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On square-type Finsler metrics of vanishing flag curvature

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Abstract. In this paper, we construct a family of Finsler metrics, called square-type Finsler metrics. We obtain the flag curvature of this metric. Then we find a necessary and sufficient condition under which the flag curvature of square-type Finsler metrics becomes zero.

Keywords: Square metric, Numata Finsler structure, flag curvature.

1. **Introduction**

In [7], Numata has shown that the Finsler metrics $F(x, y) = \alpha + \beta$ are of scalar curvature, where $\alpha := \sqrt{a_{ij}(y)y^i y^j}$ is a positive definite locally Minkowski norm and $\beta := b_i(x)y^i$ is a closed 1-form. Consider the case that $a_{ij} = \delta_{ij}$ and $b = df$, where f is a smooth function on $\mathbb{R}^n$. If necessary, scale f so that the open set

$$M := \{x \in \mathbb{R}^n | \sqrt{\delta^{ij} f_{x^i} f_{x^j}} < 1\}$$

is nonempty. Then a straightforward calculation reveals that F is of scalar curvature on M [1][2].

In [6], Dual considered the Numata-type Finsler metrics and showed that the flag curvature of the Numata-type metrics admit a non-trivial prolongation to the one-dimensional case, which has a relation with the Schwarzian derivative of the diffeomorphisms associated with these metrics. This motivates us to construct a new class of Finsler metrics with the same properties.

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Let (M,F) be a Finsler manifold. In [4], Bouarroudj construct a 1-cocycle on $Diff(M)$ with values in the space of differential operators acting on sections of some bundles, by means of the Finsler function F. As an operator, it has several expressions in terms of the Chern or Berwald connection, although its cohomology class does not depend on them. He showed that this cocycle is closely related to the conformal Schwarzian derivatives introduced by him in [5].

In 1929, Berwald construct an interesting family of projectively flat Finsler metrics on the unit ball $\mathbb{B}^n$ which as follows

$$F=\frac{\Big(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\Big)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}. \tag{1.1}$$

He showed that this class of metrics has constant flag curvature [3]. Berwald's metric can be expressed as

$$F=\frac{(\alpha+\beta)^2}{\alpha}, \tag{1.2}$$

where

$$\alpha=\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{(1-|x|^2)^2},\quad \beta=\frac{\langle x,y\rangle}{(1-|x|^2)^2}.$$

An Finsler metric in the form (1.2) is called a square metric.

2. **Preliminaries**

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x\in M$, and by $TM=\cup_{x\in M}T_xM$ the tangent bundle of M. A *Finsler metric* on M is a function $F:TM\to[0,\infty)$ which has the following properties: (i) F is C^∞ on $TM_0:=TM\setminus\{0\}$; (ii) F is positive-homogeneous of degree 1; (iii) for each $y\in T_xM$, the quadratic form

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2(y+su+tv)\right]|_{s,t=0},\quad u,v\in T_xM. \tag{2.1}$$

is positive definite.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}.$$

where $G^i(x,y)$ are local functions on TM_0 given by

$$G^i=\frac{1}{4}g^{il}\Big\{\frac{\partial g_{jl}}{\partial x^k}+\frac{\partial g_{lk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^l}\Big\}y^jy^k.$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic in M.

The second variation of geodesics gives rise to a family of linear maps $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i{}_k(y) u^k \frac{\partial}{\partial x^i}$, where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{2.2}$$

$\mathbf{R}_y$ is called the Riemann curvature in the direction y.

A flag curvature is a geometrical invariant that generalizes the sectional curvature of Riemannian geometry. Let $x \in M$, $0 \neq y \in T_xM$ and $V := V^i \frac{\partial}{\partial x^i}$. Flag curvature is obtained by carrying out the following computation at the point $(x, y) \in TM_0$, and viewing y, V as section of π^*TM:

$$\mathbf{K}(y, V) := \frac{V^i (y^j R_{jikl} y^l) V^k}{\mathbf{g}(y, y)\mathbf{g}(V, V) - [\mathbf{g}(y, V)]^2}, \tag{2.3}$$

where $\mathbf{g}$ is a Riemannian metric on π^*TM. If $\mathbf{K}$ is independent of the transverse edge V , we say that our Finsler space has scalar flag curvature. Denote this scalar by $\lambda = \lambda(x, y)$. When $\lambda(x, y)$ has no dependence on either x or y, then Finsler manifold is said to be of constant flag curvature.

3. **Numata Finsler structures**

Numata [7] has introduced that metrics of the form $F(x, y) = \sqrt{q_{ij}(y) y^i y^j} + f_{x^i} y^i$ on TM where $M \subset R^n$, with $(g_{ij}) > 0$ and $db = 0$. We are going to put this structure on the square metric.
The square is an (α, β)-metric in the following form

$$F = \frac{(\alpha + \beta)^2}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on a manifold M.

An important class of Finsler metrics can be expressed in the form $F = (\alpha + \beta)^2/\alpha$, where $\alpha := \sqrt{q_{ij}(y) y^i y^j}$ is a locally Minkowski norm and $\beta := b_i(x) y^i$ is a 1-form on M. This class of Finsler metrics is called square-type metric [7].

Let us consider the special case of square-type Finsler metric such that $q_{ij} = \delta_{ij}$ and $b = df$ where $f \in C^\infty(M)$. Then

$$F(x, y) = \frac{(\sqrt{q_{ij}(y) y^i y^j} + f_{x^i} y^i)^2}{\sqrt{q_{ij}(y) y^i y^j}}, \tag{3.1}$$

where

$$M := \{x \in \mathbb{R}^n | \sqrt{\delta_{ij} f_{x^i} f_{x^j}} < 1\} \tag{3.2}$$

is nonempty set.

Differentiating (3.1) with respect to y^i and y^j yields

$$g_{ij} = \frac{(\alpha-\beta)(\alpha+\beta)^3}{\alpha^4} q_{ij} + \frac{6(\alpha+\beta)^2}{\alpha^2} b_i b_j + \frac{2(\alpha-2\beta)(\alpha+\beta)^2}{\alpha^3}\Big(\alpha_i b_j + \alpha_j b_i\Big) - \frac{2\beta(\alpha-2\beta)(\alpha+\beta)^2}{\alpha^4}\alpha_i \alpha_j.$$

Then we get

$$det(g_{ij}) = \Big(\frac{(\alpha-\beta)(\alpha+\beta)^3}{\alpha^4}\Big)^n \quad \frac{\alpha^2 + 2\alpha^2 b^2 - 3\beta^2}{(\alpha-\beta)^2}\ det(q_{ij}).$$

Since (g_{ij}) ispositive definite, we conclude that

$$\alpha^2 + 2\alpha^2 b^2 - 3\beta^2 > 0. \tag{3.3}$$

Also, we have

$$g^{ij} = \frac{\alpha^4}{(\alpha-\beta)(\alpha+\beta)^3}\Bigg[q^{ij} - \frac{2\alpha^2}{\alpha^2 + 2\alpha^2 b^2 - 3\beta^2} b^i b^j + \frac{2(\alpha-2\beta)(\alpha\beta + 2\alpha^2 b^2 - \beta^2)}{(\alpha+\beta)(\alpha^2 + 2\alpha^2 b^2 - 3\beta^2)}\alpha^i \alpha^j - \frac{2\alpha(\alpha-2\beta)}{\alpha^2 + 2\alpha^2 b^2 - 3\beta^2}\Big(\alpha^i b^j + \alpha^j b^j\Big)\Bigg].$$

By a simple calculations for square metric, we get

$$G^i = \big[p\ y^i + q\ b^i\ \big] f_{00}, \tag{3.4}$$

where $f_{00} = f_{x^i x^j} y^i y^j$ and

$$p := \frac{(\alpha-2\beta)}{\alpha^2 + 2\alpha^2 b^2 - 3\beta^2},$$

$$q := \frac{\alpha^2}{\alpha^2 + 2\alpha^2 b^2 - 3\beta^2}.$$

Now, we are going to compute the flag curvature. By definition, we have

$$F^2 \mathbf{K}(\ell, V) = \frac{V_i (F^2 R^i_k) V^k}{\mathbf{g}(V,V) - [\mathbf{g}(\ell, V)]^2}.$$

Suppose that the transverse edges V are $\mathbf{g}-$orthogonal to the flagpole y. Then $\mathbf{g}(\ell, V) = 0$, and we get

$$F^2 \mathbf{K}(\ell, V) = \frac{V_i (F^2 R^i_k) V^k}{\mathbf{g}(V,V)}. \tag{3.5}$$

From (2.2) and (3.4) and using

$$\mathbf{g}(V,V)=V_iV^i,\qquad \mathbf{g}(V,V)=V_iy^i=0$$

we have

$$\begin{aligned}F^2\mathbf{K}(\ell,V)=&\Big[p^2+2f^jp_{y^j}-p_{x^j}y^j\Big]\\&+\frac{V_iV^k}{\mathbf{g}(V,V)}\Big[2(q_{x^k}-pq_{y^k})f^i+(-q_{y^j}q_{y^k}+2qq_{y^jy^k})f^if^j\\&\qquad -q_{y^jy^k}y^jf^i+2qf^i_k-q_{y^k}f^i_0\Big],\end{aligned}\tag{3.6}$$

By the above information, we prove the following.

Theorem 3.1. *Let (M,F) be the square-type Finsler structure, where F is given by (3.1) and $M\subset S^1$ is defined by (3.2). Then the flag curvature of F is vanishing if and only if f is a linear function or the following condition holds*

$$Af_{0k}V^k+3\beta f_{00}(f_kV^k)=0.$$

Proof. By straightforward calculations, one can obtains

$$\begin{aligned}F^2\mathbf{K}(\ell,V)=&\ \frac{1}{A^3}\Big[3\alpha^4+9\alpha^2\beta^2-12\alpha^3\beta+6\alpha^4b^2+12\alpha^2\beta^2b^2-12\alpha^3\beta b^2\\&-30\beta^4+24\alpha\beta^3\Big]f_{00}^2-\frac{(\alpha-2\beta)}{A}f_{000}+\frac{4\alpha^2(\alpha-2\beta)}{A^2}f_{00}f_{0j}f^j\\&+\frac{V_iV^k}{\mathbf{g}(V,V)}\frac{1}{A^4}\Big[(A^i_k)_2\alpha^2+(A^i_k)_3\alpha^3+(A^i_k)_4\alpha^4+(A^i_k)_5\alpha^5+(A^i_k)_6\alpha^6\\&+(A^i_k)_7\alpha^7+(A^i_k)_8\alpha^8\Big],\end{aligned}$$

where

$$\begin{aligned}&A:=\alpha^2+2\alpha^2b^2-3\beta^2,\\&f_{000}=f_{x^ix^jx^k}y^iy^jy^k,\\&f_{0j}=f_{x^ix^j}y^i,\end{aligned}$$

and

$$\begin{aligned}(A^i_k)_2:=&-18\beta^4\Big[6\beta f^if_{00}f_{0k}-3\beta f^if_{00}f_{k0}-3\beta f^if_{000}f_k+5f^if_{00}^2f_k+3\beta^2f^i_0f_{0k}\\&-3\beta f^i_0f_{00}f_k-3\beta^2f^i_kf_{00}\Big],\\(A^i_k)_3:=&\,36\beta^3f^if_{00}\Big[\beta f_{0k}-f_{00}f_k\Big],\\(A^i_k)_4:=&\,6\beta^2\Big[-6\beta^2f^jf^if_{00}f_{jk}+6\beta^2f^jf^if_{0j}f_{0k}+6\beta f^jf^if_{00}f_{0j}f_k+6b^2\beta f^if_{00}f_{0k}\\&-12b^2f^if_{00}^2f_k+10\beta(1+2b^2)f^if_{00}f_{0k}-6\beta(1+2b^2)f^if_{00}f_{k0}\\&-6\beta(1+2b^2)f^if_{000}f_k+9\beta^2(1+2b^2)f^i_0f_{0k}-6\beta(1+2b^2)f^i_0f_{00}f_k\\&-9\beta^2(1+2b^2)f^i_kf_{00}+16b^2f^if_{00}^2f_k+8f^if_{00}^2f_k)\Big],\end{aligned}$$

$$
\begin{aligned}
(A^i_k)_5 &:= -12\beta(1+2b^2)f^i f_{00}\Big[2\beta f_{0k} - f_{00}f_k\Big],\\
(A^i_k)_6 &:= -2(1+2b^2)\Big[-12\beta^2 f^j f^i f_{00}f_{jk} + 12\beta^2 f^j f^i f_{0j}f_{0k} + 6\beta f^j f^i f_{00}f_{0j}f_k\\
&\quad +6b^2\beta f^i f_{00}f_{0k} + 6b^2 f^i f^2_{00}f_k + 4(1+2b^2)\beta f^i f_{00}f_{0k} - 3\beta(1+2b^2)f^i f_{00}f_{k0}\\
&\quad -3\beta(1+2b^2)f^i f_{000}f_k - 3(1+2b^2)f^i f^2_{00}f_k + 9\beta^2(1+2b^2)f^i_0 f_{0k}\\
&\quad -3\beta(1+2b^2)f^i_0 f_{00}f_k - 9\beta^2(1+2b^2)f^i_k f_{00}\Big],
\end{aligned}
$$

$$
\begin{aligned}
(A^i_k)_7 &:= 4(1+2b^2)^2 f^i f_{00}f_{0k},\\
(A^i_k)_8 &:= 2(1+2b^2)^2\Big[-2f^j f^i f_{00}f_{jk} + 2f^j f^i f_{0j}f_{0k} + (1+2b^2)f^i_0 f_{0k}\\
&\quad -(1+2b^2)f^i_k f_{00}\Big].
\end{aligned}
$$

Now, suppose that the following holds

$$
\begin{aligned}
V_iV^k\Big[(A^i_k)_2\alpha^2 + (A^i_k)_3\alpha^3 + (A^i_k)_4\alpha^4 + (A^i_k)_5\alpha^5 + (A^i_k)_6\alpha^6\\
+(A^i_k)_7\alpha^7 + (A^i_k)_8\alpha^8\Big] = 0. \quad (3.7)
\end{aligned}
$$

By decomposition of the rational and irrational parts in (3.7), we get the following

$$
V_iV^k\Big[(A^i_k)_2 + (A^i_k)_4\alpha^2 + (A^i_k)_6\alpha^4 + (A^i_k)_8\alpha^6\Big] = 0, \qquad (3.8)
$$

$$
V_iV^k\Big[(A^i_k)_3 + (A^i_k)_5\alpha^2 + (A^i_k)_7\alpha^4\Big] = 0. \qquad (3.9)
$$

By (3.9) we get

$$
4A(f^iV_i)f_{00}\big(Af_{0k}V^k + 3\beta f_{00}(f_kV^k)\big) = 0
$$

By (3.3) we know that $A = \alpha^2 + 2\alpha^2b^2 - 3\beta^2 > 0$. Then $f_{00} = 0$ or

$$
Af_{0k}V^k + 3\beta f_{00}(f_kV^k) = 0.
$$

Therefore the proof becomes complete. □

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