Journal of Finsler Geometry and its Applications
Vol. 3, No. 2 (2022), pp 29-40
DOI: 10.22098/jfga.2022.11477.1070

On conformally flat 5-th root (α, β)-metrics with relatively isotropic mean Landsberg curvature

Marzeiya Amini[a*]

[a]Department of Mathematics, Faculty of Science, University of Qom
Qom. Iran

E-mail: marzeia.amini@gmail.com

Abstract. In this paper, we study conformally flat 5-th root (α, β)-metrics. We prove that every conformally flat 5-th root (α, β)-metric with relatively isotropic mean Landsberg curvature must be either Riemannian metrics or locally Minkowski metrics.

Keywords: Conformally flat metric, (α, β)-metric, mean Landsberg curvature.

1. Introduction

Conformal Geometry and its impact on other sciences has a long and brilliant history. It has played an elegance and important role in Physical Theories. More precisely, conformal transformation of Riemannian metrics (or Riemannian curvature) have been well studied by many geometers. The well-known Weyl theorem states that the projective and conformal properties of a Finsler manifold determine the metric properties uniquely. There are many important local and global results in Riemann-Finsler conformal geometry, which in turn lead to a better understanding on Riemann-Finsler manifolds. Also, the conformal properties of a Finsler metric deserve extra attention. Let $F = F(x, y)$ and $\tilde{F} = \tilde{F}(x, y)$ be two arbitrary Finsler metrics on a manifold M. Then we say that F is conformal to $\tilde{F}$ if and only if there exists a scalar function $\sigma = \sigma(x)$ such that

$$F(x, y) = e^{\sigma(x)} \tilde{F}(x, y).$$

AMS 2020 Mathematics Subject Classification: 53B40, 53C30, 53C60

The scalar function $\sigma = \sigma(x)$ is called the conformal factor. A Finsler metric $F = F(x, y)$ on a manifold M is said to be a conformally flat metric if there exists a locally Minkowski metric $\tilde{F} = \tilde{F}(y)$ such that $F = e^{\kappa(x)}\tilde{F}$, where $\kappa = \kappa(x)$ is a scalar function on M. A new and hot issue is to characterization of conformally flat Finsler metrics. Recently, Asanov constructed a Finslerian metric function on the manifold $N = \mathbb{R} \times M$, where M is a Riemannian manifold endowed with two real functions, and showed that the tangent Minkowski spaces of such a Finsler space are conformally flat [3]. This motivated Asanov to propose a Finslerian extension of the electromagnetic field equations whose solutions are explicit images of the solutions to the ordinary Maxwell equations.

In order to find conformally flat Finsler metrics, we consider the class of m-th root Finsler metrics. Let (M, F) be an n-dimensional Finsler manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F : TM \to \mathbb{R}$ be a scalar function defined by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ such that $a_{i_1 \dots i_m}$ is symmetric in all its indices. Then F is called an m-th root Finsler metric. An m-th root Finsler metric can be regarded as a direct generalization of a Riemannian metric in the sense that the 2-th root metric is a Riemannian metric $F = \sqrt{a_{ij}(x) y^i y^j}$. The fourth root metrics $F = \sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l}$ are called the quartic metrics. The special quartic metric $F = \sqrt[4]{y^i y^j y^k y^l}$ is called Berwald-Moór metric which plays an important role in theory of space-time structure, gravitation and general relativity. For more progress, see [9], [11], [12] and [14].

In [13], Tayebi-Razgordani proved that every conformally flat weakly Einstein 4-th root (α, β)-metric on a manifold M of dimension $n \geq 3$ is either a Riemannian metric or a locally Minkowski metric. Also, they showed that every conformally flat 4-th root (α, β)-metric of almost vanishing Ξ-curvature on a manifold M of dimension $n \geq 3$ reduces to a Riemannian metric or a locally Minkowski metric. In [10], Tayebi and the author studied conformally flat 4-th root (α, β)-metric with relatively isotropic mean Landsberg curvature and proved that every conformally flat 4-th root (α, β)-metric $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$, be a on a manifold M of dimension $n \geq 3$ with relatively isotropic mean Landsberg curvature is a Riemannian or a locally Minkowski metric. The third root metrics $F = \sqrt[3]{a_{ijk}(x) y^i y^j y^k}$ are called the cubic metrics. In [1], the author studied conformally flat 3-th root (α, β)-metric with relatively isotropic mean Landsberg curvature and proved that such metrics reduces to a Riemannian or a locally Minkowski metric. In [8], Piscoran and the author studied conformally flat square-root (α, β)-metric $F = \sqrt{\alpha(\alpha + \beta)}$ with relatively isotropic mean Landsberg curvature on a manifold M of dimension $n \geq 3$, where $\alpha = a_{ij}(x) y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M. They showed that F reduces to a Riemannian metric or a locally

Minkowski metric. In this paper, we study conformally flat 5-th root (α, β)-metric with relatively isotropic mean Landsberg curvature. More precisely, we prove the following.

Theorem 1.1. *Let $F = F(x, y)$ be a conformally flat 5-th root (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} + c(x)F\mathbf{I} = 0, \tag{1.1}$$

where $c = c(x)$ is a scalar function on M. Then F reduces to a Riemannian metric or a locally Minkowski metric.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_xM$ the tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, for a non-zero vector $y \in T_xM_0 := T_xM - \{0\}$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r \partial s \partial t}\Big[F^2(y + ru + sv + tw)\Big]_{r=s=t=0},$$

where $u, v, w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. Thus $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

On the slit tangent bundle TM_0, the Landsberg curvature $\mathbf{L}_{ijk} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is defined by $L_{ijk} := C_{ijk;m}y^m$, where ";" denotes the horizontal covariant derivative with respect to F. Further, the Landsberg curvature can be expressed as following

$$L_{ijk} = -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k}. \tag{2.1}$$

A Finsler metric is called the Landsberg metric if $L_{ijk} = 0$.

The horizontal covariant derivatives of the mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ which are defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s}y^s.$$

Here, "|" denotes the horizontal covariant derivative with respect to the Berwald connection of F. The family $\mathbf{J} := \{\mathbf{J}_y\}_{y\in TM_0}$ is called the mean Landsberg curvature. Also, the mean Landsberg curvature can be expressed as following

$$J_i := g^{jk}L_{ijk} \tag{2.2}$$

In this paper, we will focus on studying special (α,β)-metrics. Let "|" denote the covariant derivative with respect to the Levi-Civita connection of α. Denote

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$$
$$s^i{}_j := a^{im}s_{mj}, \qquad r^i{}_j := a^{im}r_{mj}, \quad r_j := b^ir_{ij}, \quad s_j := b^is_{ij},$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^j := a^{jk}b_k$. We put

$$r_0 := r_iy^i, \quad s_0 := s_iy^i, \quad r_{00} := r_{ij}y^iy^j, \quad s_{i0} := s_{ij}y^j.$$

Let G^i and G^i_α denote the geodesic coefficients of F and α respectively in the same coordinate system. Then we have

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{r_{00} - 2Q\alpha s_0\}\{\Psi b^i + \Theta\alpha^{-1}y^i\}, \tag{2.3}$$

where

$$\begin{aligned}
Q &:= \frac{\phi'}{\phi - s\phi'},\\
\Theta &:= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']},\\
\Psi &:= \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}.
\end{aligned}$$

For more details, see [5]. Let

$$\begin{aligned}
&\Delta := 1 + sQ + (b^2 - s^2)Q',\\
&\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',\\
&\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{\frac{3}{2}}}\right]',\\
&h_j := b_j - \alpha^{-1}sy_j :
\end{aligned}$$

By (2.1), (2.2), (2.3), the mean Landsberg curvature of the (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is given by

$$\begin{aligned} J_j =& \frac{1}{2\alpha^4\Delta}\Bigg\{ \frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](s_0 + r_0)h_j \\ &+ \frac{\alpha^2}{b^2-s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j \\ &+ \alpha\Big[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} \\ &+ \alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\Bigg\}. \end{aligned}$$

Here, $y_j = a_{ij}y^i$. See [4] and [6].

3. Proof of Theorem 1.1

in this section, we are going to prove Theorem 1.1. To prove it, we need the following.

Lemma 3.1. ([4]) For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the mean Cartan torsion is given by

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi')h_i. \tag{3.1}$$

In [4], the following was proved.

Lemma 3.2. ([4]) An (α, β)-metric F is a Riemannian metric if and only if $\Phi = 0$.

In order to prove Theorem 1.1, we need the following.

Lemma 3.3. *Let $F = \sqrt[5]{a_{ijklm}y^iy^jy^ky^ly^m}$ be a 5-th root metric which admits an (α, β)-metric on a manifold M of dimension $n \geq 3$. Then it can be written in the form*

$$F = \sqrt[5]{c_1\alpha^5 + c_2\beta^3\alpha^2 + c_3\beta^5}$$

by choosing suitable non-degenerate quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one-form $\beta = b_i(x)y^i$, where c_1, v_2 and c_3 are real constants.

Proof. The proof is computational and straightforward. By the same argument used in [7], we get the proof. We omit the proof. □

In [4], the following formula obtained

$$J_j + c(x)FI_j = -\frac{1}{2\alpha^4\Delta}\Bigg\{\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](s_0 + r_0)h_j$$
$$+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j + \alpha\Big[-\alpha^2 Q' s_0 h_j$$
$$+\alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j)$$
$$-(r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta} + c(x)\alpha^4\Phi(\phi - s\phi_{\prime})h_j\Bigg\}. \quad (3.2)$$

Also, we remark the following key lemma.

Lemma 3.4. ([2]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Then F is locally Minkowskian if and only if α is a flat Riemannian metric and β is parallel with respect to α.

Also, the following holds.

Lemma 3.5. ([4]) If $\phi = \phi(s)$ satisfies $\Psi_1 = 0$, then F is Riemannian.

Now, assume that $F = \alpha\phi(s)$, $s = \beta/\alpha$, is a conformally flat Finsler metric, that is, F is conformally related to a Minkowski metric $\tilde{F}$. Then there exists a scalar function $\sigma = \sigma(x)$ on the manifold, so that $\tilde{F} = e^{\sigma(x)}F$. It is easy to see that $\tilde{F} = \tilde{\alpha}\phi(\tilde{s})$, $\tilde{s} = \tilde{\beta}/\tilde{\alpha}$. We have $\tilde{\alpha} = e^{\sigma(x)}\alpha$ and $\tilde{\beta} = e^{\sigma(x)}\beta$ which are equivalent to

$$\tilde{a_{ij}} = e^{2\sigma(x)}a_{ij}, \qquad \tilde{b_i} = e^{\sigma(x)}b_i.$$

Let "$\|$" denote the covariant derivative with respect to the Levi-Civita connection of $\tilde{\alpha}$. Put $\sigma_i := \partial\sigma/\partial x^i$ and $\sigma^i := a^{ij}\sigma_j$. The Christoffel symbols Γ^i_{jk} of α and the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of $\tilde{\alpha}$ are related by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}.$$

Hence, one can obtain

$$\tilde{b}_{i\|j} = \frac{\partial\tilde{b_i}}{\partial x^j} - \tilde{b_s}\tilde{\Gamma}^i_{jk} = e^{\sigma}(b_{i|j} - b_j\sigma_i + b_r\sigma^r a_{ij}). \quad (3.3)$$

By Lemma 3.40, for Minkowski metric $\tilde{F}$, we have $\tilde{b}_{i\|j} = 0$. Thus

$$b_{i|j} = b_j\sigma_i - b_r\sigma^r a_{ij}, \tag{3.4}$$

$$r_{ij} = \frac{1}{2}(\sigma_i b_j + \sigma_j b_i) - b_r\sigma^r a_{ij}, \tag{3.5}$$

$$s_{ij} = \frac{1}{2}(\sigma_i b_j + \sigma_j b_i), \tag{3.6}$$

$$r_j = -\frac{1}{2}(b_r\sigma^r)b_j + \frac{1}{2}\sigma_j b^2, \tag{3.7}$$

$$s_j = \frac{1}{2}(b_r\sigma^r)b_j - \sigma_j b^2, \tag{3.8}$$

$$r_{i0} = \frac{1}{2}[\sigma_i\beta + (\sigma_r y^r)b_i] - \sigma_r b^r y_i, \tag{3.9}$$

$$s_{i0} = \frac{1}{2}[\sigma_i\beta + (\sigma_r y^r)b_i]. \tag{3.10}$$

Further, we have

$$r_{00} = (\sigma_r y^r)\beta - (\sigma_r y^r)\alpha^2, \tag{3.11}$$

$$r_0 = \frac{1}{2}(\sigma_r y^r)b^2 - \frac{1}{2}(\sigma_r b^r)\beta, \tag{3.12}$$

$$s_0 = \frac{1}{2}(\sigma_r y^r)\beta - \frac{1}{2}(\sigma_r y^r)b^2. \tag{3.13}$$

By (3.13), the conformally flat (α, β)-metrics satisfying $r_0 + s_0 = 0$ which is equivalent to the length of β with respect to α being a constant. We take an orthonormal basis at any point x with respect to α such that $\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}$ and $\beta = by^1$, where $b := \|\beta_x\|_\alpha$. By using the same coordinate transformation $\psi : (s, u^A) \longrightarrow (y^i)$ in T_xM, we get

$$y_1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \quad 2 \le A \le n, \tag{3.14}$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^n (u^A)^2}$. We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{3.15}$$

Put $\bar{\sigma_0} := \sigma_A u^A$. Then, by (3.5)-(3.9), (3.14) and (3.15) we have

$$r_{00} = -b\sigma_1\bar{\alpha}^2 + \frac{bs\bar{\sigma_0}\bar{\alpha}}{\sqrt{b^2 - s^2}}, \tag{3.16}$$

$$r_0 = \frac{1}{2}b^2\bar{\sigma_0} = -s_0, \tag{3.17}$$

$$r_{10} = \frac{1}{2} b\bar{\sigma}_0, \tag{3.18}$$

$$r_{A0} = \frac{1}{2} \frac{\sigma_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}} - (b\sigma_1) u_A, \tag{3.19}$$

$$s_1 = 0, \tag{3.20}$$

$$s_A = -\frac{1}{2}\sigma_A b^2, \tag{3.21}$$

$$s_{10} = \frac{1}{2} b\bar{\sigma}_0, \tag{3.22}$$

$$s_{A0} = \frac{1}{2} \frac{\sigma_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}}, \tag{3.23}$$

$$h_1 = b - \frac{s^2}{b}, \tag{3.24}$$

$$h_A = -\frac{\sqrt{b^2 - s^2} s u_A}{b\bar{\alpha}}. \tag{3.25}$$

Proof of Theorem 1.1: We remark that $\tilde{b} = constant$. If $\tilde{b} = 0$, then $F = e^{k(x)}\tilde{\alpha}$ is a Riemannian metric. Now, let F is not Riemannian metric. Assume that F is a conformally flat (α, β)-metric with relatively isotropic mean Landsberg curvature. By (3.2) and $r_0 + s_0 = 0$, we get

$$\frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j + \alpha\Big[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0)$$
$$+ \alpha^2 \Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0) y_j\Big]\frac{\Phi}{\Delta} + c(x)\alpha^4\Phi(\phi - s\phi')h_j = 0. \tag{3.26}$$

Letting $j = 1$ in (3.26), we have

$$\frac{\alpha^2}{b^2 - s_2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_1 + \alpha\Big[-\alpha^2 Q' s_0 h_1 + \alpha Q(\alpha^2 s_1 - y_1 s_0)$$
$$+ \alpha^2 \Delta s_{10} + \alpha^2(r_{10} - 2\alpha Q s_1) - (r_{00} - 2\alpha Q s_0) y_1\Big]\frac{\Phi}{\Delta}$$
$$+ c(x)\alpha^4\Phi(\phi - s\phi')h_1 = 0. \tag{3.27}$$

Putting (3.15)-(3.24) into (3.27) and multiplying the result with $2\Delta(b^2 - s^2)^{5/2}$ implies that

$$2b^2(b^2 - s^2)^{3/2}\Delta(b\Phi\phi c - b\Phi s\phi' c - \Psi_1\sigma_1)\bar{\alpha}^4$$
$$+ b^2(b^2 - s^2)\bar{\sigma}_0(b^4\Phi Q' - b^2\Phi\Delta - b^2\Phi Q' s^2)$$
$$+ 2b^2\Psi_1\Delta Q + b^2\Phi + b^2\Phi Q s + 2\Psi_1\Delta s)\bar{\alpha}^3 = 0. \tag{3.28}$$

From (3.28), we get

$$\Delta[b\Phi\phi c - b\Phi s\phi' c - \Psi_1\sigma_1] = 0, \tag{3.29}$$

$$\bar{\sigma}_0(b^4\Phi Q' - b^2\Phi\Delta - b^2\Phi Q's^2) + 2b^2\Psi_1\Delta Q + b^2\Phi + b^2\Phi Qs + 2\Psi_1\Delta s) = 0. \tag{3.30}$$

Note that $\Delta = Q'(b^2 - s^2) + sQ + 1$. Simplifying (3.30) yields

$$(b^2\Psi_1\Delta Q + \Psi_1\Delta s)\bar{\sigma_0} = 0,$$

that is

$$\Psi_1\Delta(b^2Q + s)\bar{\sigma_0} = 0. \tag{3.31}$$

Letting $j = A$ in (3.26), we have

$$\begin{aligned}&\frac{\alpha^2}{b^2 - s^2}\left[\Psi_1 + s\frac{\Phi}{\Delta}\right](r_{00} - 2\alpha Qs_0)h_A + \alpha\big[-\alpha^2Q's_0h_A + \alpha Q(\alpha^2 s_A - y_A s_0)\\&+\alpha^2\Delta s_{A0} + \alpha^2(r_{A0} - 2\alpha Q s_A) - (r_{00} - 2\alpha Q s_0)y_A\big]\frac{\Phi}{\Delta}\\&+ c(x)\alpha^4\Phi(\phi - s\phi')h_A = 0.\end{aligned} \tag{3.32}$$

Putting (3.15)-(3.24) into (3.32) and by using the similar method used in the case of $j = 1$, we get

$$(s\Delta + s + b^2Q)b^2\Phi\sigma_A\bar{\alpha}^2 - \big[(s\Delta + s + b^2Q)b^2\Phi + 2s(b^2Q + s)\Psi_1\Delta\big]\bar{\sigma}_0 u_A = 0, \tag{3.33}$$

$$s(b^2 - s^2)[b(\phi - s\phi')\Phi c - \Psi_1\sigma_1]\Delta u_A = 0. \tag{3.34}$$

It is easy to see that (3.34) is equivalent to (3.29). Further, multiplying (3.33) with u^A implies that

$$s(b^2Q + s)\Psi_1\Delta\bar{\sigma}_0\bar{\alpha}^2 = 0. \tag{3.35}$$

It is easy to see that (3.35) is equivalent to (3.31). In summary, conformally flat (α, β)-metrics with relatively isotropic mean Landsberg curvature satisfy (3.29) and (3.31). According to (3.31), we have some cases as follows:

Case (i): If $b^2Q + s = 0$, then we have $\phi = \kappa\sqrt{b^2 - s^2}$, which is a contradiction with the assumption that $\phi = \phi(s)$ is $\phi(s) = \sqrt[5]{a_1 + a_2s^3 + a_3s^5}$. Then we have $b^2Q + s \neq 0$.

Case (ii): If $\Psi_1 = 0$, then by Lemma 3.5, F is Riemannian.

Case (iii): If $\Psi_1 \neq 0$, then $\sigma_A = 0$. In the following, we prove that if $\Psi_1 \neq 0$, then by (3.29) one can get $\sigma_1 = 0$.

Now, assume that

$$\phi = \sqrt[5]{a_1 + a_2 s^3 + a_3 s^5}, \qquad a_1 \neq 0,\ a_2 \neq 0. \tag{3.36}$$

Here a_1, a_2, a_3 are numbers independent of s and $a_i \neq 0, i = 1, 2, 3$. Simplifying (3.29) and multiplying it by Δ^2, we get

$$\Big\{[-s\Phi + (b^2 - s^2)\Phi']\Delta - \frac{3}{2}(b^2 - s^2)\Phi\Delta'\Big\}\sigma_1 - b\Delta^2\Phi(\phi - s\phi')c = 0. \tag{3.37}$$

Putting (3.36) into (3.37) and by using maple program, we can obtain the following

$$\begin{aligned}
&\bigg(15c\Big(a_2(5a_3 + 8a_2)s^6 - 15a_1a_3s^5 + (a_3 + a_2)a_1s^3 + 3a_1a_2s + 5a_1^2\Big)^2\Big(25a_1a_2a_3^2s^{12}\\
&+ \big(8na_2 + (n+1)b^2a_3\big)a_2^3s^{11} - \frac{(3n-5)(b^2a_2^4 - 625a_1^2a_3^2)}{5}s^9 + \frac{15(n-22)b^2a_1a_2^3}{4}s^6\\
&- \frac{25\big(2(7n-2)a_2 + (n+2)b^2a_3\big)a_1a_2a_3}{4}s^{10} + \frac{25\big((3-3n)a_2 + b^2a_3(n-12)\big)a_1a_2^2}{2}s^8\\
&- \frac{125a_1^2a_3\big((13-n)a_2 + (n-1)b^2a_3\big)}{2}s^7 - \frac{125\big((-3n-3)a_2 + b^2a_3(n-9)\big)a_1^2a_2}{4}s^5\\
&- \frac{125a_1^3a_3(n+5)}{8}s^4 - \frac{15(n+3)b^2a_1^2a_2^2}{4}s^3 + \frac{125\big(b^2a_3 - (n+3)a_2\big)a_1^3}{2}s^2 + \frac{25b^2a_1^3a_2}{4}\Big)b\phi\bigg)\\
&+ (a_2s^3 + \frac{5a_1}{2})^7(a_3s^5 + a_2s^3 + a_1)\bigg\{\frac{1}{(2a_2s^3 + 5a_1)^2}\bigg(\bigg\{\frac{1}{(2a_2s^3 + 5a_1)^2}(3(a_2^4a_3s^{12}\\
&- a_1a_2^2a_3^2b^4ns^{11} - 144a_2^5b^4s^{10} - 160a_2^5b^2s^{12} + 13600a_1a_2^3a_3b^2s^{11} + 28750a_1^2a_2a_3^2s^{12}\\
&+ 50a_1a_2^2a_3^2ns^{13} + 80a_2^4a_3s^{12} - 200a_1a_2^2a_3^2b^4s^{11} + 1000a_1a_2^2a_3^2b^2s^{13} + 500a_1a_2^2a_3^2s^{15}\\
&+ 224a_2^5b^2ns^{12} + 1800a_1a_2^3a_3b^4s^9 - 3400a_2^3a_3b^2s^{11} + 1400a_1a_2^3a_3s^{13} + 240a_2^5b^4s^{10}\\
&- 1250a_1^2a_2a_3^2b^4s^8 + 17500a_1^2a_2a_3^2b^2s^{10} - 7500a_1^2a_2a_3^2s^{12} - 11800a_1a_2^3a_3b^4s^9\\
&- 2000a_1a_2^3a_3s^{13} + 50000a_1^2a_2a_3^2b^4s^8 - 81250a_1^2a_2a_3^2b^2s^{10} - 81250a_1^2a_2a_3^2b^2s^{10}\\
&+ 75000a_1^2a_2^2a_3b^2ns^8 - 18437a_1^3a_3^2b^2s^7 + 240a_1^3a_3^2b^2s^7 - 5650a_1^3a_2a_3b^2s^5\\
&+ 2880a_1a_2^4b^4ns^7 - 6700a_1a_2^4b^2ns^9 + 4200a_1a_2^4ns^{11} - 37500a_1^2a_2^2a_3b^4ns^6\\
&- 4050a_1^2a_2^2a_3s^{10} - 972a_1a_2^4b^4s^7 + 1460a_1a_2^4s^9 - 450a_1a_2^4s^{11} + 8750a_1^3a_3^2b^4s^5\\
&+ 9370a_1^3a_3^2s^9 + 7900a_1^2a_2^2a_3s^6 - 160a_1^2a_2^2a_3b^2s^8 + 810a_1^2a_2^2a_3s^{10} - 870a_1^3a_3^2b^4s^5\\
&- 150a_1^3a_3^2s^9 - 720a_1^2a_2^3b^4s^4 + 110a_1^2a_2^3b^2s^6 - 300a_1^2a_2^3s^8 + 310a_1^3a_2a_3b^4s^3\\
&+ 180a_1^3a_2a_3ns^7 + 100a_1^2a_2^3b^4s^4 - 4060a_1^2a_2^3b^2s^6 + 250a_1^2a_2^3s^8 - 620a_1^3a_2a_3b^4s^3\\
&- 156a_1^4a_3ns^4 + 120a_1^4a_2b^2 + 850a_1^3a_2a_3b^2s^5 - 860a_1^3a_2a_3s^7 + 250a_1^3a_2^2b^4ns\\
&- 7375a_1^3a_2^2b^2ns^3 + 4500a_1^3a_2^2ns^5 + 100a_1^4a_3b^2ns^2 - 75a_1^4a_3s^4\\
&+ 120a_1^3a_2^2b^4s - 315a_1^3a_2^2b^2s^3 + 10a_1^3a_2^2s^5 - 200a_1^4a_3b^4 + 100a_1^4a_3b^2s^2
\end{aligned}$$

$$- 15a_1^4a_2ns^2 + 50a_1^4a_2b^2 - 55a_1^4a_2s^2)s)\Big\}(10a_2a_3b^2s^6 - 6a_2^2s^4 + 16a_2^2s^6 + 10a_1a_3s^3$$
$$- 75a_1a_3s^5 + 30a_1a_2b^2s + 5a_1a_2s^3 + 25a_1^2)\Big) + \frac{1}{2(2a_2s^3+5a_1)^7}(27(8a_2^3b^2s^6$$
$$- 100a_1a_2a_3b^2s^5 + 50a_1a_2a_3s^7 - 140a_1a_2^2b^2s^3 + 10a_1a_2^2s^5 - 65a_1^2a_3s^4$$
$$+ 50a_1^2a_2 - 75a_1^2a_2s^2)(40a_2^3a_3b^2ns^{11} - 250a_1a_2a_3^2b^2ns^{10} + 40a_2^3a_3b^2s^{11}$$
$$- 500a_1a_2a_3^2b^2s^{10} + 250a_1a_2a_3^2s^{12} - 24a_2^4b^2ns^9 + 64a_2^4ns^{11} + 500a_1a_2^2a_3b^2ns^8$$
$$- 700a_1a_2^2a_3ns^{10} + 40a_2^4s^9 - 2500a_1^2a_3^2b^2s^7 + 1875a_1^2a_3^2s^9 - 1200a_1a_2^2a_3b^2s^8$$
$$+ 1000a_1a_2^2a_3s^{10} + 250a_1^2a_2a_3ns^7 + 75a_1^2a_2^2ns^5 - 1250a_1^2a_2a_3b^2ns^5$$
$$+ 250a_1^2a_3^2b^2s^7 - 3125a_1^2a_3^2s^9 + 150a_1a_2^3s^6 - 60a_1a_2^3ns^8 - 150a_1^2a_2^2b^2s^3$$
$$- 660a_1a_2^3b^2s^6 + 750a_1a_2^3s^8 + 2250a_1^2a_2a_3b^2s^5 - 3250a_1^2a_2a_3s^7$$
$$- 625a_1^3a_3ns^4 - 450a_1^2a_2^2b^2s^3 + 375a_1^2a_2^2s^5 + 2500a_1^3a_3b^2s^2 - 3125a_1^3a_3s^4$$
$$- 125a_1^3a_2ns^2 + 250a_1^3a_2b^2 - 375a_1^3a_2s^2)(b^2 - s^2)\Big\}\sigma_1 = 0,$$

where $\phi = \sqrt[5]{a_3s^5 + a_2s^3 + a_1}$. By simplifying the above relation we get the following

$$(\Pi_1 s^{48} + \Pi_2 s^{47} + ... + \Pi_{49})\sqrt[5]{a_3s^5 + a_2s^3 + a_1}$$
$$+(\zeta_1 s^{50} + \zeta_2 s^{49} + ... + \zeta_{51})\sigma_1 = 0 \quad (3.38)$$

where Π_i $(1 \leq i \leq 49)$ and ζ_k $(1 \leq k \leq 53)$ are polynomials of $a_1,\ a_2,\ a_3,\ b,\ c$, and $\zeta_{51} = b^4a_1^{10}a_2$. Equation (3.38) is equivalent to the following two equations

$$\Pi_1 s^{48} + \Pi_2 s^{47} + ... + \Pi_{48}s + \Pi_{49} = 0, \quad (3.39)$$
$$(\zeta_1 s^{50} + \zeta_2 s^{49} + ... + \zeta_{50}s + \zeta_{51})\sigma_1 = 0, \quad (3.40)$$

From (3.40), we have $\sigma_1 = 0$ or

$$\zeta_1 s^{50} + \zeta_2 s^{49} + ... + \zeta_{50}s + \zeta_{51} = 0.$$

$\zeta_1 s^{50} + \zeta_2 s^{49} + ... + \zeta_{50}s + \zeta_{51} \neq 0$, because $b \neq 0,\ a_1 \neq 0,\ a_2 \neq 0$ then $\zeta_{51} \neq 0$. This implies that $\sigma_1 = 0$. Together with $A = 0$, it follows that σ is a constant, which means that F is a locally Minkowski metric. This completes the proof. □

References

1. M. Amini, *On conformally flat cubic (α, β)-metrics*, Journal of Finsler Geometry and its Applications, **2**(1) (2021), 75-85.
2. P.L. Antonelli, R.S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic Publishers, Dordrecht, 1993.
3. G.S. Asanov, *Finslerian metric functions over product $R \times M$ and their potential applications*, Rep. Math. Phys. **41**(1998), 117-132.

4. X. Cheng, H. Wang and M. Wang, *(α,β)-metrics with relatively isotropic mean Landsberg curvature*, Publ. Math. Debrecen, **72**(2008), 475-485.
5. S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientiflc, Singapore, 2005.
6. B. Li and Z. Shen, *On a class of weak Landsberg metrics*, Science in China Series A, **50**(2007), 75-85.
7. M. Matsumoto and S. Numata, *On Finsler space with a cubic metric*, Tensor (N. S). **33**(2) (1979), 153-162.
8. L-I Piscoran and M. Amini, *On conformally flat square-root (α,β)-metrics*, Journal of Finsler Geometry and its Applications, **2**(2) (2021), 89-102.
9. A. Tayebi, *On generalized 4-th root metrics of isotropic scalar curvature*, Mathematica Slovaca, **68**(2018), 907-928.
10. A. Tayebi and M. Amini, *Conformally flat 4-th root (α,β)-metrics with relatively isotropic mean Landsberg curvature*, Math. Analysis. Convex. Optimization, **1**(2) (2020), 25-34.
11. A. Tayebi and B. Najafi, *On m-th root Finsler metrics*, J. Geom. Phys. **61**(2011), 1479-1484.
12. A. Tayebi and B. Najafi, *On m-th root metrics with special curvature properties*, C. R. Acad. Sci. Paris, Ser. I. **349**(2011), 691-693.
13. A. Tayebi and M. Razgordani, *On conformally flat fourth root (α,β)-metrics*, Differ. Geom. Appl. **62**(2019), 253-266.
14. A. Tayebi and M. Shahbazi Nia, *A new class of projectively flat Finsler metrics with constant flag curvature* $\mathbf{K}=1$, Differ. Geom. Appl. **41**(2015), 123-133.

Received: 08.09.2022
Accepted: 15.12.2022