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Superconnections and distributions

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Abstract. The use of a distribution $\mathcal{D}$ allows the presence of geometric structures such as almost product structure, so that the equivalent of these structures can be seen in tangent supermanifolds. We define associated adapted linear superconnections and find all linear superconnections on the supermanifold $\mathcal{M}$ adapted to $\mathcal{D}$.

Keywords: Supermanifold, distribution, almost product supermanifold, linear superconnection.

1. **Introduction**

In this paper we study relationships between distributions and linear connections on a supermanifold. There are many papers that have studied the relationship between linear connections and distributions on a manifold. Given a distribution and an affine connection on a manifold, considering a product for vector fields allows one to test for geodesic invariance in the same way one uses the Lie bracket to test for integrability. If the affine connection does not restrict to the distribution, one can define its restriction and in the process generalise the notion of the second fundamental form for submanifolds, see [3].

The existence of an adapted linear connection to the distribution $\mathcal{D}$ can be used to study foliated Riemannian manifolds. In [1], the author presented the

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Vranceanu connection on a foliated Riemannian manifold and showed that a totally geodesic foliation with bundle-like metric is characterized by means of this connection.

In the graded case, the situation is different and with the presence of odd variables, the above results are not necessarily obtained.
A supermanifold (or a graded manifold) of dimension (r, s) is a ringed space $\mathcal{M} = (M, \mathcal{O}_M)$ where M is a topological space (Hausdorff, countable base) and the structural sheaf $\mathcal{O}_M$ is a sheaf of super $R-$algebras with unity, locally isomorphic to $R^{r,s}$. Let U be an open subset of $x \in M$, we denote the local coordinates of $\mathcal{M}$ in $(U, \mathcal{O}_M(U))$ by $(x^a), a = -s, ..., -1, 1, ..., r$. The coordinates $(x^a), a = 1, ..., r$ are called the even coordinates and the $(x^a), a = -s, ..., -1$ are called the odd coordinates, (see [4] and [5] for more information).

If $\mathcal{M}$ is a supermanifold, one can then define vector fields on $\mathcal{M}$ as derivations of the sheaf $\mathcal{O}_M$. The derivations form a sheaf of modules over the structure sheaf $\mathcal{O}_{\mathcal{M}}$, called the tangent sheaf of $\mathcal{M}$, denoted by $\mathcal{T}_{\mathcal{M}}$. The dual of $\mathcal{T}_{\mathcal{M}}$ is denoted by $\Omega^1_{\mathcal{M}}$ and is called the cotangent sheaf of a supermanifold $\mathcal{M}$, [2].

Let $(M, \mathcal{O}_M)$ be an $(n+p, m+q)-$dimensional paracompact supermanifold and $\mathcal{TM}$ be the tangent superbundle of $\mathcal{M}$ (for more details see [2] and [6]). A distribution over $\mathcal{M}$ is a graded subsheaf $\mathcal{D}$ of $\mathcal{T}_{\mathcal{M}}$ which is locally a direct factor [6].

The use of a distribution $\mathcal{D}$ allows the presence of geometric structures such as almost product structure, so that the equivalent of these structures can be seen in tangent supermanifolds. We will define associated adapted linear superconnections and find all linear connections on the supermanifold $\mathcal{M}$ adapted to $\mathcal{D}$.

2. **Linear superconnections which restrict to a distribution**

Given a distribution $\mathcal{D}$ over $\mathcal{M}$ and $x \in M$, there exists an open neighborhood U of x and another subsheaf $\mathcal{D}' \subset \mathcal{TM}$ so that for all $y \in U$

$$\mathcal{T}_{\mathcal{M},y} = \mathcal{D}_y \oplus \mathcal{D}'_y. \tag{2.1}$$

If we denote by $\mathcal{P}$ and $\mathcal{P}'$ the projection morphisms of $\mathcal{T}_{\mathcal{M}}$ on $\mathcal{D}$ and $\mathcal{D}'$ respectively, then we have the obvious relations

$$\mathcal{P}^2 = \mathcal{P},$$
$$\mathcal{P}'^2 = \mathcal{P}',$$
$$\mathcal{P} + \mathcal{P}' = Id.$$

An almost product structure on a supermanifold is a tensor field F which satisfies the condition

$$F^2 = Id.$$

From the definition of two morphisms $\mathcal{P}$ and $\mathcal{P}'$, we see that

$$F = \mathcal{P} - \mathcal{P}', \tag{2.2}$$

is an almost product structure on $\mathcal{M}$. So $(\mathcal{M}, \mathcal{D}, \mathcal{D}')$ will be called an almost product supermanifold. Note that the the projection morphisms $\mathcal{P}$, $\mathcal{P}'$ and F are even morphisms.

Let $\mathcal{D}$ be a distribution and $x \in \mathcal{M}$, there is an open subset U over which any set of generators

$$\Big\{\chi_i | i = -m, \dots, -1, 1, \dots, n\Big\}$$

of the module $\mathcal{D}(U)$ can be enlarged to a set

$$\Big\{\chi_A\Big\} = \Big\{\chi_i, \chi_\alpha | i = -m, \dots, -1, 1, \dots, n, \quad \alpha = -q, \dots, -1, 1, \dots, p\Big\}$$

of free generators of $Der\mathcal{O}_M(U)$.

Definition. A superconnection on a supermanifold $\mathcal{M}$ is an even morphism

$$\nabla : \mathcal{T}_\mathcal{M} \mapsto \Omega^1_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M}$$

satisfying the following condition

$$\nabla(f.v) = df \otimes v + f.\nabla(v),$$

where $f \in \mathcal{O}_\mathcal{M}(U)$ and $v \in \mathcal{T}_\mathcal{M}(U)$, (see [2]).

Let $\mathcal{D}$ be an $(n, m)-$distribution on an $(n + p, m + q)-$dimensional supermanifold $\mathcal{M} = (M, \mathcal{O}_M)$. A superconnection $\overline{\nabla}$ on $\mathcal{M}$ is said to be adapted to $\mathcal{D}$ if

$$\overline{\nabla}_X Y \in \mathcal{D}, \text{ for all } X \in Der\mathcal{O}_M, Y \in \mathcal{D}.$$

A superconnection $\overline{\nabla}$ on $(\mathcal{M}, \mathcal{D}, \mathcal{D}')$ is said to be an adapted superconnection if it is adapted to both distributions $\mathcal{D}$ and $\mathcal{D}'$ i.e., for each $X, Y \in Der\mathcal{O}_M$, we have

$$\overline{\nabla}_X \mathcal{P}Y \in \mathcal{D}, \quad \overline{\nabla}_X \mathcal{P}'Y \in \mathcal{D}'. \tag{2.3}$$

Here, we prove the following.

Theorem 2.1. *Given two superconnections* (∇, ∇')*, where* ∇ *and* ∇' *on* $\mathcal{D}$ *and* $\mathcal{D}'$ *respectively, there exists an adapted linear superconnection* $\overline{\nabla}$ *on* $(\mathcal{M}, \mathcal{D}, \mathcal{D}')$.

Proof. It suffices to consider

$$\overline{\nabla}_X Y = \nabla_X \mathcal{P}Y + \nabla'_X \mathcal{P}'Y, \tag{2.4}$$

where $X, Y \in Der\mathcal{O}_M$. □

F is parallel with respect to a linear superconnection $\overline{\nabla}$ on $\mathcal{M}$ if we have

$$(\tilde{\nabla}_X F)Y = \tilde{\nabla}_X FY - F(\tilde{\nabla}_X Y) = 0, \quad \forall X, Y \in Der\mathcal{O}_M. \tag{2.5}$$

The same definition applies for $\mathcal{P}$ and $\mathcal{P}'$.

Theorem 2.2. *Let $\overline{\nabla}$ be a linear superconnection on the almost product supermanifold $(\mathcal{M}, \mathcal{D}, \mathcal{D}')$. Then the following assertions are equivalent:*

(i) *$\overline{\nabla}$ is an adapted linear superconnection.*
(ii) *The almost product structure F is parallel with respect to $\overline{\nabla}$.*
(iii) *The projection morphisms $\mathcal{P}$ and $\mathcal{P}'$ are parallel with respect to $\overline{\nabla}$.*

Proof. $(i \to ii)$ Let $\overline{\nabla}$ be an adapted linear superconnection on the almost product supermanifold $(\mathcal{M}, \mathcal{D}, \mathcal{D}')$. Then there exists a pair (∇, ∇'), where ∇ and ∇' are linear superconnections on $\mathcal{D}$ and $\mathcal{D}'$ respectively, such that

$$\begin{aligned}(\overline{\nabla}_X F)Y &= \overline{\nabla}_X FY - F(\overline{\nabla}_X Y) \\ &= \overline{\nabla}_X(\mathcal{P}Y - \mathcal{P}'Y) - F(\nabla_X \mathcal{P}Y + \nabla'_X \mathcal{P}'Y) \\ &= (\nabla_X \mathcal{P}Y - \nabla'_X \mathcal{P}'Y) - (\nabla_X \mathcal{P}Y - \nabla'_X \mathcal{P}'Y) \\ &= 0.\end{aligned}$$

$(ii \to i)$ Let F be parallel with respect to $\overline{\nabla}$. If $Y \in \mathcal{D}$ then we have

$$FY = Y$$

and

$$\overline{\nabla}_X FY = \overline{\nabla}_X Y = F(\overline{\nabla}_X Y).$$

If $\overline{\nabla}_X Y = Z + Z'$, $Z \in \mathcal{D}$ and $Z' \in \mathcal{D}'$, then

$$Z + Z' = Z - Z' \Longrightarrow Z' = 0$$

i.e.,

$$\overline{\nabla}_X Y = Z \in \mathcal{D}.$$

Similarly, if $Y \in \mathcal{D}'$, we have

$$\overline{\nabla}_X Y = Z \in \mathcal{D}'.$$

Then $\overline{\nabla}$ is an adapted linear superconnection.

$(ii \to iii)$ Let the almost product structure F is parallel with respect to $\overline{\nabla}$, then

$$\begin{aligned}0 &= \overline{\nabla}_X FY - F(\overline{\nabla}_X Y) \\ &= (\overline{\nabla}_X \mathcal{P}Y - \overline{\nabla}_X \mathcal{P}'Y) - (\mathcal{P}\overline{\nabla}_X Y - \mathcal{P}'\overline{\nabla}_X Y) \\ &= (\overline{\nabla}_X \mathcal{P}Y - \mathcal{P}\overline{\nabla}_X Y) - (\overline{\nabla}_X \mathcal{P}'Y - \mathcal{P}'\overline{\nabla}_X Y).\end{aligned}$$

Thus we get

$$\overline{\nabla}_X \mathcal{P}Y - \mathcal{P}\overline{\nabla}_X Y = 0$$

and

$$\overline{\nabla}_X \mathcal{P}' Y - \mathcal{P}' \overline{\nabla}_X Y = 0.$$

$(iii \to i)$ Since projection morphisms $\mathcal{P}$ and $\mathcal{P}'$ are parallel with respect to $\overline{\nabla}$, we have

$$\overline{\nabla}_X \mathcal{P} Y - \mathcal{P} \overline{\nabla}_X Y = 0$$

and

$$\overline{\nabla}_X \mathcal{P}' Y - \mathcal{P}' \overline{\nabla}_X Y = 0.$$

Thus

$$\overline{\nabla}_X \mathcal{P} Y = \mathcal{P} \overline{\nabla}_X Y \in \mathcal{D}$$

and

$$\overline{\nabla}_X \mathcal{P}' Y = \mathcal{P}' \overline{\nabla}_X Y \in \mathcal{D}'.$$

This completes the proof. □

Now, let ∇ and ∇' be linear superconnections on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Locally on $U \subset M$ we put

$$\begin{aligned}
&(a)\ \nabla_{\chi_j} \chi_i = \Gamma_{ij}^k \chi_k, \\
&(b)\ \nabla_{E_\alpha} \chi_i = \Gamma_{i\alpha}^k \chi_k, \\
&(c)\ \nabla'_{\chi_j} \chi_\alpha = {\Gamma'}_{\alpha j}^{\beta} \chi_\beta, \\
&(d) \nabla'_{\chi_\gamma} \chi_\alpha = {\Gamma'}_{\alpha\gamma}^{\beta} \chi_\beta.
\end{aligned}$$

Thus an adapted linear superconnection $\overline{\nabla}$ on $\mathcal{M}$ is locally given by

$$\begin{aligned}
&(a)\ \overline{\nabla}_{\chi_j} \chi_i = \Gamma_{ij}^k \chi_k, \\
&(b)\ \overline{\nabla}_{\chi_\alpha} \chi_i = \Gamma_{i\alpha}^k \chi_k, \\
&(c)\ \overline{\nabla}_{\chi_j} \chi_\alpha = {\Gamma'}_{\alpha j}^{\beta} \chi_\beta, \\
&(d)\ \overline{\nabla}_{\chi_\gamma} \chi_\alpha = {\Gamma'}_{\alpha\gamma}^{\beta} \chi_\beta.
\end{aligned}$$

Also, the torsion tensor field T^* of the linear superconnection $\overline{\nabla}$ is given by

$$T^*(X,Y) = \overline{\nabla}_X Y - (-1)^{|X||Y|} \overline{\nabla}_Y X - [X,Y], \quad \forall X, Y \in Der\mathcal{O}_M. \tag{2.6}$$

Now, by using (2.1) if we set

$$\begin{aligned}
(a)\ \ T^*(\chi_j, \chi_i) &= T_{ij}^k \chi_k + {T'}_{ij}^{\alpha} \chi_\alpha, \\
(b)\ \ T^*(\chi_\alpha, \chi_i) &= -(-1)^{|\chi_i||\chi_\alpha|} T^*(\chi_i, \chi_\alpha) \\
&= T_{i\alpha}^k \chi_k + {T'}_{i\alpha}^{\beta} \chi_\beta \\
&= -(-1)^{|\chi_i||\chi_\alpha|} (T_{\alpha i}^k \chi_k + {T'}_{\alpha i}^{\beta} \chi_\beta), \\
(c)\ \ T^*(\chi_\gamma, \chi_\alpha) &= T_{\alpha\gamma}^k \chi_k + {T'}_{\alpha\gamma}^{\beta} \chi_\beta,
\end{aligned}$$

then we can obtain all components of T^* as in the following theorem.

Theorem 2.3. *Let $\overline{\nabla} = (\Gamma^k_{iA}, {\Gamma'}^{\beta}_{\alpha A})$ be an adapted linear superconnection on the almost product supermanifold $(\mathcal{M}, \mathcal{D}, \mathcal{D}')$. Then the local components of its torsion tensor field with respect to a nonholonomic frame field $\{\chi_i, \chi_\alpha\}$ are given by*

$$\begin{aligned}
&(a)\ \ T^k_{ij} = \Gamma^k_{ij} - (-1)^{|\chi_i||\chi_j|}\Gamma^k_{ji} - V^k_{ij},\\
&(b)\ \ {T'}^{\alpha}_{ij} = -{V'}^{\alpha}_{ij},\\
&(c)\ \ T^k_{i\alpha} = -(-1)^{|\chi_i||\chi_\alpha|}T^k_{\alpha i} = \Gamma^k_{i\alpha} - V^k_{i\alpha},\\
&(d)\ \ {T'}^{\beta}_{\alpha i} = -(-1)^{|\chi_i||\chi_\alpha|}{T'}^{\beta}_{i\alpha} = {\Gamma'}^{\beta}_{\alpha i} - {V'}^{\beta}_{\alpha i},\\
&(e)\ \ T^k_{\alpha\beta} = -V^k_{\alpha\beta},\\
&(f)\ \ {T'}^{\beta}_{\alpha\gamma} = {\Gamma'}^{\beta}_{\alpha\gamma} - (-1)^{|\chi_\alpha||\chi_\gamma|}{\Gamma'}^{\beta}_{\gamma\alpha} - {V'}^{\beta}_{\alpha\gamma}.
\end{aligned} \tag{2.7}$$

Proof. It is sufficient to let

$$\begin{aligned}
(a)\ \ \mathcal{P}[\chi_j, \chi_i] &= V^k_{ij}\chi_k,\\
(b)\ \ \mathcal{P}[\chi_\beta, \chi_\alpha] &= V^k_{\alpha\beta}\chi_k,\\
(c)\ \ \mathcal{P}[\chi_\alpha, \chi_i] &= -(-1)^{|\chi_i||\chi_\alpha|}\mathcal{P}[\chi_i, \chi_\alpha]\\
&= V^k_{i\alpha}\chi_k\\
&= -(-1)^{|\chi_i||\chi_\alpha|}V^k_{\alpha i}\chi_k,
\end{aligned}$$

and

$$\begin{aligned}
(d)\ \ \mathcal{P}'[\chi_j, \chi_i] &= {V'}^{\beta}_{ij}\chi_\beta,\\
(e)\ \ \mathcal{P}'[\chi_\gamma, \chi_\alpha] &= {V'}^{\beta}_{\alpha\gamma}\chi_\beta\\
(f)\ \ \mathcal{P}'[\chi_i, \chi_\alpha] &= -(-1)^{|\chi_i||\chi_\alpha|}\mathcal{P}'[\chi_\alpha, \chi_i]\\
&= {V'}^{\beta}_{\alpha i}\chi_\beta\\
&= -(-1)^{|\chi_i||\chi_\alpha|}{V'}^{\beta}_{i\alpha}\chi_\beta.
\end{aligned}$$

Then we get the proof. □

For any $X, Y, Z \in Der\mathcal{O}_M$, the curvature tensor field R^* of $\overline{\nabla}$ is given by the following formula

$$R^*(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - (-1)^{|X||Y|}\overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z. \tag{2.8}$$

To compute the component of R^* we suppose that

$$\begin{aligned}
(a)\ R^*(\chi_k, \chi_j)\chi_i &= R^h_{ijk}E_h,\\
(b)\ R^*(\chi_k, \chi_\alpha)\chi_i &= -(-1)^{|\chi_\alpha||\chi_k|}R^*(\chi_\alpha, \chi_k)\chi_i\\
&= R^h_{i\alpha k}E_h\\
&= -(-1)^{|\chi_k||\chi_\alpha|}R^h_{ik\alpha}E_h,\\
(c)\ R^*(\chi_\beta, \chi_\alpha)\chi_i &= R^h_{i\alpha\beta}E_h,
\end{aligned}$$

and

$$
\begin{aligned}
&(d)\ \ R^*(\chi_k,\chi_j)\chi_\alpha = {R'}^{\beta}_{\alpha jk}\chi_\beta,\\
&(e)\ \ R^*(\chi_k,\chi_\gamma)\chi_\alpha = -(-1)^{|\chi_k||\chi_\gamma|}R^*(\chi_\gamma,\chi_k)\chi_\alpha\\
&\qquad\qquad\qquad\qquad = {R'}^{\beta}_{\alpha\gamma k}\chi_\beta\\
&\qquad\qquad\qquad\qquad = -(-1)^{|\chi_k||\chi_\gamma|}{R'}^{\beta}_{\alpha k\gamma}\chi_\beta,\\
&(f)\ \ R^*(E_\mu,\chi_\gamma)\chi_\alpha = {R'}^{\beta}_{\alpha\gamma\mu}\chi_\beta.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(a)\ \ R^h_{ijk} &= \Gamma^h_{ij||k} + (-1)^{|\chi_k|(|E_s|+|\chi_i|+|\chi_j|)}\Gamma^s_{ij}\Gamma^h_{sk} - (-1)^{|\chi_k||\chi_j|}\Gamma^h_{ik||j}\\
&\quad - (-1)^{|\chi_i||\chi_j|+|\chi_j||E_s|}\Gamma^s_{ik}\Gamma^h_{sj} - V^s_{jk}\Gamma^h_{is} - {V'}^{\alpha}_{jk}\Gamma^h_{i\alpha},
\end{aligned}
$$

$$
\begin{aligned}
(b)\ \ R^h_{i\alpha k} &= \Gamma^h_{i\alpha||k} + (-1)^{|\chi_k|(|E_s|+|\chi_i|+|\chi_\alpha|)}\Gamma^s_{i\alpha}\Gamma^h_{sk} - (-1)^{|\chi_k||\chi_\alpha|}\Gamma^h_{ik|\alpha}\\
&\quad - (-1)^{|\chi_i||\chi_\alpha|+|\chi_\alpha||E_s|}\Gamma^s_{ik}\Gamma^h_{s\alpha} - V^s_{\alpha k}\Gamma^h_{is} - {V'}^{\varepsilon}_{\alpha k}\Gamma^h_{i\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
(c)\ \ R^h_{i\alpha\beta} &= \Gamma^h_{i\alpha|\beta} + (-1)^{|\chi_\beta|(|E_s|+|\chi_i|+|\chi_\alpha|)}\Gamma^s_{i\alpha}\Gamma^h_{s\beta} - (-1)^{|\chi_\beta||\chi_\alpha|}\Gamma^h_{i\beta|\alpha}\\
&\quad - (-1)^{|\chi_i||\chi_\alpha|+|\chi_\alpha||E_s|}\Gamma^s_{i\beta}\Gamma^h_{s\alpha} - V^s_{\alpha\beta}\Gamma^h_{is} - {V'}^{\varepsilon}_{\alpha\beta}\Gamma^h_{i\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
(d)\ \ {R'}^{\ \beta}_{\alpha jk} &= {\Gamma'}^{\ \beta}_{\alpha j||k} + (-1)^{|\chi_k|(|\chi_\alpha|+|\chi_j|+|E_\varepsilon|)}{\Gamma'}^{\ \varepsilon}_{\alpha j}{\Gamma'}^{\ \beta}_{\varepsilon k} - (-1)^{|\chi_k||\chi_j|}{\Gamma'}^{\ \beta}_{\alpha k||j}\\
&\quad - (-1)^{|\chi_j||\chi_\alpha|+|\chi_j||E_\varepsilon|}{\Gamma'}^{\ \varepsilon}_{\alpha k}{\Gamma'}^{\ \beta}_{\varepsilon j} - V^{\ s}_{jk}{\Gamma'}^{\ \beta}_{\alpha s} - {V'}^{\ \varepsilon}_{jk}{\Gamma'}^{\ \beta}_{\alpha\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
(e)\ \ {R'}^{\beta}_{\alpha\gamma k} &= {\Gamma'}^{\ \beta}_{\alpha\gamma||k} + (-1)^{|\chi_k|(|\chi_\alpha|+|\chi_\gamma|+|E_\varepsilon|)}{\Gamma'}^{\ \varepsilon}_{\alpha\gamma}{\Gamma'}^{\ \beta}_{\varepsilon k} - (-1)^{|\chi_k||\chi_\gamma|}{\Gamma'}^{\ \beta}_{\alpha k|\gamma}\\
&\quad - (-1)^{|\chi_\gamma||\chi_\alpha|+|\chi_\gamma||E_\varepsilon|}{\Gamma'}^{\ \varepsilon}_{\alpha k}{\Gamma'}^{\ \beta}_{\varepsilon\gamma} - V^{\ s}_{\gamma k}{\Gamma'}^{\ \beta}_{\alpha s} - {V'}^{\ \varepsilon}_{\gamma k}{\Gamma'}^{\ \beta}_{\alpha\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
(f)\ \ {R'}^{\ \beta}_{\alpha\gamma\mu} &= {\Gamma'}^{\ \beta}_{\alpha\gamma|\mu} + (-1)^{|E_\mu|(|\chi_\alpha|+|\chi_\gamma|+|E_\varepsilon|)}{\Gamma'}^{\ \varepsilon}_{\alpha\gamma}{\Gamma'}^{\ \beta}_{\varepsilon\mu} - (-1)^{|E_\mu||\chi_\gamma|}{\Gamma'}^{\ \beta}_{\alpha\mu|\gamma}\\
&\quad - (-1)^{|\chi_\gamma||\chi_\alpha|+|\chi_\gamma||E_\varepsilon|}{\Gamma'}^{\ \varepsilon}_{\alpha\mu}{\Gamma'}^{\ \beta}_{\varepsilon\gamma} - V^{\ s}_{\gamma\mu}{\Gamma'}^{\ \beta}_{\alpha s} - {V'}^{\ \varepsilon}_{\gamma\mu}{\Gamma'}^{\ \beta}_{\alpha\varepsilon}.
\end{aligned}
$$

Now, we are ready to state and prove the general form of all adapted linear superconnections on $(\mathcal{M},\mathcal{D},\mathcal{D}')$.

Theorem 2.4. *Let $(\mathcal{M},\mathcal{D},\mathcal{D}')$ be an almost product supermanifold and $\tilde{\nabla}$ be a linear superconnection on $\mathcal{M}$. Then all the adapted linear superconnections on $\mathcal{M}$ are given by*

$$\nabla_X Y = \mathcal{P}\tilde{\nabla}_X\mathcal{P}Y + \mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y + \mathcal{P}S(X,\mathcal{P}Y) + \mathcal{P}'S(X,\mathcal{P}'Y), \tag{2.9}$$

where X, Y are graded vector fields and S is an arbitrary tensor field of type (1, 2) on $\mathcal{M}$.

Proof. Let $Y \in \mathcal{D}$, then $\mathcal{P}S(X,\mathcal{P}Y) \in \mathcal{D}$, $\mathcal{P}\tilde{\nabla}_X\mathcal{P}Y \in \mathcal{D}$ and

$$\mathcal{P}'Y = 0.$$

We can also obtain $Y \in \mathcal{D}'$ in a similar way. It is easy to check that ∇ is an adapted linear superconnection on $\mathcal{M}$.

Conversely, suppose that ∇ is an adapted linear superconnection on $\mathcal{M}$. For graded vector fields X and Y, if we put

$$\nabla_X Y - \tilde{\nabla}_X Y = S(X,Y), \tag{2.10}$$

then we have

$$\mathcal{P}'(\nabla_X\mathcal{P}Y) = \mathcal{P}'\Big(\tilde{\nabla}_X\mathcal{P}Y + S\big(X,\mathcal{P}Y\big)\Big) = 0,$$

and

$$\mathcal{P}(\nabla_X\mathcal{P}'Y) = \mathcal{P}\Big(\tilde{\nabla}_X\mathcal{P}'Y + S\big(X,\mathcal{P}'Y\big)\Big) = 0.$$

Thus we see that

$$\begin{aligned}\nabla_X Y =&\tilde{\nabla}_X Y + S(X,Y)\\ =&(\mathcal{P}+\mathcal{P}')\Big(\tilde{\nabla}_X(\mathcal{P}+\mathcal{P}')Y\Big) + \Big(\mathcal{P}+\mathcal{P}')S\big(X,(\mathcal{P}+\mathcal{P}')Y\Big)\\ =&\mathcal{P}\tilde{\nabla}_X\mathcal{P}Y + \mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y + \mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y + \mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y\\ &+\mathcal{P}S(X,\mathcal{P}Y) + \mathcal{P}'S(X,\mathcal{P}Y) + \mathcal{P}S(X,\mathcal{P}'Y) + \mathcal{P}'S(X,\mathcal{P}'Y)\end{aligned}$$

which implies that

$$\nabla_X Y = \mathcal{P}\tilde{\nabla}_X\mathcal{P}Y + \mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y + \mathcal{P}S(X,\mathcal{P}Y) + \mathcal{P}'S(X,\mathcal{P}'Y).$$

This completes the proof. □

Next, for two graded vector fields X and Y, we define

$$S^\circ(X,Y) = \mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y + \mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y,$$

and

$$S^*(X,Y) = \mathcal{P}\Big([\mathcal{P}'X,\mathcal{P}Y] - \tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y\Big) + \mathcal{P}'\Big([\mathcal{P}X,\mathcal{P}'Y] - \tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y\Big).$$

Both S° and S^* are tensor fields of type (1, 2) on $\mathcal{M}$:

$$\begin{aligned}S^\circ(X_1f+X_2,Y) &= \mathcal{P}'\tilde{\nabla}_{X_1f+X_2}\mathcal{P}Y + \mathcal{P}\tilde{\nabla}_{X_1f+X_2}\mathcal{P}'Y\\ &= \mathcal{P}'\Big((-1)^{|X_1||f|}f\tilde{\nabla}_{X_1}\mathcal{P}Y + \tilde{\nabla}_{X_2}\mathcal{P}Y\Big)\\ &\quad+\mathcal{P}\Big((-1)^{|X_1||f|}f\tilde{\nabla}_{X_1}\mathcal{P}'Y + \tilde{\nabla}_{X_2}\mathcal{P}'Y\Big).\end{aligned}$$

Then

$$\begin{aligned}S^\circ(X_1f+X_2,Y) =&(-1)^{|X_1||f|}f\Big(\mathcal{P}'\tilde{\nabla}_{X_1}\mathcal{P}Y + \mathcal{P}\tilde{\nabla}_{X_1}\mathcal{P}'Y\Big)\\ &+\Big(\mathcal{P}'\tilde{\nabla}_{X_2}\mathcal{P}Y + \mathcal{P}\tilde{\nabla}_{X_2}\mathcal{P}'Y\Big).\end{aligned}$$

Thus we get

$$\begin{aligned}S^{\circ}(X_1f+X_2,Y)=&\Big((-1)^{|X_1||f|}f\mathcal{P}'\tilde{\nabla}_{X_1}\mathcal{P}Y+\mathcal{P}'(X_1(f)\mathcal{P}Y)\Big)\\&+\Big((-1)^{|X_1||f|}f\mathcal{P}\tilde{\nabla}_{X_1}\mathcal{P}'Y+\mathcal{P}(X_1(f)\mathcal{P}'Y)\Big)\\&+\Big(\mathcal{P}'\tilde{\nabla}_{X_2}\mathcal{P}Y+\mathcal{P}\tilde{\nabla}_{X_2}\mathcal{P}'Y\Big)\\=&\Big(\mathcal{P}'\tilde{\nabla}_{X_1}f\mathcal{P}Y+\mathcal{P}\tilde{\nabla}_{X_1}f\mathcal{P}'Y\Big)+\Big(\mathcal{P}'\tilde{\nabla}_{X_2}\mathcal{P}Y+\mathcal{P}\tilde{\nabla}_{X_2}\mathcal{P}'Y\Big)\end{aligned}$$

which give us

$$S^{\circ}(X_1f+X_2,Y)=S^{\circ}(X_1,fY)+S^{\circ}(X_2,Y).$$

Also, we have

$$\begin{aligned}S^{\circ}(X,Y_1f+Y_2)=&\mathcal{P}'\tilde{\nabla}_X\mathcal{P}(Y_1f+Y_2)+\mathcal{P}\tilde{\nabla}_X\mathcal{P}'(Y_1f+Y_2)\\=&(-1)^{|f||Y_1|}\mathcal{P}'\tilde{\nabla}_Xf\mathcal{P}Y_1+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y_2\\&+(-1)^{|f||Y_1|}\mathcal{P}\tilde{\nabla}_Xf\mathcal{P}'Y_1+\mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y_2\\=&(-1)^{|f||Y_1|}\mathcal{P}'\Big(X(f)\mathcal{P}Y_1+(-1)^{|X||f|}f\tilde{\nabla}_X\mathcal{P}Y_1\Big)+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y_2\\&+(-1)^{|f||Y_1|}\mathcal{P}\Big(X(f)\mathcal{P}'Y_1+(-1)^{|X||f|}f\tilde{\nabla}_X\mathcal{P}'Y_1\Big)+\mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y_2\\=&0+(-1)^{|f||Y_1|+|X||f|}f\mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y_1+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y_2\\&+0+(-1)^{|f||Y_1|+|X||f|}f\mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y_1+\mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y_2\\=&\Big(\mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y_1+\mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y_1\Big)f+\Big(\mathcal{P}'\tilde{\nabla}_X\mathcal{P}Y_2+\mathcal{P}\tilde{\nabla}_X\mathcal{P}'Y_2\Big)\end{aligned}$$

which implies that

$$S^{\circ}(X,Y_1f+Y_2)=S^{\circ}(X,Y_1)f+S^{\circ}(X,Y_2).$$

Also, we have

$$\begin{aligned}S^{*}(Xf,Y)=&\mathcal{P}\Big([\mathcal{P}'(Xf),\mathcal{P}Y]-\tilde{\nabla}_{\mathcal{P}'(Xf)}\mathcal{P}Y\Big)+\mathcal{P}'\Big([\mathcal{P}(Xf),\mathcal{P}'Y]-\tilde{\nabla}_{\mathcal{P}(Xf)}\mathcal{P}'Y\Big)\\=&\mathcal{P}\Big(\mathcal{P}'(Xf)(\mathcal{P}Y)-(-1)^{|Xf||Y|}\mathcal{P}Y\big(\mathcal{P}'(Xf)\big)\Big)-\mathcal{P}\Big((-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y\Big)\\&+\mathcal{P}'\Big(\mathcal{P}(Xf)(\mathcal{P}'Y)-(-1)^{|Xf||Y|}\mathcal{P}'Y(\mathcal{P}(Xf))\Big)-\mathcal{P}'\Big((-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y\Big)\end{aligned}$$

$$\begin{aligned}
=&\mathcal{P}\Big((-1)^{|X||f|}f\mathcal{P}'X(\mathcal{P}Y)\\
&-(-1)^{|Xf||Y|+|X||f|}\big(\mathcal{P}Y(f)\mathcal{P}'X+(-1)^{|Y||f|}f\mathcal{P}Y(\mathcal{P}'X)\big)\Big)\\
&-\mathcal{P}\Big((-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y+((\mathcal{P}'X)(f))\mathcal{P}Y-((\mathcal{P}'X)(f))\mathcal{P}Y\Big)\\
&+\mathcal{P}'\Big((-1)^{|X||f|}f\mathcal{P}X(\mathcal{P}'Y)\\
&-(-1)^{|Xf||Y|+|X||f|}((\mathcal{P}'Y(f))\mathcal{P}X+(-1)^{|Y||f|}f\mathcal{P}'Y(\mathcal{P}X))\Big)\\
&-\mathcal{P}'\Big((-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y+((\mathcal{P}X)(f))\mathcal{P}'Y-((\mathcal{P}X)(f))\mathcal{P}'Y\Big)\\
=&\mathcal{P}\Big(((\mathcal{P}'X)(f))\mathcal{P}Y+(-1)^{|X||f|}f(\mathcal{P}'X)(\mathcal{P}Y)-(-1)^{|X||fY|}f\mathcal{P}Y(\mathcal{P}'X)\Big)\\
&-\mathcal{P}\Big((-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y+((\mathcal{P}'X)(f))\mathcal{P}Y\Big)\\
&+\mathcal{P}'\Big(((\mathcal{P}X)(f))\mathcal{P}'Y+(-1)^{|X||f|}f\mathcal{P}X(\mathcal{P}'Y)-(-1)^{|X||fY|}f\mathcal{P}'Y(\mathcal{P}X))\Big)\\
&-\mathcal{P}'\Big((-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y+((\mathcal{P}X)(f))\mathcal{P}'Y\Big)\\
=&\mathcal{P}\Big([\mathcal{P}'(X),\mathcal{P}(fY)]-\tilde{\nabla}_{\mathcal{P}'X)}\mathcal{P}(fY)\Big)+\mathcal{P}'\Big([\mathcal{P}X,\mathcal{P}'(fY)]-\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'(fY)\Big).
\end{aligned}$$

It follows that

$$S^*(Xf,Y)=S^*(X,fY).$$

Also, we have

$$\begin{aligned}
S^*(X,Yf)&=\mathcal{P}\Big([\mathcal{P}'X,\mathcal{P}(Yf)]-\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}(Yf)\Big)+\mathcal{P}'\Big([\mathcal{P}X,\mathcal{P}'(Yf)]-\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'(Yf)\Big)\\
&=\mathcal{P}\Big(\mathcal{P}'X(\mathcal{P}(Yf))-(-1)^{|X||Yf|}\mathcal{P}(Yf)(\mathcal{P}'X)-(-1)^{|Y||f|}\tilde{\nabla}_{\mathcal{P}'X}f\mathcal{P}Y\Big)\\
&\quad+\mathcal{P}'\Big(\mathcal{P}X(\mathcal{P}'(Yf))-(-1)^{|X||Yf|}(\mathcal{P}'(Yf))(\mathcal{P}X)-(-1)^{|Y||f|}\tilde{\nabla}_{\mathcal{P}X}f\mathcal{P}'Y\Big)\\
&=\mathcal{P}\Big((-1)^{|Y||f|}\mathcal{P}'X(f)\mathcal{P}Y+(-1)^{|X||f|+|Y||f|}f\mathcal{P}'X(\mathcal{P}Y)\\
&\quad-(-1)^{|X||Yf|+|Y||f|}f\mathcal{P}Y(\mathcal{P}'X)\Big)\\
&\quad+\mathcal{P}'\Big((-1)^{|Y||f|}\mathcal{P}X(f)\mathcal{P}'Y+(-1)^{|X||f|+|Y||f|}f\mathcal{P}X(\mathcal{P}'Y)\\
&\quad-(-1)^{|X||Yf|+|Y||f|}f\mathcal{P}'Y(\mathcal{P}X)\Big)\\
&\quad-(-1)^{|Y||f|}\mathcal{P}\Big(\mathcal{P}'X(f)\mathcal{P}Y+(-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y\Big)\\
&\quad-(-1)^{|Y||f|}\mathcal{P}'\Big(\mathcal{P}X(f)\mathcal{P}'Y+(-1)^{|X||f|}f\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
S^*(X,Yf)&=\Big(\mathcal{P}[\mathcal{P}'X,\mathcal{P}Y]\Big)f-\Big(\mathcal{P}\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y\Big)f+\Big(\mathcal{P}'[\mathcal{P}X,\mathcal{P}'Y]\Big)f-\Big(\mathcal{P}'\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y\Big)f\\
&=S^*(X,Y)f.
\end{aligned}$$

By direct calculations we deduce that

$$(a)\ \mathcal{P}S^{\circ}(X,\mathcal{P}Y)=0,$$
$$(b)\ \mathcal{P}'S^{\circ}(X,\mathcal{P}'Y)=0,$$

and

$$(a)\ \mathcal{P}S^{*}(X,\mathcal{P}Y)=\mathcal{P}\Big([\mathcal{P}'X,\mathcal{P}Y]-\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y\Big),$$
$$(b)\ \mathcal{P}'S^{*}(X,\mathcal{P}'Y)=\mathcal{P}'\Big([\mathcal{P}X,\mathcal{P}'Y]-\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y\Big).$$

Finally, we obtain two adapted linear superconnections ∇° and $\overline{\nabla}$

$$\begin{aligned}\nabla^{\circ}_X Y &= \mathcal{P}\tilde{\nabla}_X\mathcal{P}Y+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y+\mathcal{P}S^{\circ}(X,\mathcal{P}Y)+\mathcal{P}'S^{\circ}(X,\mathcal{P}'Y)\\ &= \mathcal{P}\tilde{\nabla}_X\mathcal{P}Y+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\overline{\nabla}_X Y =&\mathcal{P}\tilde{\nabla}_X\mathcal{P}Y+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y+\mathcal{P}S^{*}(X,\mathcal{P}Y)+\mathcal{P}'S^{*}(X,\mathcal{P}'Y)\\ =&\mathcal{P}\tilde{\nabla}_X\mathcal{P}Y+\mathcal{P}'\tilde{\nabla}_X\mathcal{P}'Y+\mathcal{P}\Big([\mathcal{P}'X,\mathcal{P}Y]-\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}Y\Big)\\ &+\mathcal{P}'\Big([\mathcal{P}X,\mathcal{P}'Y]-\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}'Y\Big)\end{aligned}$$

which yields

$$\begin{aligned}\overline{\nabla}_X Y =&\mathcal{P}\tilde{\nabla}_{X-\mathcal{P}'X}\mathcal{P}Y+\mathcal{P}'\tilde{\nabla}_{X-\mathcal{P}X}\mathcal{P}'Y+\mathcal{P}[\mathcal{P}'X,\mathcal{P}Y]+\mathcal{P}'[\mathcal{P}X,\mathcal{P}'Y]\\ =&\mathcal{P}\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}Y+\mathcal{P}'\tilde{\nabla}_{\mathcal{P}'X}\mathcal{P}'Y+\mathcal{P}[\mathcal{P}'X,\mathcal{P}Y]+\mathcal{P}'[\mathcal{P}X,\mathcal{P}'Y]\end{aligned} \tag{2.12}$$

for graded vector fields X,Y.

If $\overline{\nabla}$ is torsion-free then for each i,j,α and β, we have $[E_j,E_i]\in\mathcal{D}$ and $[E_\beta,E_\alpha]\in\mathcal{D}'$. This means that distributions $\mathcal{D}$ and $\mathcal{D}'$ are involutive. So, we conclude the following.

Corollary 2.5. *The following hold:*

(i) Two adapted linear superconnections ∇° and $\overline{\nabla}$, defined in (2.11) *and* (2.12)*, coincide if and only if they have the same torsion tensor fields.*

(ii) If $\overline{\nabla}$ is torsion-free then distributions $\mathcal{D}$ and $\mathcal{D}'$ are involutive.

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