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On conformal vector fields on Einstein Finsler manifolds

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Abstract. In this paper, we study conformal vector fields on Finsler manifolds. Let $(M, \mathbf{g})$ be an Einstein-Finsler manifold of dimension $n \geq 2$. Suppose that V is conformal vector field on M. We find a condition under which V reduces to a concircular vector field.

Keywords: Finsler metric, Einstein manifold, geodesic circle, concircular transformation, concircular vector field.

1. Introduction

A geodesic circle in an Euclidean space is a straight line or a circle with finite positive radius, which can be generalized naturally to Riemannian or Finsler geometry. Firstly, in 1940, Yano introduced concircular transformations on Riemannian manifolds [28]. Exactly, a geodesic circle in a Riemannian manifold, as well as in a Finsler manifold, is a curve with constant first Frenet curvature and zero second one. In other words, a geodesic circle is a torsion free curve with constant curvature. A concircular transformation on a Riemannian manifold is a conformal transformation which preserves geodesic circles ([12], [28]). Many researchers have developed the theory of concircular transformations to different contents ([13, 14, 25]). In 1970, Vogel showed that every concircular transformation on a Riemannian manifold is conformal [26]. This notion has been extended to Finsler geometry by Agrawal and Izumi [1]. Also, a similar result is proved by Bidabad-Shen in 2012 [5]. That is, every transformation

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which preserves geodesic circles reduces to a conformal transformation. So, by the modified definition, a diffeomorphism φ, between two Finsler manifolds (M,F) and $(\tilde{M},\tilde{F})$, is said to be concircular if it maps geodesic circles to geodesic circles. Also, two Finsler metrics defined on a manifold are said to be concircular if they have the same geodesic circles.

In [4], Bidabad-Joharinad studied conformal vector fields on Finsler spaces. They showed that every vector field on a Finsler space which keeps geodesic circles (concircular vector fiels) invariant is conformal. An arbitrary vector field $V = v^i(x)\partial/\partial x^i$ on a Finsler manifold $(M,\mathbf{g})$ is said to be concircular if

$$\mathcal{L}_{\hat{V}}\mathbf{g} = 2\rho\mathbf{g}, \qquad \nabla\rho + \mathbf{g}(G,\dot{\nabla}\rho) = \phi\mathbf{g}, \tag{1.1}$$

where, ∇ and $\dot{\nabla}$ are the Cartan horizontal and vertical covariant derivatives respectively. Also, $\rho = \rho(x)$ is a real function on M called characteristic function of V. Here, $\hat{V}$ is the complete lift of V, i.e., $\hat{V} = v^i(x)\partial/\partial x^i + y^j(\partial_j v^i)\partial/\partial y^i$. They find a necessary and sufficient condition for a vector field to keep geodesic circles invariant. This leads to a significant definition of concircular vector fields on a Finsler space. They classified complete Finsler spaces admitting a special conformal vector fields.

The Liouville theorem explains that every conformal transformation between two open neighborhoods of n-dimensional Euclideann-space ($n \geq 3$) is a combination of inversion and similarity. In [6], Brinkmann proved that a conformal transformation of an Einstein metric on the Riemannian manifold $(M,\mathbf{g})$ remains Einstein if and only if the gradient of the conformal characteristic function ρ of this transformations satisfies the ODE: $D_iD_j\rho + k\rho g_{ij} = 0$, where D is the Levi-Civita connection and $k = k(x)$ is a constant equal to the scalar curvature. In [28], Yano proved that the ODE holds for the characteristic function of a conformal transformation if and only if this transformation, which he called concircular, leaves invariant geodesic circles.

An n-dimensional Finsler manifold (M,F) is called an Einstein manifold if $\mathbf{Ric} = (n-1)k(x)F^2$, where $k = k(x)$ is a scalar function on M. In this paper, we study conformal vector fields on Einstein Finsler manifold and prove the following.

Theorem 1.1. *Let (M,F) be an Einstein Finsler manifold. Then every conformal vector field V on M reduces to a concircular vector field* (1.1) *if and only if there are scalar functions $\lambda = \lambda(x)$ and $\Psi = \Psi(x,y)$ on M and TM, respectively, such that the following hold:*

$$2(n-2)(\lambda g_{ij} - \dot{A}^r{}_{ij}\rho_r) + \Big[\Psi + 4(n-1)k\rho\Big]g_{ij} + \Psi_{ij}F^2 + 2(\Psi_i y_j + \Psi_j y_i) = 0, \tag{1.2}$$

$$\rho^r C^k{}_{ri} = 0. \tag{1.3}$$

where ${C^k}_{ri}$ *and* ${\dot{A}^r}_{ij}$ *denote the Cartan torsion and Landsberg curvature of* F. *Here,* $y_i := FF_{y^i}$, $\rho_r := \partial\rho/\partial x^r$, $\rho^r := g^{ri}\rho_i$, $\Psi_i := \Psi_{y^i}$ *and* $\Psi_{ij} := \Psi_{y^i y^j}$.

2. **Preliminary**

Let M be an $n-$dimensional C^∞ manifold. We denote by $\pi : TM \to M$ the bundle of tangent vectors and by $\pi_0 : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A Finsler structure on M is a function $F : TM \to [0, \infty)$, with the following properties:

i) F is C^∞ on $TM_0 := TM - \{0\}$;
ii) $F(x, y)$ is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y)$, $\forall\lambda > 0$, where we denote an element of TM by (x, y);
iii) The Hessian matrix of $F^2/2$ is positive definite on TM_0;

$$g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

A Finsler manifold (M, g) is a pair of a differential manifold M and a tensor field $g = (g_{ij})$ on TM which defined by a Finsler structure F.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

The spray of a Finsler structure F is a vector field on TM as:

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}. \tag{2.1}$$

where G^i are called the spray (or geodesic) coefficients

$$G^i = \frac{1}{4}g^{il}\Big\{F^2_{x^m y^l}y^m - F^2_{x^l}\Big\} \tag{2.2}$$

and $(g^{ij}) := (g_{ij})^{-1}$. The geodesics of F are characterized by the second order differential equation:

$$\frac{d^2c^i}{dt^2} + 2G^i(c(t), \dot{c}(t)) = 0.$$

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := {B^i}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$${B^i}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature of the Finsler metric F. We call a Finsler metric F a Berwald metric, if $\mathbf{B}=0$.

For $y \in T_xM$, define the Landsberg curvature $\dot{\mathbf{A}}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\dot{\mathbf{A}}_y(u,v,w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w),y\big).$$

F is called a Landsberg metric if $\dot{\mathbf{A}}_y = 0$. By definition, every Berwald metric is a Landsberg metric.

We denote here by G^i_j the coefficients of nonlinear connection on TM, where

$$G^i_j := \frac{\partial G^i}{\partial y^j}.$$

By means of this nonlinear connection tangent space can be split into the horizontal and vertical subspaces with the corresponding basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, which are related to the typical bases of TM, $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$, by

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G^j_i\frac{\partial}{\partial y^j},$$

where

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\Big(\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{kl}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^i}\Big).$$

The components of the hh-curvature of Chern connection are expressed here by

$$R^i_{jkl} = \frac{\delta\Gamma^i_{jl}}{\delta x^k} + \frac{\delta\Gamma^i_{jk}}{\delta x^l} - \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk}.$$

Here, we denote by ∇ and $\dot{\nabla}$, horizontal and vertical covariant derivatives in Cartan connection.

The following relations between Chern connection ∇ and Berwald connections D hold

$$\begin{aligned} D_iY^k &= \nabla_iY^k + \nabla_0C^k_{ir}Y^r = \nabla_iY^k + L^k_{ir}Y^r \\ D_iY_k &= \nabla_iY_k + \nabla_0C_{kir}Y^r = \nabla_iY^k - L_{kir}Y^r \\ &= \nabla_iY^k - L^r_{ki}Y_r \end{aligned} \tag{2.3}$$

The Riemann curvature $R_y : T_pM \to T_pM$ is a linear transformations on tangent spaces, which is defined by

$$R_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i} \tag{2.4}$$

$$R^i_k := 2\frac{\partial G^i}{\partial x^i} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.5}$$

For a two-dimensional plane $P \subset T_pM$ and $y \in T_pM \setminus \{0\}$ such that $P = span\{y, u\}$, the pair $\{P, y\}$ is called a flag in T_pM. The flag curvature $\mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y\big(u, R_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

We say that F is of scalar curvature if for any $y \in T_pM \setminus \{0\}$ the flag curvature $\mathbf{K}(P, y) = \lambda(y)$ is independent of P containing y. This is equivalent to the following condition in a local coordinate system (x^i, y^i) in TM:

$$R^i_k = \lambda F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If λ is a constant, then F is said to be of constant curvature.

Let $(M, \mathbf{g})$ be a Finsler manifold. A vector field $V = v^i(x)\frac{\partial}{\partial x^i}$ on M is said to be concircular if

$$\begin{aligned} &\mathcal{L}_{\hat{V}}\mathbf{g} = 2\rho\mathbf{g}, \\ &\nabla\rho + \mathbf{g}(G, \dot{\nabla}\rho) = \phi\mathbf{g}, \end{aligned} \tag{2.6}$$

where ∇ and $\dot{\nabla}$ are the Cartan horizontal and vertical covariant derivatives, respectively, and ϕ is a smooth function on M. Here,

$$\hat{V} = v^i(x)\frac{\partial}{\partial x^i} + y^j(\partial_j v^i)\frac{\partial}{\partial y^i}$$

is the complete lift of V. For the other vector fields see [18].

In a local coordinate system equation (2.6) is written in the following form

$$\nabla_k\rho_l + G^j_k\dot{\nabla}_j\rho_l = \phi g_{kl}. \tag{2.7}$$

The vector field V is said to be concircular, if its local flow preserves geodesic circles. V is said to be a conformal vector field or an infinitesimal conformal transformation, if it satisfies

$$\mathcal{L}_{\hat{V}}g_{ij} = 2\rho(x)g_{ij},$$

where $\rho(x)$ is a real function on M called characteristic function of V . If $\rho(x)$ is constant or zero, then V is said to be homothetic or Killing.

In [4], Bidabad-Joharinad studied conformal vector fields on Finsler spaces. They showed that every vector field on a Finsler space which keeps geodesic circles invariant is conformal. They find a necessary and sufficient condition for a vector field to keep geodesic circles invariant. This leads to a significant definition of concircular vector fields on a Finsler space. They classified complete Finsler spaces admitting a special conformal vector fields.

In [23], Shen-Yang found a necessary and sufficient condition for a conformal vector field to be concircular vector field. On a Finsler manifold, a conformal vector field with the conformal factor ρ is concircular if and only if ρ satisfies

$$\rho_{i|j} = \lambda g_{ij}, \quad \rho^r C^k_{ri} = 0, \tag{2.8}$$

where

$$\rho_i := \rho_{x^i}, \quad \rho^i := g^{ir}\rho_r,$$

and $\lambda = \lambda(x)$ is a scalar function on M and the symbol "|" means the horizontal covariant derivative of Cartan (or Chern) connection.

Proposition 2.1. ([4]) *Let V be a conformal vector field on the Finsler manifold $(M, \mathbf{g})$. Then the following holds*

$$\mathcal{L}_{\hat{V}}(\boldsymbol{Ric})_{ij} = -(n-2)\Big(\nabla_i\nabla_j\rho - \rho_m \dot{A}^m_{ij}\Big) - \Psi g_{ij} - y_i\Psi_j - y_j\Psi_i - \frac{1}{2}F^2\Big(\dot{\partial}_i\dot{\partial}_j\Psi\Big), \tag{2.9}$$

where $\Psi = \Psi(x,y)$ is a homogeneous and scalar function on TM of degree zero in y.

Now, we can prove the main result of this paper.

Proof of Theorem 1.1: If V is conformal vector field it satisfies

$$\mathcal{L}_{\hat{V}} g_{ij} = 2\rho(x) g_{ij},$$

where $\rho = \rho(x)$ is a real function on M called characteristic function of V. On the other hand according to the Proposition 2.1 we have (2.9). Contracting (2.9) with $y^i y^j$ implies that

$$\mathcal{L}_{\hat{V}}(\mathbf{Ric}) = -(n-2)\Big(\nabla_0\nabla_0\rho(x)\Big) - \Psi F^2. \tag{2.10}$$

(M, F) is Einstein-Finsler manifold, that is

$$\mathbf{Ric} = (n-1)k(x)F^2$$

for scalar function $k = k(x)$. Thus

$$\mathbf{Ric}_{ij} = (n-1)k(x)g_{ij}. \tag{2.11}$$

Taking a Lie derivative of (2.11) along $\hat{V}$ yields

$$\begin{aligned}\mathcal{L}_{\hat{V}}(\mathbf{Ric}_{ij}) &= (n-1)\Big\{(\mathcal{L}_{\hat{V}} k(x))g_{ij} + k(x)\mathcal{L}_{\hat{V}} g_{ij}\Big\} \\ &= (n-1)\Big\{V.k(x)g_{ij} + 2k(x)\rho(x)g_{ij}\Big\}.\end{aligned} \tag{2.12}$$

By (2.10) and (2.12), we get

$$-(n-2)(\nabla_0\nabla_0\rho) - \Psi F^2 = (n-1)\Big\{V.k(x) + 2k(x)\rho(x)\Big\}F^2. \tag{2.13}$$

(2.13) is equal to following

$$2(n-1)k(x)\rho(x)F^2 + (n-1)V.k(x)F^2 + \Psi F^2 + (n-2)(\nabla_0\nabla_0\rho) = 0. \quad (2.14)$$

Taking twice vertical derivative of (2.14) implies that

$$2(n-2)D_iD_j\rho + \Psi g_{ij} + 4(n-1)k(x)\rho(x)g_{ij} + \Psi_{ij}F^2 + 2\Psi_i y_j + 2\Psi_j y_i = 0. \quad (2.15)$$

According to (2.3), the following holds

$$D_iD_j\rho = \nabla_i\rho_j - L^r_{ij}\rho_r. \quad (2.16)$$

By putting (2.16) in (2.15) we obtain

$$2(n-2)(\nabla_i\rho_j - L^r_{ij}\rho_r) + \Psi g_{ij} + 4(n-1)k(x)\rho(x)g_{ij} + \Psi_{ij}F^2 + 2\Psi_i y_j + 2\Psi_j y_i = 0. \quad (2.17)$$

By considering (2.8), we get the proof. □

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