13)] \times \alpha$ implies that

$$\sum_{k=1}^{4} A^i_{(2k-1)}\alpha^{2k} = H^i_{00}(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2j}) + \alpha\bar{\alpha}\bar{s}^i_0(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}). \qquad (3.14)$$

Then

$$\sum_{k=1}^{4} A^i_{(2k)}\alpha^{2k} = H^i_{00}(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}) + \alpha\bar{\alpha}\bar{s}^i_0(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2(j-1)}). \qquad (3.15)$$

We split the proof into two cases:

case(1): If $\bar{\alpha} \neq \mu(x)\alpha$, then from (3.15) we see that $\alpha\bar{\alpha}\bar{s}^i_0$ is a homogeneous polynomial with respect to y. Therefore $\bar{s}^i_0 = 0$, i.e., $\bar{\beta}$ is closed.

case(2): If $\bar{\alpha} = \mu(x)\alpha$, then (3.14) and (3.15) reduce to following

$$\sum_{k=1}^{4} A^i_{(2k-1)}\alpha^{2k} = H^i_{00}(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2j}) + \mu\alpha^2\bar{s}^i_0(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}) \tag{3.16}$$

and

$$\sum_{k=1}^{4} A^i_{(2k)}\alpha^{2k} = H^i_{00}(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}) + \mu\alpha^2\bar{s}^i_0(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2(j-1)}). \tag{3.17}$$

By (3.17), it follows that $B_0 H^i_{00} = \beta^7 H^i_{00}$ has the factor α^2. Thus H^i_{00} has the factor α^2, and we can conclude that for each i there exists a scalar function $\sigma^i(x)$ on M such that

$$H^i_{00} = \sigma^i(x)\alpha^2.$$

Thus (3.16) and (3.17) reduce to following

$$\sum_{k=1}^{4} A^i_{(2k-1)}\alpha^{2k} = \sigma^i(x)\alpha^2(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2j}) + \mu\alpha^2\bar{s}^i_0(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}) \tag{3.18}$$

and

$$\sum_{k=1}^{4} y_i A^i_{(2k)}\alpha^{2k} = y_i\sigma^i(x)\alpha^2(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}) + \mu\alpha^2\bar{s}^i_0(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2(j-1)}). \tag{3.19}$$

By multiplying (3.18) and (3.19) with y_i, we have

$$\sum_{k=1}^{4} y_i A^i_{(2k-1)}\alpha^{2k} = y_i\sigma^i(x)\alpha^2(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2j}) \tag{3.20}$$

and

$$\sum_{k=1}^{4} y_i A^i_{(2k)}\alpha^{2k} = y_i\sigma^i(x)\alpha^2(\sum_{j=0}^{3} B_{(2j)}\alpha^{2j}). \tag{3.21}$$

By (3.21), it follows that $-y_i A^i_2 + (y_i\sigma^i(x))B_0 = -6\lambda\beta^4\alpha^2(\beta s_0 + r_{00}) + \beta^7(y_i\sigma^i(x))$ has the factor α. Thus we get

$$y_i\sigma^i(x) = 0.$$

Then by (3.20) and (3.21), we have

$$\sum_{k=1}^{4} y_i A^i_{(2k-1)}\alpha^{2k} = 0 \tag{3.22}$$

and

$$\sum_{k=1}^{4} y_i A^i_{(2k)}\alpha^{2k} = 0. \tag{3.23}$$

By (3.22), we can see $y_i A_1^i + y_i A_3^i \alpha^2$ has the factor α^4. Thus $(-3\lambda+1)\beta^5 r_{00}$ has the factor α^2. Since $n \geq 3$, then $-3\lambda + 1 \neq 0$ and r_{00} must has the factor α^2. Then we have

$$r_{00} = \eta(x)\alpha^2.$$

From (3.23), we deduce that $y_i A_2^i + y_i A_4^i \alpha^2$ has the factor α^4. Thus $(3\lambda-1)\beta^5 s_0$ has the factor α^2. Since $n \geq 3$ so $3\lambda - 1 \neq 0$ and

$$s_0 = 0.$$

By (3.19), it follows that $A_8^i \alpha^8 = -8b^4 s_0^i$ has the factor β, i.e., for each i there exists a scalar function $\xi^i(x)$ such that

$$-8b^4 s_0^i = \xi^i(x)\beta. \tag{3.24}$$

Multiplying (3.24) with a_{ik} and differentiating with respect to y^j implies that

$$-8b^4 s_{kj} = \xi_k(x) b_j, \tag{3.25}$$

where $\xi_k := \xi^i a_{ik}$. Contracting (3.25) with $y^k y^j$ yields

$$(\xi_k(x) y^k)\beta = 0. \tag{3.26}$$

Thus $\xi_k(x) y^k = 0$, and so

$$\xi_j = 0.$$

This implies that

$$s_{kj} = 0.$$

Since $A_1^i = -3\lambda\beta^5\eta\alpha^2 y^i$, then (3.18) reduces to following

$$\begin{aligned}\Big[-3\lambda\beta^5\eta y^i + \sum_{k=2}^{4} A^i_{(2k-1)}\alpha^{2(k-2)} - \sigma^i(x)\Big(\sum_{j=1}^{2} B_{(2j-1)}\alpha^{2(j-1)}\Big) \\ - \mu\bar{s}_0^i\Big(\sum_{j=1}^{3} B_{(2j)}\alpha^{2(j-1)}\Big)\Big]\alpha^2 = \mu\bar{s}_0^i\beta^7\end{aligned} \tag{3.27}$$

By (3.27), it follows that

$$\bar{s}_0^i = 0.$$

Thus in any case, $\bar{\beta}$ is closed. It is well known that the Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed. Since both F and $\bar{F}$ are Douglas metric, then we get the proof. □

4. **Proof of Theorem 1.2**

In this section, we are going to prove the Theorem 1.2. Then, we prove the Corollaries 1.3, 1.4 and 1.5.

Proof of Theorem 1.2: First we prove the necessity. Since Douglas tensor is an invariant under projective change between two Finsler metrics. If F is projectively related to $\bar{F}$, then they have the same Douglas tensor. By Theorem 1.1, we obtain that both F and $\bar{F}$ are Douglas metrics. It is well known that infinite series metric $F = \frac{\beta^2}{\beta-\alpha}$ with $b^2 \neq 10$ is a Douglas metric if and only if β is parallel with respect to α. Then

$$b_{i|j} = 0. \tag{4.1}$$

Plugging (4.1) into (2.2) and considering (3.1) implies that

$$G^i = G^i_\alpha. \tag{4.2}$$

On the other hand, it is proved that the Randers metric $\bar{F} = \bar{\alpha}+\bar{\beta}$ is a Douglas metric if and only if

$$\bar{s}_{ij} = 0. \tag{4.3}$$

By putting (4.3) in (2.2) with (3.2), we get

$$\bar{G}^i = G^i_{\bar{\alpha}} + \frac{\bar{r}_{00}}{2(\bar{\alpha}+\bar{\beta})} y^i. \tag{4.4}$$

Since F is projectively related to $\bar{F}$, then

$$G^i = \bar{G}^i + Py^i, \tag{4.5}$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively and $P = P(x, y)$ is positively homogeneous scalar function on TM_0 of degree one with respect to y. By (4.2), (4.4) and (4.5), it follows that

$$G^i_\alpha - G^i_{\bar{\alpha}} = \Big[\frac{\bar{r}_{00}}{2(\bar{\alpha}+\bar{\beta})} + P\Big] y^i. \tag{4.6}$$

The left-hand side of the above equation is a quadratic form, then there exists a one form $\theta = \theta_i(x)y^i$ on M such that,

$$\frac{\bar{r}_{00}}{2(\bar{\alpha}+\bar{\beta})} + P = \theta. \tag{4.7}$$

Then by (4.6) and (4.7), we have

$$G^i_\alpha = G^i_{\bar{\alpha}} + \theta y^i. \tag{4.8}$$

This means that α is projectively related to $\bar{\alpha}$. Thus by (4.1), (4.3) and (4.8), we get the proof of necessity. Now, we prove the sufficiency. Since $\bar{\beta}$ is closed, then it suffice to prove that F is projectively related to $\bar{\alpha}$. Plugging (4.1) into (2.2) with (3.1) yields (4.2). Plugging (4.3) into (2.2) with (3.2) yields (4.4). From (4.3), (4.4) and (4.8) we have $G^i = G^i_{\bar{\alpha}} + \theta y^i$. Then F is projectively related to $\bar{F}$. □

Proof of Corollary 1.3: By Theorem 1.2, the necessity is obvious. For the sufficiency, from Theorem 1.2, we know that β is a closed 1-form, i.e., $s_{ij} = 0$. In [11], Lee-Lee proved that the infinite series metric $F = \frac{\beta^2}{\beta-\alpha}$ is a weakly Berwald metric if and only if $r_{ij} = s_j = 0$. Thus $b_{i|j} = 0$, i.e., β is parallel with respect to α. $\square$

Remark 4.1. *Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form $dV_{BH} = \sigma(x)dx$ is defined by*

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\Big(c(t), \dot{c}(t)\Big)\Big]|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. The S-curvature is a scalar function on TM, which was introduced by Shen to study volume comparison in Riemann-Finsler geometry [22]*. Thus, it follows that the S-curvature $\mathbf{S}(y)$ measures the rate of change of the distortion on (T_xM, F_x) in the direction $y \in T_xM$.*

In [6]*, Cheng-Shen a non-Riemannian (α, β)-metric F of non-Randers type $\phi \neq k_1\sqrt{1 + k_2s^2} + k_3s$ is of isotropic S-curvature, $\mathbf{S} = (n + 1)cF$, where $c = c(x)$ is a scalar function on M, if and only if one of the following holds*
(a) β satisfies

$$r_{ij} = \varepsilon\{b^2a_{ij} - b_ib_j\}, \quad s_j = 0, \tag{4.9}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n + 1)k\frac{\phi\Delta^2}{b^2 - s^2} \tag{4.10}$$

where k is a constant. In this case, $\mathbf{S} = (n + 1)cF$ with $c = k\varepsilon$.
(b) β satisfies

$$r_{ij} = 0, \quad s_j = 0. \tag{4.11}$$

In this case, $\mathbf{S} = 0$, regardless of choices of a particular ϕ (see Theorem 1.2).

By a direct computation, we can obtain a formula for mean Cartan torsion of an (α, β)-metric as follows

$$I_i = -\frac{\Phi}{2\Delta\phi\alpha^2}(\phi - s\phi')(\alpha b_i - sy_i).$$

Then the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β)-metrics. Hence, we suppose that $\Phi \neq 0$.

In the proof of Corollary 1.3, we remark that an infinite series metric $F = \frac{\beta^2}{\beta-\alpha}$ is weakly Berwaldian if and only if $r_{ij} = s_j = 0$ [11]*. Then, we conclude the following.*

Corollary 4.2. *For an infinite series metric,* $\mathbf{S} = 0$ *if and only if* $\mathbf{E} = 0$.

The authors guess that if they replace the condition $\mathbf{E} = 0$ *with the condition of isotropic E-curvature* $\mathbf{E} = \frac{n+1}{2} cF^{-1}\mathbf{h}$, *where* $c = c(x)$ *is a scalar function on* M *and* $\mathbf{h}$ *is the angular metric, then the Corollary 1.3 is hold.*

Proof of Corollary 1.4: In [7], Cheng-Shen proved that a Randers metric $F = \bar{\alpha} + \bar{\beta}$ has isotropic S-curvature, $\bar{\mathbf{S}} = (n+1)c\bar{F}$, if and only if

$$\bar{e}_{00} = 2c(\bar{\alpha}^2 - \bar{\beta}^2),$$

where $\bar{e}_{ij} := \bar{r}_{ij} + \bar{b}_i \bar{s}_j + \bar{b}_j \bar{s}_i$, $\bar{e}_{00} = \bar{e}_{ij} y^i y^j$ and $c = c(x)$ is a scalar function on M. In [26], Tang proved that for a Randers metric, $\bar{\mathbf{H}} = 0$ if and only if $\bar{\mathbf{S}} = 0$. Thus $\bar{\mathbf{H}} = 0$ implies that

$$\bar{e}_{ij} = 0.$$

By assumption, $\bar{\beta}$ is closed and then

$$\bar{s}_{ij} = 0.$$

This implies that

$$\bar{r}_{ij} = 0$$

and then $\bar{\beta}$ is parallel. In [10], Hashiguchi-Ichijyō showed that for a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, if $\bar{\beta}$ is parallel then F is Berwaldian. On the other hand, by assumption we have $G^i = \bar{G}^i + Py^i$, where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively. Then

$$G^i{}_{jkl} = P_{jkl} y^i + P_{jk} \delta^i{}_l + P_{jl} \delta^i{}_k + P_{kl} \delta^i{}_j. \tag{4.12}$$

Taking a trace of (4.12) implies that

$$P_{jk} = \frac{2}{n+1} E_{jk}. \tag{4.13}$$

Plugging (4.13) in (4.12) yields

$$B^i{}_{jkl} = \frac{2}{n+1} \{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l} y^i\}.$$

This means that F is a Douglas metric. On the other hand, Lee-Park proved that every Douglas infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ with $b^2 \neq 10$ on a manifold M of dimension $n > 2$ is Berwaldian [12]. Then by assumptions, we can conclude that the infinite series metric is a Berwald metric. This completes the proof. □

Proof of Corollary 1.5: As we explain in the Corollary 1.4, it is proved that the infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ is a Douglas metric if and only if β is parallel with respect to α, i.e., $b_{i|j} = 0$. In [9], Hashiguchi-Ichijyō showed that a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed. Then by Theorem 1.2, we get the proof. □

By Theorem 1.2, we can conclude the following.

Corollary 4.3. *Let $F = \frac{\beta^2}{\beta-\alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two (α, β)-metrics on a manifold M of dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms with $b^2 \neq 10$. Then F is projectively related to $\bar{F}$ if and only if F is Berwald metric and $\bar{F}$ is Douglas metric and the corresponding Riemannian metrics α and $\bar{\alpha}$ are projectively related.*

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