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Some volume comparison theorems on Finsler manifolds of weighted Ricci curvature bounded below

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Abstract. This paper mainly studies the volume comparison in Finsler geometry under the condition that the weighted Ricci curvature Ric_∞ has a lower bound. By using the Laplacian comparison theorems of distance function, we characterize the growth ratio of the volume coefficients. Further, some volume comparison theorems of Bishop-Gromov type are obtained.

Keywords: volume comparison, the weighted Ricci curvature, Laplacian comparison theorem, distance function, volume coefficient.

1. Introduction

Finsler geometry is a natural generalization and extension of Riemannian geometry. Similar to Riemannian case, Finsler manifolds with Ricci curvature bounded below are always of some amazing properties. At the same time, volume comparison on Finsler manifolds play an important role in the studies of geometry and topology of Finsler manifolds. For example, Z. Shen proved the Gromov-Bishop volume comparison theorem in Finsler geometry under the condition that $\mathrm{Ric} \geq (n-1)\lambda$. As an application, he obtained some precompactness and finiteness theorems for Finsler manifolds([6]). B. Wu and Y. Xin obtained a similar volume comparison theorem under the conditions that

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$\mathrm{Ric} \leq c < 0$ and $\|\mathbf{S}\|_x = \sup_{X \in T_xM \setminus 0} \frac{\mathbf{S}(X)}{F(X)} \leq \Lambda$ ([10]). Later, Ohta proved a new version of the Gromov-Bishop volume comparison theorem on Finsler manifolds with weighted Ricci curvature bounded below, $\mathrm{Ric}_N \geq K$ for some $K \in R$ and $N \in [n, \infty)$([5]). Further, the first author and Z. Shen establish a relative volume comparison of Bishop-Gromov type on Finsler manifolds. As the applications, they obtain an upper bound for volumes of the Finsler manifolds and a theorem of Bonnet-Myers type on Finsler manifolds with weighted Ricci curvature bounded below, $\mathrm{Ric}_\infty \geq K > 0$ when S-curvature is bounded([2]). On the other hand, Q. Xia obtains a volume comparison theorem under the assumption that $\mathrm{Ric}_N \geq K$ for $N \in [n, \infty)$ and $K \in \mathbb{R}$. Based on this, Q. Xia proves the existence of two types of optimal (p, q)-Sobolev inequalities on compact Finsler manifolds ([11]).

In this paper, our main aim is to derive some new volume comparison on Finsler manifolds with weighted Ricci curvature bounded below, $\mathrm{Ric}_\infty \geq K > 0$. In order to introduce our main theorems clearly, we first give some necessary notations. Let (M, F, m) be an n-dimensional Finsler manifold with a smooth measure m and $x \in M$. Let $\Omega_x := M \setminus (\{x\} \cup Cut(x))$ be the cut-domain on M, where $Cut(x)$ is the cut locus of x, which has zero Hausdorff measure. Then, for any $z \in \Omega_x$, we can choose the geodesic polar coordinates (r, ξ) centered at x such that $r(z) = F(v)$ and $\xi^\alpha(z) = \xi^\alpha(\frac{v}{F(v)})$, where $v = \exp_x^{-1}(z) \in T_xM \setminus \{0\}$. By Gauss's lemma, the unit radial coordinate vector $\frac{\partial}{\partial r}$ and the coordinate vectors $\frac{\partial}{\partial \xi^\alpha}$ for $1 \leq \alpha \leq n-1$ are mutually vertical with respect to $g_{\nabla r}$ ([1], Lemma 6.1.1). Therefore, we can write the volume form as $dm|_{\exp_x(r\xi)} = \sigma(x, r, \xi) dr d\xi$, where $\xi \in I_x = \{\xi \in T_xM \mid F(\xi) = 1\}$ and $z = \exp_x(r\xi)$. Besides, in the following, $B(x, R)$ denotes the geodesic ball of radius R at the center $x \in M$. We also denote the geodesic sphere of radius r at the center $x \in M$ by $S(x, r)$.

Our first main theorem is as follows.

Theorem 1.1. *Let (M, F, m) be an n-dimensional Finsler manifold with a smooth volume form dm. Assume that $\mathrm{Ric}_\infty \geq K > 0$ and $|\mathbf{S}| \leq \delta$. Then, along any minimizing geodesic starting from the center x of $B(x, R)$ and for any $0 < r_1 < r_2 < \min\left\{R, \frac{\pi}{2}\sqrt{\frac{n-1}{K}}\right\}$, we have*

$$\frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq \left(\frac{r_2}{r_1}\right)^{n-1} e^{(r_2 - r_1)\delta}. \tag{1.1}$$

Further, for any $0 < r_1 < r_2 < \min\left\{R, \frac{\pi}{2}\sqrt{\frac{n-1}{K}}\right\}$, we have

$$\frac{m(B(x, r_2))}{m(B(x, r_1))} \leq \left(\frac{r_2}{r_1}\right)^{n} e^{r_2 \delta}. \tag{1.2}$$

In Theorem 1.1, if (M, F, m) is a forward complete Finsler manifold, by Theorem 3.2 (that is, Theorem 4.1 in [2]), we know that

$$R \leq \frac{\pi}{\sqrt{K}}\Big(\frac{\delta}{\sqrt{K}} + \sqrt{\frac{\delta^2}{K} + n - 1}\Big).$$

Further, we can get the following theorem.

Theorem 1.2. *Let (M, F, m) be an n-dimensional Finsler manifold with a smooth volume form dm. Assume that $\mathrm{Ric}_\infty \geq K > 0$ and $|\tau| \leq k$. Then, along any minimizing geodesic starting from the center x of $B(x, R)$ and for any $0 < r_1 < r_2 < \min\left\{R, \frac{\pi}{4}\sqrt{\frac{n-1}{K}}\right\}$, we have*

$$\frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq \left(\frac{r_2}{r_1}\right)^{n+4k-1}. \tag{1.3}$$

Further, for any $0 < r_1 < r_2 < \min\left\{R, \frac{\pi}{4}\sqrt{\frac{n-1}{K}}\right\}$, we have

$$\frac{m(B(x, r_2))}{m(B(x, r_1))} \leq \left(\frac{r_2}{r_1}\right)^{n+4k}. \tag{1.4}$$

Here, τ denotes the distortion of F.

More generally, we have the following theorem.

Theorem 1.3. *Let (M, F, m) be an n-dimensional Finsler manifold with a smooth volume form dm. Assume that $\mathrm{Ric}_\infty \geq K > 0$. Then, along any minimizing geodesic starting from the center x of $B(x, R)$ and for any $0 < r_0 < r_1 < r_2 < R$, we have*

$$\frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq e^{\frac{1}{2}K(r_2^2 - r_1^2)} e^{(r_2 - r_1)(m_0 + K r_0)}. \tag{1.5}$$

Further, for any $0 < r_0 < r_1 < r_2 < R$, we have

$$\frac{m(B(x, r_2))}{m(B(x, r_1))} \leq \frac{erf(\sqrt{\frac{K}{2}}r_2) - erf(\sqrt{\frac{K}{2}}r_0)}{erf(\sqrt{\frac{K}{2}}r_1) - erf(\sqrt{\frac{K}{2}}r_0)} e^{(r_2 - r_0)(m_0 + K r_0)}, \tag{1.6}$$

where $m_0 := \sup_{z \in r^{-1}(r_0)} \Delta r(z)$ characterizes the mean curvature of the geodesic sphere $S(x, r_0)$ and $erf(x)$ denotes the Gaussian error function, $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta$.

For some detailed discussions about Gaussian error function $erf(x)$, please see [4].

The paper is organized as follows. In Section 2, we will give some necessary definitions and notations. In Section 3, we will give some necessary and important lemmas and a theorem. Then we will give the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3 in Section 4.

2. **Preliminaries**

In this section, we briefly review some basis definitions and notations in Finsler geometry. For more details, we refer to [1,3].

Let M be an n-dimensional smooth manifold. For a point $x \in M$, denote by T_xM the tangent space of M at x. The tangent bundle TM of M is the union of tangent spaces with a natural differential structure,

$$TM = \bigcup_{x\in M} T_xM. \tag{2.1}$$

Denote the elements in TM by (x, y) with $y \in T_xM$. Let $TM_0 := TM\backslash\{0\}$ and $\pi : TM\backslash\{0\} \to M$ be the natural projective map. The pull-back π^*TM is an n-dimensional vector bundle on TM_0. A Finsler metric on manifold M means a function $F : TM \longrightarrow [0, \infty)$ on the tangent bundle satisfying the following properties:

(1) F is C^∞ on $TM\backslash\{0\}$;
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $(x, y) \in TM$ and all $\lambda > 0$;
(3) F is strongly convex, that is, the matrix $(g_{ij}(x, y)) = \left(\frac{1}{2}(F^2)_{y^iy^j}\right)$ is positive definite for any nonzero $y \in T_xM$.

Such a pair (M, F) is called a Finsler manifold and $g := g_{ij}(x, y)dx^i \otimes dx^j$ is called the fundamental tensor of F.

Let (M, F, m) be an n-dimensional Finsler manifold equipped with a measure m on M. Write the volume form dm of m as $dm = \sigma(x)dx^1dx^2 \cdots dx^n$. Define

$$\tau(x, y) := \ln \frac{\sqrt{\det (g_{ij}(x, y))}}{\sigma(x)}. \tag{2.2}$$

We call τ the *distortion* of F. It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM\backslash\{0\}$, let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Set

$$\mathbf{S}(x, y) := \frac{d}{dt} [\tau(\sigma(t), \dot\sigma(t))] |_{t=0}. \tag{2.3}$$

$\mathbf{S}$ is called the *S-curvature* of F ([3,6]).

Let Y be a C^∞ geodesic field on an open subset $U \subset M$ and $\hat g = g_Y$. Let

$$dm := e^{-\psi}\mathrm{Vol}_{\hat g}, \quad \mathrm{Vol}_{\hat g} = \sqrt{det\left(g_{ij}\left(x, Y_x\right)\right)}dx^1 \cdots dx^n.$$

It is easy to see that ψ is given by

$$\psi(x) = \ln \frac{\sqrt{\det\left(g_{ij}\left(x, Y_x\right)\right)}}{\sigma(x)} = \tau\left(x, Y_x\right),$$

which is just the distortion along Y_x at $x \in M$ ([3]). Let $y := Y_x \in T_xM$ (that is, Y is a geodesic extension of $y \in T_xM$). Then, by the definitions of the S-curvature and the Hessian ([6,7]), we have

$$\mathbf{S}(x,y) = y[\tau(x,Y_x)] = d\psi(y),$$
$$\dot{\mathbf{S}}(x,y) = y[\mathbf{S}(x,Y)] = y[Y(\psi)] = \mathrm{Hess}\psi(y),$$

where $\dot{\mathbf{S}}(x,y) := \mathbf{S}_{|m}(x,y)y^m$ and "$|$" denotes the horizontal covariant derivative with respect to the Chern connection. Further, the weighted Ricci curvatures are defined as follows.

$$\mathrm{Ric}_N(y) = \mathrm{Ric}(y) + \mathrm{Hess}\psi(y) - \frac{d\psi(y)^2}{N-n}, \tag{2.4}$$
$$\mathrm{Ric}_\infty(y) = \mathrm{Ric}(y) + \mathrm{Hess}\psi(y). \tag{2.5}$$

We define a map $\mathcal{L} : TM \to T^*M$ by

$$\mathcal{L}(y) := \begin{cases} g_y(y,\cdot), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

It is easy to see that

$$F(x,y) = F^*(x,\mathcal{L}(y)). \tag{2.6}$$

Thus $\mathcal{L}$ is a norm-preserving transformation ([2,7]). We call $\mathcal{L}$ the Legendre transformation on Finsler manifold (M,F). Given a smooth function u on M, the differential du_x at any point $x \in M$,

$$du_x = \frac{\partial u}{\partial x^i}(x)dx^i$$

is a linear function on T_xM. We define the gradient vector $\nabla u(x)$ of u at $x \in M$ by $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_xM$. In a local coordinate system, we can express ∇u as

$$\nabla u(x) = \begin{cases} g^{*ij}(x,du)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}, & x \in M_u, \\ 0, & x \in M \setminus M_u, \end{cases} \tag{2.7}$$

where $M_u = \{x \in M \mid du(x) \neq 0\}$ ([2,7]).

Associated with the measure m on M, we may decompose the volume form dm of m as $dm = \mathrm{e}^\Phi dx^1dx^2\cdots dx^n$. Then the divergence of a differentiable vector field V on M is defined by

$$\mathrm{div}_m V := \frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \Phi}{\partial x^i}, \quad V = V^i\frac{\partial}{\partial x^i}. \tag{2.8}$$

One can also define $\mathrm{div}_m V$ in the weak form by following divergence formula:

$$\int_M \phi \ \mathrm{div}_m V \ dm = -\int_M d\phi(V) \ dm \tag{2.9}$$

for all $\phi \in \mathcal{C}_c^\infty(M)$.

Now we define the Finsler Laplacian Δu of $u \in H^1_{\mathrm{loc}}(M)$ by

$$\Delta u := \mathrm{div}_m(\nabla u). \tag{2.10}$$

Equivalently, we can define Laplacian Δu on the whole M in the weak sense by

$$\int_M \phi \ \Delta u \ dm := -\int_M d\phi(\nabla u) dm \tag{2.11}$$

for all $\phi \in \mathcal{C}_c^\infty(M)$. From (2.10), Finsler Laplacian is a nonlinear elliptic differential operator of the second order. Moreover, since the gradient vector field ∇u is merely continuous on $M \backslash M_u$, even when $u \in \mathcal{C}^\infty(M)$, it is necessary to introduce the Laplacian in the weak form as (2.11).

Next, we define

$$s_c(t) := \begin{cases} \frac{1}{\sqrt{c}} \sin(\sqrt{c}t) & c > 0, \\ t & c = 0, \\ \frac{1}{\sqrt{-c}} \sin(\sqrt{-c}t) & c < 0. \end{cases}$$

Obviously, $s_c(t)$ is the solution of the differential equation $f''+cf = 0$ satisfying $f(0) = 0$ and $f'(0) = 1$. Further, we define

$$ct_c(t) := \frac{s'_c(t)}{s_c(t)} = \begin{cases} \sqrt{c} \cot(\sqrt{c}t) & c > 0, \\ \frac{1}{t} & c = 0, \\ \sqrt{-c} \coth(\sqrt{-c}t) & c < 0. \end{cases}$$

Let A be a closed subset in a Finsler manifold (M, F). Let

$$r(x) := d(A, x).$$

$r(x)$ is locally Lipschitz function. Thus they are differentiable almost everywhere. We have the following

Lemma 2.1. ([7,8]) *Let $r(x) := d(A, x)$ for a closed subset $A \subseteq M$. Then, on an open subset $U \subset M$, we have*

$$F^* (x, dr) = F (x, \nabla r) = 1, \quad x \in U$$

In general, a Lipschitz function f on a Finsler manifold (M, F) is called a distance function if the following identity holds almost everywhere on M ([7,8])

$$F (x, \nabla f_x) = 1.$$

Particularly, for any point $p \in M$, we have the distance function $r(x) = d(p, x)$.

Let (M, F, m) be an n-dimensional Finsler manifold with a smooth measure m and $x \in M$. Let $\Omega_x := M \backslash (\{x\} \cup Cut(x))$ be the cut-domain on M. For any $z \in \Omega_x$, we can choose the geodesic polar coordinates (r, ξ) centered at x for z such that $r(z) = F(v)$ and $\xi^\alpha(z) = \xi^\alpha(\frac{v}{F(v)})$, where $v = \exp_x^{-1}(z) \in T_xM \backslash \{0\}$. Actually, $z = \exp_x(r\xi)$. A basic fact is that the distance function $r = d(x, z)$ satisfies the following ([7,8])

$$\nabla r|_z = \frac{\partial}{\partial r}|_z. \tag{2.12}$$

Further, we can write the volume form as $dm|_{\exp_x(r\xi)} = \sigma(x,r,\xi)drd\xi$, where $\xi \in I_x = \{\xi \in T_xM \mid F(\xi) = 1\}$. Then, for geodesic ball $B(x,R)$ of radius R at the center $x \in M$, the volume of $B(x,R)$ is

$$m(B(x,R)) = \int_{B(x,R)} dm = \int_{B(x,R)\cap\Omega_x} dm = \int_0^R dr \int_{\mathcal{D}_x(r)} \sigma(x,r,\xi)d\xi, \tag{2.13}$$

where $\mathcal{D}_x(r) = \{\xi \in I_x \mid r\xi \in \exp_x^{-1}(\Omega_x \cap B(x,R))\}$. Obviously, for any $0 < s < t < R$, $\mathcal{D}_x(t) \subseteq \mathcal{D}_x(s)$.

3. **Some important theorems and lemmas**

In order to prove the main results in this paper, we first introduce some necessary lemmas and a theorem that we will need later. Firstly, we need the following lemma for the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.1. (Laplacian comparison, [2,12]) *Let (M,F,m) be an n-dimensional Finsler manifold with weighted Ricci curvature satisfying* $\mathrm{Ric}_\infty \geq (n-1)c$. *Then the following bound on Δr holds.*

(a) *If $\mathbf{S} \geq -\delta$, then the following holds on $\Omega_p \cap B(p,r_o)$,*

$$\Delta r \leq \frac{d}{dt}\Big[\ln \chi(t)\Big]|_{t=r(x)},$$

where $\chi(t) = [s_c(t)]^{n-1}e^{\delta t}$, $0 < t < r_o$. Here $r_o = +\infty$ when $c \leq 0$ and $r_o = \frac{\pi}{2\sqrt{c}}$ when $c > 0$.

(b) *If the distortion τ satisfies that $|\tau| \leq k$, then the following holds on $\Omega_p \cap B(p,r_o)$,*

$$\Delta r \leq \frac{d}{dt}\Big[\ln \chi(t)\Big]|_{t=r(x)},$$

where $\chi(t) := [s_c(t)]^{n+4k-1}$, $0 < t < r_o$. Here $r_o := +\infty$ when $c \leq 0$ and $r_o = \pi/(4\sqrt{c})$ when $c > 0$.

The following theorem is valuable for determining the upper bound of radius of the balls in Theorem 1.1.

Theorem 3.2. ([2]) *Let (M,F,m) be an n-dimensional forward complete Finsler manifold with smooth volume form $dm = \phi(x)dm_{BH}$. Assume*

$$Ric_\infty \geq K > 0, \qquad |\mathbf{S}| \leq \delta.$$

Then the diameter of the manifold M is bounded.

$$Diam(M) \leq \frac{\pi}{\sqrt{K}}\Big(\frac{\delta}{\sqrt{K}} + \sqrt{\frac{\delta^2}{K} + n - 1}\Big).$$

The following lemma is necessary for the proof of Theorem 1.3.

Lemma 3.3. ([2,9,12]) *Let (M, F, m) be an n-dimensional Finsler manifold with weighted Ricci curvature satisfying $Ric_\infty \geq K$. Then, for any $p \in M$, the following holds whenever the distance function $r(x) := d_F(p, x)$ is smooth and $r(x) > r_0$*

$$\Delta r \leq \frac{d}{dt}[\ln \chi(t)]\bigg|_{t=r(x)},$$

where $\chi(t) = e^{m_0(t-r_0)-\frac{1}{2}K(t-r_0)^2}$ ($r_0 < t < \infty$) and $m_0 := \sup_{x\in r^{-1}(r_0)} \Delta r(x)$ characterizes the mean curvature of the geodesic sphere $S(p, r_0)$ ([7,8]).

4. **Proof of the Main Theorems**

In this section, we will give the proofs of our main results.

Proof of Theorem 1.1. Firstly, let $\eta : [0, r] \to M$ be the minimizing geodesic from $\eta(0) = x$ to $\eta(r) = z$, where $r = d(x, z)$. By using the geodesic polar coordinates (r, ξ) centered at x and by (2.12), the Laplacian of the distance function r satisfies ([7,8])

$$\Delta r = \frac{\partial}{\partial r} \log \sigma(x, r, \xi). \tag{4.1}$$

By the assumption, $Ric_\infty \geq K > 0$, that is, $Ric_\infty \geq (n-1)c > 0$, where $c = \frac{K}{n-1} > 0$. According to Lemma 3.1(a), we have

$$\Delta r \leq \frac{d}{dt}\left[\ln \chi_0(t)\right]|_{t=r}.$$

where $\chi_0 = [s_c(t)]^{n-1}e^{\delta t}, 0 < t < r_0$ and $r_0 = \frac{\pi}{2\sqrt{c}} = \frac{\pi}{2}\sqrt{\frac{n-1}{K}}$. Thus we can get

$$\Delta r \leq \delta + (n-1)ct_c(r) = \delta + \sqrt{(n-1)K}\cot\left(\sqrt{\frac{K}{n-1}}r\right),$$

where $0 < r < \frac{\pi}{2}\sqrt{\frac{n-1}{K}}$.

By

$$\cot r = \frac{1}{r} - \frac{1}{3}r - \frac{1}{45}r^3 - \frac{2}{945}r^5 - \cdots \leq \frac{1}{r}, \quad 0 < r < \pi. \tag{4.2}$$

we have $\sqrt{(n-1)K}\cot(\sqrt{\frac{K}{n-1}}r) \leq \frac{n-1}{r}$. Then we obtain the following

$$\Delta r \leq \delta + \frac{n-1}{r}.$$

Therefore, we can get

$$\frac{\partial}{\partial r}\log \sigma(x, r, \xi) \leq \delta + \frac{n-1}{r},$$

Integrating in r on both sides of the above inequality from r_1 to r_2 , we get

$$\log \frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq (n-1)\log\frac{r_2}{r_1} + (r_2 - r_1)\delta,$$

which implies that

$$\frac{\sigma(x,r_2,\xi)}{\sigma(x,r_1,\xi)} \le \left(\frac{r_2}{r_1}\right)^{n-1} e^{(r_2-r_1)\delta}.$$

This is just (1.1). Further, for any $0 < s < r_1 < t < r_2 < \min\left\{R, \frac{\pi}{2}\sqrt{\frac{n-1}{K}}\right\}$, we have

$$\sigma(x,t,\xi)s^{n-1} \le t^{n-1}\sigma(x,s,\xi)e^{(t-s)\delta} \le t^{n-1}\sigma(x,s,\xi)e^{r_2\delta}.$$

Now, integrating in t from r_1 to r_2, we get

$$s^{n-1}\int_{r_1}^{r_2}\sigma(x,t,\xi)dt \le \frac{1}{n}(r_2^n - r_1^n)\sigma(x,s,\xi)e^{r_2\delta}.$$

Then, integrating on both sides of above inequality with respect to s from 0 to r_1 yields

$$\frac{1}{n}r_1^n\int_{r_1}^{r_2}\sigma(x,t,\xi)dt \le \frac{1}{n}(r_2^n - r_1^n)\int_0^{r_1}\sigma(x,s,\xi)e^{r_2\delta}ds.$$

Further,

$$\begin{aligned}\int_{r_1}^{r_2}dt\int_{D_x(t)}\sigma(x,t,\xi)d\xi &\le \frac{r_2^n-r_1^n}{r_1^n}e^{r_2\delta}\int_0^{r_1}ds\int_{D_x(t)}\sigma(x,s,\xi)d\xi\\ &\le \frac{r_2^n-r_1^n}{r_1^n}e^{r_2\delta}\int_0^{r_1}ds\int_{D_x(s)}\sigma(x,s,\xi)d\xi.\end{aligned}$$

By (2.13), we have

$$m(B(x,r_2)) - m(B(x,r_1)) \le \frac{r_2^n-r_1^n}{r_1^n}e^{r_2\delta}m(B(x,r_1)).$$

Therefore,

$$\frac{m(B(x,r_2))}{m(B(x,r_1))} \le 1 + \left(\frac{r_2}{r_1}\right)^n e^{r_2\delta} - e^{r_2\delta} \le \left(\frac{r_2}{r_1}\right)^n e^{r_2\delta}.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, let $\eta : [0,r] \to M$ be the minimizing geodesic from $\eta(0) = x$ to $\eta(r) = z$. By using the geodesic polar coordinates (r,ξ) centered at x and by (2.12), the Laplacian of the distance function r satisfies (4.1).

By the assumption that $Ric_\infty \ge K > 0$ and by Lemma 3.1(b), we have

$$\Delta r \le \frac{d}{dt}\left[\ln\chi_0(t)\right]|_{t=r}.$$

Here $\chi_0 = [s_c(t)]^{n+4k-1}$, $0 < t < r_0$, $c = \frac{K}{n-1} > 0$ and $r_0 = \frac{\pi}{4\sqrt{c}} = \frac{\pi}{4}\sqrt{\frac{n-1}{K}}$. Thus we have

$$\Delta r \le (n+4k-1)\sqrt{c}\cot(\sqrt{c}r).$$

By (4.2) again, we obtain

$$\Delta r \leq (n + 4k - 1)\frac{1}{r}.$$

Therefore, we have

$$\frac{\partial}{\partial r} \log \sigma(x, r, \xi) \leq (n + 4k - 1)\frac{1}{r}$$

Integrating in r on both sides of the above inequality from r_1 to r_2, we can get

$$\log \frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq (n + 4k - 1) \log \frac{r_2}{r_1},$$

Then we get

$$\frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq \left(\frac{r_2}{r_1}\right)^{n+4k-1}.$$

Further, for any $0 < s < r_1 < t < r_2 < \min\left\{R, \frac{\pi}{4}\sqrt{\frac{n-1}{K}}\right\}$, we have

$$\sigma(x, t, \xi)s^{n+4k-1} \leq \sigma(x, s, \xi)t^{n+4k-1}.$$

Now, by the same process in the proof of the Theorem 1.1, we can obtain the following

$$m(B(x, r_2)) - m(B(x, r_1)) \leq m(B(x, r_1))\frac{{r_2}^{n+4k} - {r_1}^{n+4k}}{{r_1}^{n+4k}},$$

which means that

$$\frac{m(B(x, r_2))}{m(B(x, r_1))} \leq \left(\frac{r_2}{r_1}\right)^{n+4k}.$$

This completes the proof of Theorem 1.2. □

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. By $Ric_\infty \geq K > 0$ and Lemma 3.3, we have

$$\Delta r \leq \frac{d}{dt}\left[\ln \chi_0(t)\right]\Big|_{t=r},$$

where $\chi_0(t) = e^{m_0(t-r_0)-\frac{1}{2}K(t-r_0)^2}$, $r_0 < t < \infty$. Hence, in this case, we have

$$\Delta r \leq [m_0 - K(t - r_0)]\big|_{t=r} = m_0 + K(r_0 - r).$$

Therefore, by (4.1), we have

$$\frac{\partial}{\partial r} \log \sigma(x, r, \xi) \leq m_0 + K(r_0 - r).$$

Integrating in r on both sides of the above inequality from r_1 to r_2 yields

$$\log \frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq (m_0 + Kr_0)(r_2 - r_1) + \frac{1}{2}K(r_1^2 - r_2^2),$$

from which, we obtain the following

$$\frac{\sigma(x, r_2, \xi)}{\sigma(x, r_1, \xi)} \leq e^{\frac{1}{2}K(r_1^2-r_2^2)}e^{(r_2-r_1)(m_0+Kr_0)}.$$

Further, for any $0 < r_0 < s < r_1 < t < r_2 < R$, we have

$$\begin{aligned}\sigma(x,t,\xi)e^{-\frac{1}{2}Ks^2} &\leq \sigma(x,s,\xi)e^{-\frac{1}{2}Kt^2}e^{(t-s)(m_0+Kr_0)}\\ &\leq \sigma(x,s,\xi)e^{-\frac{1}{2}Kt^2}e^{(r_2-r_0)(m_0+Kr_0)}.\end{aligned}$$

Next, integrating in t from r_1 to r_2 on both sides of above inequality, we get

$$e^{-\frac{1}{2}Ks^2}\int_{r_1}^{r_2}\sigma(x,t,\xi)dt \leq \sigma(x,s,\xi)e^{(r_2-r_0)(m_0+Kr_0)}\int_{r_1}^{r_2}e^{-\frac{1}{2}Kt^2}dt.$$

Then integrating in s from r_0 to r_1, we have

$$\int_{r_0}^{r_1}e^{-\frac{1}{2}Ks^2}ds\int_{r_1}^{r_2}\sigma(x,t,\xi)dt \leq e^{(r_2-r_0)(m_0+Kr_0)}\int_{r_1}^{r_2}e^{-\frac{1}{2}Kt^2}dt\int_{r_0}^{r_1}\sigma(x,s,\xi)ds,$$

By

$$\int_0^{r_1}e^{-\frac{1}{2}Ks^2}ds = \frac{\sqrt{\pi}erf(\frac{\sqrt{K}}{\sqrt{2}}r_1)}{\sqrt{2K}}$$

and

$$\int_{r_1}^{r_2}e^{-\frac{1}{2}Kt^2}dt = \frac{\sqrt{\pi}(erf(\frac{\sqrt{K}}{\sqrt{2}}r_2)-erf(\frac{\sqrt{K}}{\sqrt{2}}r_1))}{\sqrt{2K}},$$

we can get

$$\int_{r_1}^{r_2}\sigma(x,t,\xi)dt \leq \frac{erf(\sqrt{\frac{K}{2}}r_2)-erf(\sqrt{\frac{K}{2}}r_1)}{erf(\sqrt{\frac{K}{2}}r_1)-erf(\sqrt{\frac{K}{2}}r_0)}e^{(r_2-r_0)(m_0+Kr_0)}\int_{r_0}^{r_1}\sigma(x,s,\xi)ds.$$

Further,

$$\begin{aligned}&\int_{r_1}^{r_2}dt\int_{D_x(t)}\sigma(x,t,\xi)d\xi\\ &\leq \left(\frac{erf(\sqrt{\frac{K}{2}}r_2)-erf(\sqrt{\frac{K}{2}}r_1)}{erf(\sqrt{\frac{K}{2}}r_1)-erf(\sqrt{\frac{K}{2}}r_0)}\right)e^{(r_2-r_0)(m_0+Kr_0)}\int_{r_0}^{r_1}ds\int_{D_x(s)}\sigma(x,s,\xi)d\xi.\end{aligned}$$

By (2.13), we get

$$\begin{aligned}&m(B(x,r_2))-m(B(x,r_1))\\ &\leq \left(\frac{erf(\sqrt{\frac{K}{2}}r_2)-erf(\sqrt{\frac{K}{2}}r_1)}{erf(\sqrt{\frac{K}{2}}r_1)-erf(\sqrt{\frac{K}{2}}r_0)}\right)e^{(r_2-r_0)(m_0+Kr_0)}\left(m(B(x,r_1))-m(B(x,r_0))\right)\\ &\leq \left(\frac{erf(\sqrt{\frac{K}{2}}r_2)-erf(\sqrt{\frac{K}{2}}r_1)}{erf(\sqrt{\frac{K}{2}}r_1)-erf(\sqrt{\frac{K}{2}}r_0)}\right)e^{(r_2-r_0)(m_0+Kr_0)}m(B(x,r_1)).\end{aligned}$$

Therefore,

$$\frac{m(B(x,r_2))}{m(B(x,r_1))} \leq 1 + \left(\frac{erf(\sqrt{\frac{K}{2}}r_2) - erf(\sqrt{\frac{K}{2}}r_1)}{erf(\sqrt{\frac{K}{2}}r_1) - erf(\sqrt{\frac{K}{2}}r_0)} \right) e^{(r_2 - r_0)(m_0 + Kr_0)}$$
$$\leq \left(\frac{erf(\sqrt{\frac{K}{2}}r_2) - erf(\sqrt{\frac{K}{2}}r_0)}{erf(\sqrt{\frac{K}{2}}r_1) - erf(\sqrt{\frac{K}{2}}r_0)} \right) e^{(r_2 - r_0)(m_0 + Kr_0)}.$$

This completes the proof of Theorem 1.3. □

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