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Funk-type Finsler metrics

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Abstract. In this paper, we introduce a class of Finsler metrics with interesting curvature properties. Then we find necessary and sufficient condition under which these Finsler metrics are locally dually flat and Douglas metrics.

Keywords: Locally dually flat Finsler metrics; Douglas metric; projectively flat Finsler metric.

1. Introduction

The 4-th Hilbert's Problem is related to characterize the distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are shortest paths [13]. Very soon, it turns out that there are lots of solutions to the problem. For 2- and 3-dimensions cases, one can see [1], [2], [6], [17] and [30].

Distance functions induced by a Finsler metrics are regarded as smooth ones. The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Such Finsler metrics are called projective Finsler metrics. In [12], Hamel characterized projective Finsler metrics by a system of PDE's. It is well-known that every projective Finsler metric has scalar flag curvature, namely, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function of tangent vectors. In [8] and [9], Funk classified all projective Finsler metrics with constant curvature on convex domains in $\mathbb{R}^2$. With additional conditions, he showed that the standard Riemannian metric is the unique of projectively Finsler metrics with $\mathbf{K} = 1$ on $\mathbb{S}^2$ [10]. Based

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on the mentioned research, he obtained an interesting class of non-Riemannian projectively flat Finsler metric, namely, Funk metrics.

Let Ω be a strongly convex bounded domain in $\mathbb{R}^n$. For $p, q \in \Omega$, let ℓ_{pq} denote the ray issuing from p to q passing through q. Define

$$d_f(p,q) := \ln \frac{|z-p|}{|z-q|}, \tag{1.1}$$

where $z \in \partial\Omega$ is the intersection point of ℓ_{pq} with $\partial\Omega$. Then d_f is an inner metric on Ω, which is called the *Funk metric*. The Funk metric d_f in (1.1) is regular and the induced Finsler metric F_f is determined by

$$x + \frac{y}{F_f(y)} \in \partial\Omega, \qquad y \in T_x\Omega, \tag{1.2}$$

where $\Omega \subseteq \mathrm{R}^n$ is a strongly convex bounded domain. The following holds

$$x + \frac{y}{F_f(y)} \in \partial\Omega, \quad \Longleftrightarrow \quad \left\| x + \frac{y}{F_f(y)} \right\|^2 = 1.$$

Rewriting this condition as

$$F_f^2(1 - \|x\|^2) - 2F_f \cdot \langle x, y\rangle - \|y\|^2 = 0$$

the non-negative root of this quadratic equations is given by

$$F_f(x,y) = \frac{\sqrt{\langle x,y\rangle^2 + (1-\|x\|^2)\|y\|^2} + \langle x,y\rangle}{(1-\|x\|^2)}. \tag{1.3}$$

The Funk metric is the most important metric in Finsler Geometry. It is non-reversible, positively complete and projectively flat metric of constant flag curvature $\mathbf{K} = -\frac{1}{4}$. It is easy to show that the Funk metric satisfies:

$$F_{x^k} = F F_{y^k}.$$

The Hilbert metric is regular too and its induced Finsler metric F_h is determined by

$$F_h(y) := \frac{1}{2}\Big(F_f(y) + F_f(-y)\Big). \tag{1.4}$$

Symmetrizing the metric (1.3), we obtain the Hilbert metric in the unit ball $\mathbb{B}^n$ as follows

$$F_h(x,y) = \frac{\sqrt{(1-\|x\|^2)\|y\|^2 + \langle x,y\rangle^2}}{(1-\|x\|^2)}. \tag{1.5}$$

Observe that this is a Riemannian metric. It is reversible, complete and projectively flat with constant flag curvature $\mathbf{K} = -1$.

In [27], Shen conjectured that there exist non-trivial positively complete, projectively flat Finsler metrics of constant curvature $\mathbf{K} = 0$. In [28], he proved the existence of projectively flat Finsler metrics of curvature $\mathbf{K} = 0$ by constructing a projectively flat and R-flat spray using the Funk metrics. This fact proves the importance of Funk metrics and shows that it deserve to study these metrics for more deep progress in Finsler geometry.

In this paper, inspired by the related PDE for Funk metrics, we are going to study a family of Finsler metrics that satisfy an special partial differential equation.

Definition. *Let (M, F) be a Finsler manifold. Then F is called a Funk-type Finsler metric if it satisfies the following PDE*

$$F_{x^k} = gFF_{y^k} + hF^2F_{y^m y^k}x^m, \tag{1.6}$$

where $g = g(r)$ and $h = h(r)$ are two functions and

$$r := F_{y^j}x^j.$$

Some interesting metrics belong to this class of Finsler metric. Obviously, the Funk metrics satisfy (1.6) with $g = 1$ and $h = 0$. Thus, these Finsler metrics can be viewed as a generalization of Funk metric. Many well-known Finsler metrics belong to this class.

We have the following interesting Riemannian metrics defined on $\mathbb{B}^n$

$$F = \frac{\sqrt{|y|^2 - |x|^2|y|^2 + < x, y >^2)}}{\sqrt{1 - |x|^2}},$$
$$\bar{F} = \frac{|y|}{\sqrt{1 - |x|^2}}.$$

F has constant curvature $\mathbf{K} = 1$. It is easy to show that

$$F_{x^k} = F_0 F F_{y^k},$$
$$\bar{F}_{x^k} = \bar{F}_0 \bar{F} \bar{F}_{y^k} + \bar{F}^2 \bar{F}_{y^m y^k} x^k,$$

where $F_0 := F_{y^m}x^m$ and $\bar{F}_0 := \bar{F}_{y^m}x^m$.

Let (M, F) be a Finsler manifold. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}$$

are called the spray coefficients. The special forms of the spray coefficients of a Finsler metric can define some interesting classes of Finsler metrics as follows:

(i) A Finsler metric $F = F(x, y)$ on $\mathcal{U}$ is projective if and only if its geodesic coefficients G^i are in the form

$$G^i(x, y) = P(x, y)y^i,$$

where $P : \mathcal{U} = \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one.

(ii) F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k.$$

(iii) As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [3]. F is called a Douglas metric if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i.$$

(iv) A Finsler metric $F = F(x,y)$ on a manifold is locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j}.$$

In this paper, we consider the class of Funk-type Finsler metrics. We find the necessary and sufficient condition under which a Funk-type Finsler metric is projectively flat, locally dually flat and Douglas metric. More precisely, we prove the following.

Theorem 1.1. *Let $F = F(x,y)$ be a Finsler metric satisfies (1.6) on a manifold M. Then the following hold*

(a) *F is a projectively flat Finsler metric if and only if $g'(r) = 2h(r)$;*
(b) *F is a locally dually flat Finsler metric if and only if $g'(r) = 3h(r)$;*
(c) *F is a Douglas metric if and only if*

$$g'(r) - 2h(r) = \frac{H_{00}}{F^2} \tag{1.7}$$

where $H_{00}(x,y) := H_{ij}(x)y^i y^j$ is a homogeneous polynomial in (y^i) of degree two.

Taking a trace of Berwald curvature $\mathbf{B}$ give us the mean Berwald curvature $\mathbf{E}$. A Finsler metric F on an n-dimensional manifold M has isotropic mean Berwald curvature if

$$\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h},$$

where $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric and $c = c(x)$ is a scalar function on M. It is easy to see that the Funk metric has isotropic mean Berwald curvature $c = 1/2$. It is interesting to find the form of mean Berwald curvature of Funk-type Finsler metric. Then, we prove the following.

Theorem 1.2. *Let $F = F(x,y)$ be a special Funk-type Finler metric in $\mathbb{R}^n$ such that satisfies*

$$F_{x^k} = \big(t(n-1)r + q\big)FF_{y^k} - tF^2F_{y^m y^k}x^m,$$

where t and q are real constants. Then F has isotropic mean Berwald curvature.

2. **Preliminary**

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$; (iii) The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positively defined on TM_0

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then the pair (M, F) is called a Finsler manifold.

For a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l}\Big\}, \quad y \in T_xM. \tag{2.1}$$

$\mathbf{G}$ is called the associated spray to (M, F). F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for all $x \in M$.

Define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For $y \in T_xM_0$, define $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{E}_y(u, v) := E_{ij}(y) u^i v^j,$$

where

$$E_{ij} := \frac{1}{2} B^m{}_{ijm},$$

$u = u^i \frac{\partial}{\partial x^i}|_x$ and $v = v^i \frac{\partial}{\partial x^i}|_x$. The non-Riemannian quantity $\mathbf{E}$ is called the mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$.

For a two-dimensional plane $P \subset T_xM$ and $y \in T_xM_0$, the flag mean Berwald curvature $\mathbf{E}(P, y)$ is defined by

$$\mathbf{E}(P, y) := \frac{F^3(x, y)\mathbf{E}_y(u, u)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2},$$

where $P := span\{y, u\}$. F is called of isotropic mean Berwald curvature if for any flag (P, y), the following holds

$$\mathbf{E}(P, y) = \frac{n+1}{2} c \quad \Longleftrightarrow \quad E_{ij} = \frac{n+1}{2} c F_{y^i y^j} \quad \Longleftrightarrow \quad E_{ij} = \frac{n+1}{2} c F^{-1} h_{ij}, \tag{2.2}$$

where $c = c(x)$ is a scalar function on M. The Funk metrics have isotropic mean Berwald curvature with $c = \frac{1}{2}$.

The Douglas metrics are extension of Berwald metrics, which introduced by Douglas as a projective invariant in Finsler geometry. A Finsler metric is called a Douglas metric if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i,$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ is a scalar function on M and $P = P(x,y)$ is a homogeneous function of degree one with respect to y on TM_0 (see [20], [21] and [22]).

Also, by using the Berwald and mean Berwald curvatures of F, one can define the Douglas curvature $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\Big\}.$$

The Finsler metric F satisfies $\mathbf{D} = 0$ is called a Douglas metric. Equivalently, a Finsler metric is a Douglas metric if and only if $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

A distance function on a set U is a function $d : U \times U \to \mathbb{R}$ with the following properties

(a): $d(p,q) \geq 0$ and equality holds if and only if $p = q$;
(b): $d(p,q) \leq d(p,r) + d(r,q)$.

A distance function on a convex domain $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projective* (or *rectilinear*) if straight lines are shortest paths. The Hilbert's Fourth Problem is to characterize projective distance functions.

A distance function d on a manifold M is said to be smooth if it is induced by a Finsler metric F on M,

$$d(p,q) := \inf_c \int_0^1 F(\dot{c}(t))dt,$$

where the infimum is taken over all curves $c(t)$, $0 \leq t \leq 1$, joining $p = c(0)$ to $q = c(1)$. Thus smooth distance functions can be studied using calculus.

Now we start to discuss smooth projective distance functions, or projective Finsler metrics on an open domain $\mathcal{U} \subset \mathbb{R}^n$. First, let us use the following notations. The local coordinates of a tangent vector $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_p \in T_x\mathcal{U}$ will be denoted by (x,y). Hence all quantities are functions of $(x,y) \in \mathcal{U} \times \mathbb{R}^n$. It is known that a Finsler metric $F = F(x,y)$ on $\mathcal{U}$ is projective if and only if its geodesic coefficients G^i are in the form

$$G^i(x,y) = P(x,y)y^i,$$

where $P : \mathcal{U} = \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. The scalar function $P = P(x, y)$ is called the projective factor of F. The following lemma plays an important role for studying the projective Finsler metrics.

Lemma 2.1. (Rapcsák [18]) *Let $F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. $F(x, y)$ is projective on $\mathcal{U}$ if and only if it satisfies*

$$F_{x^k y^l} y^k = F_{x^l}. \tag{2.3}$$

In this case, the projective factor $P(x, y)$ is given by

$$P = \frac{F_{x^k} y^k}{2F}. \tag{2.4}$$

In [12], Hamel proved that a Finsler metric $F = F(x, y)$ on $\mathcal{U} \subset \mathbb{R}^n$ is projective if and only if

$$F_{x^k y^l} = F_{x^l y^k}. \tag{2.5}$$

Thus (2.4) and (2.3) are equivalent.

Let $F(x, y)$ be a projective Finsler metric on $\mathcal{U} \subset \mathbb{R}^n$ and $P(x, y)$ its projective factor. Let us put

$$\Xi := P^2 - P_{x^k} y^k. \tag{2.6}$$

The Riemann curvature $\mathbf{R_y} = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.7}$$

Plugging $G^i = P y^i$ into (2.7) yields

$$R^i{}_k = \Xi\, \delta^i_k + \tau_k\, y^i, \tag{2.8}$$

where

$$\tau_k = 3(P_{x^k} - P P_{y^k}) + \Xi_{y^k}. \tag{2.9}$$

See [24] for more discussion. It follows from (2.8) and (2.9) that the Ricci curvature $\mathbf{Ric} := R^k{}_k$ is given by

$$\mathbf{Ric} = (n-1)\Xi. \tag{2.10}$$

By the symmetry property that

$$g_{ji} R^i{}_k = g_{ki} R^i{}_j,$$

one can show that

$$R^i{}_k = \Xi\Big\{\delta^i_k - F^{-1} F_{y^k} y^i\Big\}. \tag{2.11}$$

Comparing (2.9) and (2.11), we obtain

$$P_{x^k} - P P_{y^k} = -\frac{(\Xi F)_{y^k}}{3F}. \tag{2.12}$$

From (2.10) and (2.11), we immediately obtain the following

Lemma 2.2. *For a locally projectively flat Finsler metric F on an n-manifold M, the flag curvature and the Ricci curvature are related by*

$$\mathbf{K}(P, \mathbf{y}) = \frac{1}{n-1} \frac{\mathbf{Ric}(\mathbf{y})}{F^2(\mathbf{y})}, \qquad \mathbf{y} \in P \subset T_pM.$$

It follows from Lemma 2.2 that a locally projectively flat Finsler metric has constant Ricci curvature if and only if it has constant flag curvature.

In [23], Shen studied projectively flat Finsler metrics of constant flag curvature. He found the following lemma to determine the local metric structure of projective Finsler metrics with constant curvature.

Lemma 2.3. *Let $F = F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then F is projective if and only if there is a positively homogeneous function with degree one, $P = P(x, y)$, and a positively homogeneous function of degree zero, $\lambda(x, y)$, on $T\mathcal{U} = \mathcal{U} \times \mathbb{R}^n$ such that*

$$F_{x^k} = (PF)_{y^k} \tag{2.13}$$

$$P_{x^k} = PP_{y^k} - \frac{1}{3F}(\lambda F^3)_{y^k}. \tag{2.14}$$

In this case, $P = \frac{1}{2}F^{-1}F_{x^k}y^k$ and F is of scalar curvature $\mathbf{K} = \lambda$.

There is another important notion in Finsler geometry, that is locally dually flat Finsler metrics. A Finsler metric $F = F(x, y)$ on a manifold is locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j} \tag{2.15}$$

Dually flat Finsler metrics on an open subset in $\mathbb{R}^n$ can be characterized by a simple PDE.

Lemma 2.4. [26] *A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is locally dually flat if and only if it satisfies the following equations:*

$$\left[F^2\right]_{x^m y^k} y^m = 2\left[F^2\right]_{x^k}$$

In this case, local function $H = H(x, y)$ in (2.15) is given by

$$H = \frac{1}{6}\left[F^2\right]_{x^m} y^m.$$

3. Proof of Theorems

In this section, we are going to prove theorem 1.1. For this aim, we need the following Lemma.

Lemma 3.1. *Let $F = F(x, y)$ be a Finsler metric on a manifold M. Suppose that F satisfies (1.6). Then the following hold*

$$F_{x^m y^k} y^m - F_{x^k} = \Big(g'(r) - 2h(r)\Big) F^2 F_{y^m y^k} x^m, \tag{3.1}$$

$$\big[F^2\big]_{x^m y^k} y^m - \big[F^2\big]_{x^k} = 2g(r)F^2 F_{y^k} + 2\Big(g'(r) - 2h(r)\Big) F^3 F_{y^m y^k} x^m, \tag{3.2}$$

$$\big[F^2\big]_{x^m y^k} y^m - 2\big[F^2\big]_{x^k} = 2\Big(g'(r) - 3h(r)\Big) F^3 F_{y^m y^k} x^m, \tag{3.3}$$

$$G^i = \frac{F}{2}\Big(g(r) - rg'(r) + 2rh(r)\Big) y^i + \frac{F^2}{2}\Big(g'(r) - 2h(r)\Big) x^i. \tag{3.4}$$

Proof. Differentiating (1.6) with respect to y^j and contracting it with y^k, we get

$$F_{x^k} y^k = g(r)F^2, \tag{3.5}$$

$$F_{x^k y^j} y^k = g(r)FF_{y^j} + F^2\Big(g'(r) - h(r)\Big) F_{y^k y^j} x^k. \tag{3.6}$$

By (1.6), (3.5) and (3.6) we have

$$\big[F^2\big]_{x^j} = 2g(r)F^2 F_{y^j} + 2h(r)F^3 F_{y^k y^j} x^k, \tag{3.7}$$

$$\big[F^2\big]_{x^m y^j} y^m = 4g(r)F^2 F_{y^j} + 2\Big(g'(r) - h(r)\Big) F^3 F_{y^k y^j} x^k. \tag{3.8}$$

From (1.6), (3.6), (3.7) and (3.8) we obtain (3.1), (3.2) and (3.3). It is easy to see that

$$g^{ik} F_{y^m y^k} = g^{ik}\Big(\frac{g_{mk} - F_{y^m} F_{y^k}}{F}\Big) = \frac{\delta^i_m}{F} - \frac{F_{y^m}}{F^2} y^i \tag{3.9}$$

Then by contracting (3.2) with g^{ik} we get (3.4). □

Proof of Theorem 1.1: By Lemma 2.1 and (3.1) one can prove (a). lemma 2.4 and (3.3) suffice to prove (b). It follows from (3.4) that

$$G^i y^j - G^j y^i = \frac{F^2}{2}\Big(g'(r) - 2h(r)\Big)(x^i y^j - x^j y^i) \tag{3.10}$$

It is known that F is a Douglas metric if and only if $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. Then by (3.10) we conclude F is a Douglas metric if and only if $F^2(g'(r) - 2h(r))$ are homogeneous polynomials in (y^i) of degree two. This proves (c).

Corollary 3.2. *Let F is a Finsler metric satisfies (1.6) on manifold M. Then F is Douglas metric and locally dually flat metric if and only if*

$$h(r)=\frac{1}{3}g'(r)=\frac{H_{00}}{F^2}, \tag{3.11}$$

where $H_{00}(x,y):=H_{ij}(x)y^iy^j$ is a homogeneous polynomial in (y^i) of degree two.

Proof. According to Theorem 1.1, F is locally dually flat Finsler metric if and only if

$$g'(r)=3h(r), \tag{3.12}$$

and F is a Douglas metric if and only if

$$g'(r)-2h(r)=\frac{H_{00}}{F^2}, \tag{3.13}$$

By (3.12) and (3.13) we get (3.11). □

By Theorem 1.1, we conclude the following

Corollary 3.3. *Let $F=F(x,y)$ be a Finsler metric satisfies (4.1) on a manifold M. Then F is a projectively flat and locally dually flat Finsler metric if and only if $h=0$ and $g=constant$. In this case, F reduces to a Funk metric.*

Proof of Theorem 1.2: By (3.4) we conclude

$$\begin{aligned}\frac{\partial G^m}{\partial y^m} &= \Big(\frac{n+1}{2}\big(g(r)-rg'(r)+2rh(r)\big)+rg'(r)-2rh(r)\Big)F\\ &+ \frac{F^2}{2}\Big(g''(r)-2h'(r)\Big)F_{y^jy^k}x^jx^k. \end{aligned} \tag{3.14}$$

Substituting

$$\begin{aligned} g(r) &= t(n-1)r+q,\\ h(r) &= -t \end{aligned}$$

in (3.14) yields

$$\frac{\partial G^m}{\partial y^m}=\frac{(n+1)q}{2}F \tag{3.15}$$

Thus we conclude F is of isotropic E-curvature. □

4. **Special Funk-type Finsler Metrics**

It is interesting to study a special class of Funk-type metrics. Let (M, F) be a Finsler manifold. Then F is called s special Funk-type metric if it satisfies the following PDE

$$F_{x^k} = g\Big\{FF_{y^k} + F^2 F_{y^m y^k} x^m\Big\}, \tag{4.1}$$

where $g = g(r)$ is a scalar function and

$$r := F_{y^j} x^j.$$

By Theorem 1.1, we get the following

Corollary 4.1. *Let (M, F) be a compact Finsler manifold. Suppose that F satisfies (4.1). Then the following hold*

(a) *F is a projectively flat metric if and only if it is locally Minkowskian metric;*

(b) *F is a locally dually flat Finsler metric if and only if locally Minkowskian metric.*

Proof. By Theorem 1.1, F is a projectively flat Finsler metric if and only if

$$g'(r) = 2g(r).$$

It follows that

$$g(r) = e^{2r} g(0).$$

Letting $t \to \pm\infty$ and considering $g < \infty$ implies that $g(0) = 0$ and then $g = 0$. In this case, we get $F_{x^k} = 0$ and $P = 0$. Thus $G^i = 0$ and F reduces to a locally Minkowskian metric.

For the case (b), we use the same argument. □

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