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On conformal change of projective Ricci curvature of Kropina metrics

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Abstract. In this paper, we study and investigate the conformal change of projective Ricci curvature of Kropina metrics. Let F and $\tilde{F}$ be two conformally related Kropina metrics on a manifold M. We prove that $\widetilde{\mathbf{PRic}} = \mathbf{PRic}$ if and only if the conformal transformation is a homothety.

Keywords: Kropina metrics, Projectively Ricci curvature, Conformal transformation.

1. **Introduction**

The study of conformal geometry includes an important part of research in Riemannian and Finsler geometry. The studies actually seek to discover the relations between some important geometric quantities and their correspondences. The conformal geometry of Riemmanian metrics have been well studied by many geometers and has played an important role in physical theories. The S curvature is a non-Riemannian quantity and play an important role in Finsler geometry, which was introduced by Z. Shen [10]. In [11], Z. Shen considered the projective spray $\tilde{G}$ associated with a given spray G on an n-dimensional manifold which is defined by G and its S-curvature S as

$$\tilde{\mathbf{G}} = \mathbf{G} + \frac{2\mathbf{S}}{n+1}Y$$

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where $Y := y^i \frac{\partial}{\partial y^i}$ is vertical radial field on TM. Then $\tilde{G}$ is projectively invariant, and it is easy to see that Ricci curvature $\widetilde{Ric}$ of $\tilde{G}$ is given by

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2,$$

where " | " denotes the horizontal covariant derivative with respect to Berwald connection of G. Recently, Z. Shen defined the concept of projective Ricci curvature for a Finsler metric F in Finsler geometry as

$$\mathbf{PRic} := \widetilde{\mathbf{Ric}}, \tag{1.1}$$

A Finsler metric is called projective Ricci curvature if $\mathbf{PRic} = 0$. The concept of isotropic $PRic$ curvature is defined and some conditions that implies the Randers metric has isotropic $PRic$-curvature are investigated [6].

The class of (α, β)-metrics form a special and important class of Finsler metrics with many applications which can be expressed in the form $F = \alpha\phi(s)$, $s = \beta/\alpha$, where $\alpha := \alpha(x, y) = \sqrt{a_{ij}(x)y^iy^j}$ is a positive-definite Riemannian metric, $\beta := \beta(y) = b_i(x)y^i$ is a 1-form on M and $\phi(s)$ is a C^∞ positive function on some open interval. In particular, when $\phi(s) = 1 + s$, the Finsler metric $F = \alpha + \beta$ is called a Randers metric and when $\phi(s) = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called a Kropina metric. Kropina metrics were first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and were investigated by V.K.Kropina [7].

In this class we use some notations as follows

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \qquad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of α. Further, put

$$\begin{aligned} &r^i{}_j := a^{im}r_{mj}, \qquad s^i{}_j := a^{im}s_{mj}, \qquad r_j := b^m r_{mj}, \\ &s_j := b^m s_{mj}, \qquad q_{ij} := r_{im}s^m_{\ j}, \qquad t_{ij} := s_{im}s^m_{\ j}, \\ &q_j := b^i q_{ij} = r_m s^m_{\ j}, \qquad t_j := b^i t_{ij} = s_m s^m_{\ j}, \end{aligned}$$

where $a^{ij} := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. We will denote

$$r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad r_0 := r_iy^i, \quad s_0 := s_iy^i.$$

In this paper, we study conformal transformation of $PRic$ curvature of Kropina metrics and get the following.

Theorem 1.1. *Let F and $\tilde{F}$ be two conformally related Kropina metrics on a manifold M. Then $\tilde{\mathbf{PRic}} = \mathbf{PRic}$ if and only if the conformal transformation is a homothety.*

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \backslash \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber $\pi^*_v TM$ at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \Big\{(x, y, v)|y \in T_xM_0, v \in T_xM\Big\}.$$

A *Finsler metric* on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) $F(x, \lambda y) = \lambda F(x, y) \quad \lambda > 0$;
(iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $F^2/2$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^i y^j},$$

is positive definite.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

For a Finsler metric $F = F(x, y)$ on a manifold M, its geodesics are characterized by the system of differential equations

$$\ddot{c}^i + 2G^i(\dot{c}) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and are given by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\Big\},$$

where $y \in T_xM$ and $(g^{ij}) := (g_{ij})^{-1}$.

The Riemann curvature $R_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ of F is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

When $F(x,y) = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric,

$$R^i{}_k = R^i{}_{jkl}(x) y^j y^l,$$

where $R^i{}_{jkl}(x)$ denotes the coefficients of the usual Riemannian curvature tensor. Thus, the quantity R_y in Finsler geometry is still called the Riemann curvature.

The Ricci curvature **Ric** is defined by

$$\mathbf{Ric} := R^i{}_i.$$

By definition, the Ricci curvature is a positively homogeneous function of degree two in $y \in TM$.

For a Finsler metric F, the Busemann–Hausdorff volume form $dV_{BH} := \sigma_{BH}(x)\omega^1 \wedge \cdots \wedge \omega^n$, is defined by

$$\sigma_{BH} := \frac{Vol(B^n(1))}{Vol\Big\{(y^i) \in R^n \mid F(x, y^i \frac{\partial}{\partial x^i} \mid_x)\Big\}}.$$

Here Vol$\{.\}$ denotes the Euclidean volume function and $B^n(1)$ denotes the unit ball on R^n. When $F(x,y) = \sqrt{g_{ij}(x) y^i y^j}$ is a Riemannian metric, then

$$\sigma_{BH}(x) = \sqrt{det(g_{ij})}.$$

There is a notion of distortion $\tau = \tau(x,y)$ on TM associated with the Busemann–Hausdorff volume form

$$dV_{BH} := \sigma_{BH}(x)\omega^1 \wedge \cdots \wedge \omega^n,$$

i.e.,

$$\tau(x,y) := \ln\left[\frac{\sqrt{det(g_{ij}(x,y))}}{\sigma_{BH}(x)}\right].$$

The $S-$curvature is defined by

$$\mathbf{S}(x,y) := \frac{d}{dt}\Big[\tau\big(c(t), \dot{c}(t)\big)\Big] \mid_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. From the definition, we see that the S-curvature measures the rate of change of the distortion on (T_xM, F_x) in the direction $y \in T_xM$. For a Finsler metric F, the S-curvature is given by following:

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big[\ln \sigma_{BH}\Big]. \tag{2.1}$$

3. Some Fundamental Lemmas

An (α, β)-metric can be expressed in the form $F = \alpha\phi(s), s = \beta/\alpha$, where $\alpha := \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta := \beta(y) = b_i(x)y^i$ is a 1-form on M, and $\phi(s)$ is a C^∞ positive function on some open interval [4][9]. In particular, when $\phi(s) = 1 + s$, the Finsler metrics $F = \alpha + \beta$ is called Randers metrics, which were introduced and studied by Randers. If $\phi(s) = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called a Kropina metric. Kropina metrics were first introduced by Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and were investigated by Kropina.

By a conformal change $\tilde{F} = e^{\kappa(x)}F$ various quantities are changed as follows:

$$\tilde{\alpha} = e^{\kappa(x)}\alpha, \quad \tilde{\beta} = e^{\kappa(x)}\beta.$$

Let $\tilde{\alpha} = \sqrt{\tilde{a_{ij}}y^i y^j}$ and $\tilde{\beta} = \tilde{b}_i(x)y^i$. Then

$$\tilde{a}_{ij} = e^{2\kappa(x)}a_{ij}, \quad \tilde{a}^{ij} = e^{-2\kappa(x)}a^{ij}, \quad \tilde{b_i} = e^{\kappa(x)}b_i, \quad \tilde{b^i} = e^{-\kappa(x)}b^i.$$

Further, we have [3]

$$\tilde{b}_{i||j} = e^{\kappa(x)}\big(b_{i;j} - b_j\kappa_i + f a_{ij}\big), \tag{3.1}$$

where $\tilde{b}_{i||j}$ denote the covariant derivative of $\tilde{b}_i$ with respect to $\tilde{\alpha}$ and

$$f := b^m\kappa_m.$$

From (3.1), we get

$$\tilde{s}_{ij} = e^{\kappa(x)}\Big[s_{ij} + \frac{1}{2}(b_i\kappa_j - b_j\kappa_i)\Big], \tag{3.2}$$

$$\tilde{r}_{ij} = e^{\kappa(x)}\Big[r_{ij} - \frac{1}{2}(b_i\kappa_j + b_j\kappa_i) + f a_{ij}\Big]. \tag{3.3}$$

The following holds.

Lemma 3.1. [1] *Let $\tilde{F}$ and F be two Finsler metrics on an n-dimensional manifold M. If $\tilde{F} = e^{\kappa(x)}F$, then the relation between the geodesic coefficients $\tilde{G}^i$ and G^i is given by*

$$\tilde{G}^i = G^i + \kappa_0 y^i - \frac{F^2}{2}\kappa^i, \tag{3.4}$$

where $\kappa^i = g^{il}\kappa_l$. Further, we have

$$\tilde{G}^i{}_j = G^i{}_j + \kappa_j y^i + \kappa_0\delta^i_j - y_j\kappa^i, \tag{3.5}$$

$$\tilde{G}^i{}_{jk} = G^i{}_{jk} + \kappa_j\delta^i{}_k + \kappa_k\delta^i_j - g_{jk}\kappa^i. \tag{3.6}$$

Lemma 3.2. *Let $\tilde{F}$ and F be two (α,β)-metrics on an n-dimensional manifold M. If $\tilde{F} = e^{\kappa(x)}F$, then*

$$(a)\quad {}^{\alpha}\widetilde{\boldsymbol{Ric}} = {}^{\alpha}\boldsymbol{Ric} - \alpha^2 a^{ij}\kappa_{i;j} + (n-2)\big(\kappa_0^2 - \kappa_{0;m}y^m - \alpha^2\kappa_m\kappa^m\big), \tag{3.7}$$

$$(b)\quad \tilde{s}^m_{\ 0} = e^{-\kappa(x)}\Big[s^m_{\ 0} + \frac{1}{2}(b^m\kappa_0 - \beta\kappa^m)\Big], \tag{3.8}$$

$$(c)\quad \tilde{s}^m_{\ 0||m} = e^{-\kappa(x)}\Bigg[s^m_{\ 0;m} + (n-3)s^m_{\ 0}\kappa_m + \frac{1}{2}(b^m\kappa_0 - \beta\kappa^m)_{;m} + \frac{(n-3)}{2}(f\kappa_0 - \beta\kappa_m\kappa^m)\Bigg], \tag{3.9}$$

$$(d)\quad \tilde{t}_{00} = t_{00} + \beta\kappa_m s^m_{\ 0} - s_0\kappa_0 + \frac{1}{2}f\beta\kappa_0 - \frac{1}{4}(\beta^2\kappa_m\kappa^m + b^2\kappa_0^2), \tag{3.10}$$

$$(e)\quad \tilde{t}^m_{\ m} = e^{-2\kappa(x)}\Big[t^m_{\ m} - 2s_m\kappa^m + \frac{1}{2}(f^2 - b^2\kappa_m\kappa^m)\Big], \tag{3.11}$$

$$(f)\quad \tilde{s}_0 = s_0 + \frac{1}{2}(b^2\kappa_0 - f\beta), \tag{3.12}$$

$$(g)\quad \tilde{\rho}_0 = \rho_0, \tag{3.13}$$

$$(h)\quad \tilde{\rho}_m\tilde{s}^m_0 = e^{-\kappa(x)}\Big[\rho_m s^m_0 - \frac{1}{2(1-b^2)}(b^m\kappa_0 - \beta\kappa^m)(r_m + s_m)\Big], \tag{3.14}$$

Proof. We prove the Lemma, part by part as follows:

(a): Let $\tilde{F}$ and F be two Finsler metrics on an n-dimensional manifold M. There is a relation between the Ricci curvature $\widetilde{\mathbf{Ric}}$ and $\mathbf{Ric}$ as follows [1]:

$$\begin{aligned}\tilde{\mathbf{Ric}} \;=\;& \mathbf{Ric} + (n-2)(\kappa_0^2 - \kappa_{0;0} - F^2\kappa_m\kappa^m) - 2F^2(\kappa^m J_m) - F^2 g^{ij}\kappa_{i;j}\\ & -F^2(\kappa^m I_m)_{;0} - 2F^2\kappa_0(\kappa^m I_m) + 2F^4 I_m\kappa^j\kappa^k C^m_{jk}\\ & -F^4\kappa^j\kappa^k I_{j.k} - F^4\kappa^j\kappa^k C^s_{jm}C^m_{ks}.\end{aligned} \tag{3.15}$$

From (3.15), we get

$${}^{\alpha}\tilde{\mathbf{Ric}} = {}^{\alpha}\mathbf{Ric} - \alpha^2 a^{ij}\kappa_{i;j} + (n-2)(\kappa_0^2 - \kappa_{0;0} - \alpha^2\kappa_m\kappa^m).$$

Now, we have

$$\begin{aligned}(b): \tilde{s}^m_{\ 0} &= \tilde{s}^m_{\ r}y^r\\ &= \tilde{a}^{mi}\tilde{s}_{ir}y^r\\ &= e^{-\kappa(x)}a^{mi}[s_{ir} + \frac{1}{2}(b_i\kappa_r - b_r\kappa_i)]y^r\\ &= e^{-\kappa(x)}[s^m_{\ r} + \frac{1}{2}(b^m\kappa_r - b_r\kappa^m)]y^r\\ &= e^{-\kappa(x)}[s^m_{\ 0} + \frac{1}{2}(b^m\kappa_0 - \beta\kappa^m)].\end{aligned}$$

$$
\begin{aligned}
(c): \tilde{s}^m{}_{0||m} &= \tilde{s}^m{}_{r||m} y^r \\
&= \left(\frac{\partial}{\partial x^m}\tilde{s}^m{}_r + \tilde{s}^i{}_r \tilde{\Gamma}^m_{im} - \tilde{s}^m_i \tilde{\Gamma}^i{}_{rm}\right) y^r \\
&= -\kappa_m e^{-\kappa(x)}\Big[s^m{}_r + \frac{1}{2}\Lambda^m_r\Big] y^r + e^{-\kappa(x)}\Big[\frac{\partial}{\partial x^m} s^m{}_r + \frac{1}{2}\frac{\partial \Lambda^m_r}{\partial x^m}\Big] y^r \\
&\quad + e^{-\kappa(x)}\Big[s^i{}_r + \frac{1}{2}\Lambda^i_r\Big]\Big(G^m{}_{im} + \kappa_i \delta^m{}_m + \kappa_m \delta^m_i - a_{im}\kappa^m\Big) y^r \\
&\quad - e^{-\kappa(x)}\Big[s^m{}_i + \frac{1}{2}\Lambda^m_i\Big]\Big(G^i{}_{rm} + \kappa_m \delta^i{}_r + \kappa_r \delta^i_m - a_{rm}\kappa^i\Big) y^r \\
&= e^{-\kappa(x)}\Bigg\{ -\kappa_m\Big[s^m{}_0 + \frac{1}{2}\Lambda^m_0\Big] + \frac{\partial}{\partial x^m} s^m{}_0 + \frac{1}{2}\frac{\partial \Lambda^m_0}{\partial x^m} \\
&\quad + s^i{}_0 G^m{}_{im} + n s^i{}_0 \kappa_i + s^i{}_0 \kappa_i - s^i{}_0 \kappa_i + \frac{1}{2}\Lambda^i_r G^m{}_{im} y^r \\
&\quad + s^0{}_i \kappa^i + \frac{1}{2}\Lambda^i_r (n\kappa_i + \kappa_i - \kappa_i) y^r - s^m_i G^i{}_{rm} y^r - s^m{}_0 \kappa_m \\
&\quad - \frac{1}{2}\Lambda^m_i G^i{}_{rm} y^r - \frac{1}{2} b^m \kappa_i (\kappa_m \delta^i{}_r + \kappa_r \delta^i_m - a_{rm}\kappa^i) y^r \\
&\quad + \frac{1}{2}(b_i \kappa^m \kappa_m \delta^i{}_r + b_i \kappa^m \kappa_r \delta^i_m - b_i \kappa^m a_{rm} \kappa^i) y^r \Bigg\}
\end{aligned}
$$

where

$$
\Lambda^m_r := b^m \kappa_r - b_r \kappa^m, \qquad \Lambda^m_0 := \Lambda^m_r y^r.
$$

For (d), we have

$$
\begin{aligned}
(d): \tilde{t}_{00} &= \tilde{t}_{ij} y^i y^j \\
&= \tilde{s}_{im} \tilde{s}^m_j y^i y^j \\
&= \tilde{s}_{im} \tilde{a}^{mr} \tilde{s}_{rj} y^i y^j \\
&= \Big[s_{im} + \frac{1}{2}(b_i \kappa_m - b_m \kappa_i)\Big] a^{mr} \Big[s_{rj} + \frac{1}{2}(b_r \kappa_j - b_j \kappa_r)\Big] y^i y^j \\
&= \Big[s_{im} + \frac{1}{2}(b_i \kappa_m - b_m \kappa_i)\Big]\Big[s^m_j + \frac{1}{2}(b^m \kappa_j - b_j \kappa^m)\Big] y^i y^j \\
&= \Bigg[s_{im} s^m_j + \frac{1}{2}(s_{im} b^m \kappa_j - s_{im} b_j \kappa^m) + \frac{1}{2}(s^m_j b_i \kappa_m - s^m_j b_m \kappa_i) \\
&\quad + \frac{1}{4}(b_i \kappa_m b^m \kappa_j - b_i \kappa_m b_j \kappa^m - b_m \kappa_i b^m \kappa_j + b_m \kappa_i b_j \kappa^m)\Bigg] y^i y^j
\end{aligned}
$$

Thus

$$
\begin{aligned}
\tilde{t}_{00} &= t_{00} + \frac{1}{2}\Big(- s_0 \kappa_0 + \beta \kappa_m s^m_0 + \beta \kappa_m s^m_0 - s_0 \kappa_0\Big) \\
&\quad + \frac{1}{4}\Big(f\beta\kappa_0 - \beta^2 \kappa_m \kappa^m - b^2 \kappa_0^2 + f\beta\kappa_0\Big) \\
&= t_{00} + \beta \kappa_m s^m_0 - s_0 \kappa_0 + \frac{1}{2} f\beta\kappa_0 - \frac{1}{4}\Big(\beta^2 \kappa_m \kappa^m + b^2 \kappa_0^2\Big).
\end{aligned}
$$

Also, for (e) we obtain

$$\begin{aligned}(e): \tilde{t}^{m}_{\ m} &= \tilde{a}^{mi}\tilde{t}_{im}\\ &= e^{-2\kappa(x)}a^{mi}\tilde{t}_{im}\\ &= e^{-2\kappa(x)}a^{mi}\Big[t_{im}+\frac{1}{2}(-s_i\kappa_m - s_{ij}b_m\kappa^j)+\frac{1}{2}(s^j_{\ m}b_i\kappa_j - s^j_{\ m}b_j\kappa_i)\\ &\quad +\frac{1}{4}(b_i\kappa_j b^j\kappa_m - b_i\kappa_j b_m\kappa^j - b_j\kappa_i b^j\kappa_m + b_j\kappa_i b_m\kappa^j)\Big].\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{t}_{00} &= e^{-2\kappa(x)}\Big[t^{m}_{\ m}+\frac{1}{2}(-s_m\kappa^m - s_m\kappa^m)-\frac{1}{2}(s_m\kappa^m + s_m\kappa^m)\\ &\quad +\frac{1}{4}(f^2 - b^2\kappa_m\kappa^m - b^2\kappa_m\kappa^m + f^2)\Big]\\ &= e^{-2\kappa(x)}\Big[t^{m}_{\ m} - 2s_m\kappa^m + \frac{1}{2}(f^2 - b^2\kappa_m\kappa^m)\Big].\end{aligned}$$

Now, we try to obtain (f) as follows

$$\begin{aligned}(f): \tilde{s}_0 &= \tilde{s}_i y^i\\ &= \tilde{b}^j\tilde{s}_{ji}y^i\\ &= b^j\Big[s_{ji}+\frac{1}{2}(b_j\kappa_i - b_i\kappa_j)\Big]y^i\\ &= s_0 + \frac{1}{2}(b^2\kappa_0 - f\beta).\end{aligned}$$

In order to prove (g), we have

$$\begin{aligned}(g): \tilde{\rho}_0 &= -\frac{\tilde{r}_0+\tilde{s}_0}{1-\tilde{b}^2}\\ &= -\frac{\tilde{b}^m\tilde{r}_{m0}+\tilde{b}^m\tilde{s}_{m0}}{1-b^2}\\ &= -b^m\left[\frac{r_{m0}-\frac{1}{2}(b_m\kappa_0+\beta\kappa_m)+fa_{m0}+s_{m0}+\frac{1}{2}(b_m\kappa_0-\beta\kappa_m)}{1-b^2}\right]\end{aligned}$$

which yields

$$\begin{aligned}\tilde{\rho}_0 &= -\frac{r_0+s_0}{1-b^2}\\ &= \rho_0.\end{aligned}$$

Finally, we prove (h) as follows

$$\begin{aligned}(h): \tilde{\rho}_m\tilde{s}^m_0 &= -e^{-\kappa(x)}\frac{(r_m+s_m)\Big[s^m_{\ 0}+\frac{1}{2}(b^m\kappa_0-\beta\kappa^m)\Big]}{1-b^2}\\ &= \frac{-e^{-\kappa(x)}}{1-b^2}\Big[r_m s^m_{\ 0}+\frac{1}{2}(b^m\kappa_0-\beta\kappa^m)r_m + s_m s^m_{\ 0}+\frac{1}{2}\Lambda^m_0 s_m\Big].\end{aligned}$$

Then, we get

$$\begin{aligned} &= \frac{-e^{-\kappa(x)}}{1-b^2}\Big[q_0+t_0+\frac{1}{2}(b^m\kappa_0-\beta\kappa^m)(s_m+r_m)\Big] \\ &= e^{-\kappa(x)}\Big[\rho_m s^m_0-\frac{1}{2(1-b^2)}(b^m\kappa_0-\beta\kappa^m)(r_m+s_m)\Big]. \end{aligned}$$

This completes the proof. □

4. **Proof of Theorem 1.1**

The projective Ricci curvature of a Kropina metric is computed in [5], and it is given at below.

Lemma 4.1. *Let $F=\dfrac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M. Then the projective Ricci curvature of F is given by*

$$\begin{aligned} \boldsymbol{PRic} &= {}^{\alpha}\boldsymbol{Ric}+(n-2)\left[\frac{1}{b^2}r_{0;0}+\frac{F}{b^2}q_0+\frac{1}{b^2}s_{0;0}-\frac{1}{b^4}(r_0+s_0)^2\right] \\ &\quad +\frac{2}{b^2}q_{00}+(n-1)\Big[\frac{4}{F^2b^4}r^2_{00}-\frac{4}{b^4F}r_0r_{00}+\frac{F}{b^2}t_0\Big]-\frac{nF}{b^4}s_0r \\ &\quad -\frac{n}{b^4}rr_{00}-Fs^m{}_{0;m}+\frac{F}{b^2}b^ms_{0;m}+\frac{1}{b^2}b^mr_{00;m}-\frac{F^2}{2b^2}s^ms_m \\ &\quad +\frac{1}{b^2}(Fs_0+r_{00})r^m{}_m-\frac{F}{b^2}s^mr_{0m}-\frac{F^2}{4}t^m{}_m. \end{aligned} \tag{4.1}$$

Proof of Theorem 1.1: For $\tilde{F}$, we have

$$\begin{aligned} \widetilde{\mathbf{PRic}} &= {}^{\alpha}\widetilde{\mathbf{Ric}}+(n-2)\left[\frac{1}{\tilde{b}^2}\tilde{r}_{0||0}+\frac{\tilde{F}}{\tilde{b}^2}\tilde{q}_0+\frac{1}{\tilde{b}^2}\tilde{s}_{0||0}-\frac{1}{\tilde{b}^4}(\tilde{r}_0+\tilde{s}_0)^2\right]+\frac{2}{\tilde{b}^2}\tilde{q}_{00} \\ &\quad +(n-1)\left[\frac{4}{\tilde{F}^2\tilde{b}^4}\tilde{r}^2_{00}-\frac{4}{\tilde{b}^4\tilde{F}}\tilde{r}_0\tilde{r}_{00}+\frac{\tilde{F}}{\tilde{b}^2}\tilde{t}_0\right]-\frac{n\tilde{F}}{\tilde{b}^4}\tilde{s}_0\tilde{r}-\frac{n}{\tilde{b}^4}\tilde{r}\tilde{r}_{00} \\ &\quad -\tilde{F}\tilde{s}^m{}_{0||m}+\frac{\tilde{F}}{\tilde{b}^2}\tilde{b}^m\tilde{s}_{0||m}+\frac{1}{\tilde{b}^2}\tilde{b}^m\tilde{r}_{00||m}+\frac{1}{\tilde{b}^2}(\tilde{F}\tilde{s}_0+\tilde{r}_{00})\tilde{r}^m{}_m \\ &\quad -\frac{\tilde{F}^2}{2\tilde{b}^2}\tilde{s}^m\tilde{s}_m-\frac{\tilde{F}}{\tilde{b}^2}\tilde{s}^m\tilde{r}_{0m}-\frac{\tilde{F}^2}{4}\tilde{t}^m{}_m. \end{aligned} \tag{4.2}$$

By substituting (3.7)-(3.14) into this very equation, we obtain the relation between $P\tilde{R}ic$ and $PRic$ as follows

$$\begin{aligned} \widetilde{\mathbf{PRic}} &= \mathbf{PRic}-\alpha^2\kappa^m_{;m}+(n-2)\Big\{\kappa^2_0-\kappa_{0;0}-\alpha^2\kappa_m\kappa^m+\frac{1}{b^2}\Big[-\frac{1}{2}(b^2\kappa_0-f\beta)_{;0} \\ &\quad +\kappa_0(b^2\kappa_0-f\beta-2r_0)+\frac{1}{2}\alpha^2\big(f^2-b^2\kappa_m\kappa^m+2r_m\kappa^m\big)\Big]+\frac{F}{b^2}\Big[\frac{1}{2}(+r\kappa_0 \end{aligned}$$

$$-\beta r_m\kappa^m - b^2 s^m{}_0\kappa_m + fs_0) - \frac{1}{4}\beta(f^2 - b^2\kappa_m\kappa^m)\Big] + \frac{1}{b^2}\Big[\frac{1}{2}(b^2\kappa_0 - f\beta)_{;0}$$
$$-\kappa_0(b^2\kappa_0 - f\beta + 2s_0) - \frac{1}{2}\alpha^2(f^2 - b^2\kappa_m\kappa^m - 2s_m\kappa^m)\Big]\Big\} + \frac{2}{b^2}\Big[\frac{1}{2}\kappa_0 r_0$$
$$-\frac{1}{2}\kappa_0 s_0) - \frac{1}{2}\beta\kappa^m(r_{m0} + s_{m0}) - \frac{1}{4}(b^2\kappa_0^2 - \beta^2\kappa_m\kappa^m)\Big] + (n-1)\Big\{\frac{4}{F^2 b^4}$$
$$\times\Big[\beta\kappa_0(\beta\kappa_0 - 2r_{00}) + \alpha^2 f(2r_{00} + \alpha^2 f - 2\beta\kappa_0)\Big] + \frac{4}{b^4 F}\Big[\beta r_0\kappa_0 - \alpha^2 f r_0$$
$$+\frac{1}{2}(b^2\kappa_0 - f\beta)\big(r_{00} - \beta\kappa_0 + \alpha^2 f\big)\Big] + \frac{F}{b^2}\Big[\frac{1}{2}\big(b^2 s^m{}_0\kappa_m - fs_0 - \beta s_m\kappa^m\big)$$
$$+\frac{1}{4}(f^2\beta - b^2\beta\kappa_m\kappa^m)\Big]\Big\} - \frac{nF}{2b^4}r(b^2\kappa_0 - f\beta) - \frac{n}{b^4}(-\beta r\kappa_0 + \alpha^2 fr)$$
$$-F\Big[(n-3)s^m{}_0\kappa_m + \frac{1}{2}(b^m\kappa_0 - \beta\kappa^m)_{;m} + \frac{(n-3)}{2}(f\kappa_0 - \beta\kappa_m\kappa^m)\Big]$$
$$+\frac{F}{b^2}\Big[-fs_0 + \beta s_m\kappa^m + \frac{1}{2}(b^2\kappa_0 - f\beta)_{;m}b^m - \frac{1}{2}b^2(f\kappa_0 - \beta\kappa_m\kappa^m)\Big]$$
$$+\frac{1}{b^2}\Big[\big(2\beta\kappa_0 - \alpha^2 f - r_{00}\big)f - b^m(\beta\kappa_0 - \alpha^2 f)_{;m} - \big(2r_0 - b^2\kappa_0 + 2f\beta\big)\kappa_0$$
$$+\big(2r_{m0}\kappa^m + f\kappa_0 - \beta\kappa_m\kappa^m\big)\beta\Big] + \frac{1}{b^2}\Bigg\{\Big[(n-1)(Fs_0 + r_{00})f$$
$$+\Big[\frac{1}{2}F(b^2\kappa_0 - f\beta) - \beta\kappa_0 + \alpha^2 f\Big]\Big[r^m{}_m + (n-1)f\Big]\Bigg\}$$
$$-\frac{F^2}{2b^2}\Big[b^2\kappa_m s^m + \frac{1}{4}b^2\big(b^2\kappa_m\kappa^m - f^2\big)\Big] - \frac{F}{b^2}\Big[fs_0$$
$$-\frac{1}{2}\big(\beta\kappa_m s^m - b^2\kappa^m r_{0m} + fr_0 - b^2 f\kappa_0\big) - \frac{1}{4}(b^2\beta\kappa_m\kappa^m$$
$$+f^2\beta)\Big] - \frac{F^2}{4}\Big[-2s_m\kappa^m + \frac{1}{2}(f^2 - b^2\kappa_m\kappa^m)\Big]. \tag{4.3}$$

Taking $\widetilde{\mathbf{PRic}} = \mathbf{PRic}$ in (4.3) and multiplying both sides by $8b^4\alpha^4\beta^2$, we obtain

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0, \tag{4.4}$$

where

$$\begin{aligned} A_6 \;=\;& 2b^2\beta\Big\{(n-2)\Big[-3b^2\beta\kappa_m\kappa^m - f^2\beta + 2\beta r_m\kappa^m + 2r\kappa_0 - 2b^2 s^m{}_0\kappa_m \\ &+2fs_0 + 4\beta s_m\kappa^m\Big] + (n-1)\Big[2b^2 s^m{}_0\kappa_m + 2fs_0 - 2\beta s_m\kappa^m + 3\beta f^2 \\ &-b^2\beta\kappa_m\kappa^m + 2b^2\kappa_0 f - 2b^2 f\kappa_0\Big] - 4b^2\beta\kappa^m{}_{;m} + 2b^2\kappa_0 r^m{}_m + 2\beta f r^m{}_m \\ &-2b^2(b^m\kappa_0 - \beta\kappa^m)_{;m} + (2n-3)b^2\beta\kappa_m\kappa^m - 8fs_0 - 3\beta f^2 - 2nr\kappa_0 \\ &+2(b^2\kappa_0 - f\beta)_{;m}b^m + 4\beta b^m f_{;m} + 6\beta s_m\kappa^m - 2b^2\kappa^m r_{0m} + 2fr_0 \\ &-4(n-3)b^2 s^m{}_0\kappa_m\Big\} - 4n\beta^2 rf, \end{aligned}$$

$$
\begin{aligned}
A_4 \;=\; & 4\beta^2\Big\{2(n-2)b^2\Big[b^2(\kappa_0^2-\kappa_{0;0})-2\kappa_0(r_0+s_0)\Big]+2(n-1)\Big[2\beta^2f^2 \\
& -4\beta f r_0+b^2\beta\kappa_0 f+b^2r_{00}f\Big]+b^2\Big[-2\kappa_0(r_0+s_0)+b^2\kappa_0^2-2r_{00}f \\
& +2\beta\kappa_m(r_{m0}-s_{m0})-2b^m(\beta\kappa_0)_{;m}+2\beta f\kappa_0-2\beta\kappa_0 r^m_{\ m}-\beta^2\kappa_m\kappa^m\Big] \\
& +2n\beta r\kappa_0\Big\},
\end{aligned}
$$

$$A_2 = 16(n-1)\beta^3\Big[2\beta r_0\kappa_0+(b^2\kappa_0+3f\beta)(r_{00}-\beta\kappa_0)\Big], \tag{4.5}$$

$$A_0 = 32(n-1)\beta^5\kappa_0(\beta\kappa_0-2r_{00}). \tag{4.6}$$

Rewrite (4.4) as

$$(A_6\alpha^4+A_4\alpha^2+A_2)\alpha^2+A_0=0. \tag{4.7}$$

The above equation shows that α^2 divides $32(n-1)\beta^5\kappa_0(\beta\kappa_0-2r_{00})$. Since α^2 is irreducible and $\beta^5\kappa_0$ can factor into linear terms, we have that α^2 divides $\beta\kappa_0-2r_{00}$. Thus there exists a function $c(x)$ such that

$$\beta\kappa_0-2r_{00}=c(x)\alpha^2. \tag{4.8}$$

Substituting (4.8) into (4.7) and by (4.5), we get

$$
\begin{aligned}
&(A_6\alpha^2+A_4)\alpha^2= \\
&\quad -16(n-1)\beta^3\Big[2\beta r_0\kappa_0+(b^2\kappa_0+3f\beta)(r_{00}-\beta\kappa_0)+2c\beta^2\kappa_0\Big]
\end{aligned} \tag{4.9}
$$

which implies the following:

$$
\begin{aligned}
&A_6\alpha^2+A_4=0,\\
&2\beta r_0\kappa_0+(b^2\kappa_0+3f\beta)(r_{00}-\beta\kappa_0)+2c\beta^2\kappa_0=0.
\end{aligned} \tag{4.10}
$$

Differentiating (4.8) with respect to y^i yields

$$2cy_i=b_i\kappa_0+\beta\kappa_i-4r_{i0}. \tag{4.11}$$

Contracting (4.11) with b^i gives

$$r_0=\frac{1}{4}\Big(b^2\kappa_0+f\beta-2c\beta\Big). \tag{4.12}$$

Rewrite (4.8) as

$$r_{00}=\frac{1}{2}\Big(\beta\kappa_0-c\alpha^2\Big). \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.10), we obtain

$$-2f\beta^2\kappa_0+2c\beta^2\kappa_0-c\alpha^2b^2\kappa_0-3cf\alpha^2\beta=0. \tag{4.14}$$

Differentiating (4.14) with respect to y^i yields

$$
\begin{aligned}
&-4f\beta b_i\kappa_0-2f\beta^2\kappa_i+4c\beta b_i\kappa_0+2c\beta^2\kappa_i-cb^2\alpha^2\kappa_i-2cb^2\kappa_0y_i\\
&\quad -3cf\alpha^2b_i-6cf\beta y_i=0.
\end{aligned} \tag{4.15}
$$

Contracting (4.15) with b^i gives

$$-4f\beta b^2\kappa_0 - 2f^2\beta^2 + 4cb^2\beta\kappa_0 + 2cf\beta^2 - cfb^2\alpha^2 - 2cb^2\beta\kappa_0 \\ -3cfb^2\alpha^2 - 6cf\beta^2 = 0.$$

The above equation is equivalent to the following two equations.

$$-2fb^2\kappa_0 - f^2\beta + cb^2\kappa_0 - 2cf\beta = 0, \\ -2cfb^2\alpha^2 = 0. \tag{4.16}$$

From (4.16) we conclude that

$$f = 0, \quad \text{or} \quad c = 0.$$

Plugging $c = 0$ into (4.16) yields

$$-2f\beta(2b^2\kappa_0 + f\beta) = 0.$$

which is equivalent to

$$f = 0, \\ 2b^2\kappa_0 + f\beta = 0. \tag{4.17}$$

Differentiating (4.17) with respect to y^i yields

$$2b^2\kappa_i + fb_i = 0.$$

Contracting the above equation with b^i, we get

$$3fb^2 = 0. \tag{4.18}$$

It follows from (4.18) that $f = 0$. Hence

$$\kappa_m = 0,$$

therefore $\kappa(x)$= constant. □

References

1. S. Bácsó and X. Cheng, *Finsler conformal transformations and the curvature invariances*, Publ. Math. Debrecen, **70**(2007), 221-231.
2. X. Cheng, Y. Shen, and X. Ma, *On a class of projective Ricci flat Finsler metrics*, Publ. Math. Debrecen. **7528**(2017), 1-12.
3. X. Cheng and Z. Shen, *Finsler Geometry: An Approach via Randers Spaces*, Springer, 2012.
4. K. Kaur and G. Shanker, *On the geodesics of a homogeneous Finsler space with a special (α, β)-metric*, Journal of Finsler Geometry and its Applications, **1**(1) (2020), 26-36.
5. X. Cheng, X. Ma and Y.Shen , *On Projective Ricci Flat Kropina Metrics*, Journal of Mathematics, **37**(4) (2017), 705-713.
6. L. Ghasemnezhad, B.Rezaei and M.Gabreni, *On isotropic projective Ricci curvature of C-reducible Finsler metrics*, Turkish Journal of Mathematics, **43**(3) (2019), 1730-1741.
7. V. K. Kropina, *On projective two-dimensional Finsler spaces with a special metric*, Trudy Sem. Vektor. Tenzor. Anal. **11**(1961), 277-292.

8. V.K. Matsumoto, *On projective two-dimensional Finsler spaces with a special metric*, Trudy Sem. Vektor. Tenzor. Anal., **11**(1961), 277-292.
9. H. Sadeghi, *A special class of Finsler metrics*, Journal of Finsler Geometry and its Applications, **1**(1) (2020), 60-65.
10. Z. Shen, *Volume comparison and applications in Riemann-Finsler geometry*, Advances in Math. 128 (1997), 306-328.
11. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.
12. A. Tayebi and T. Tabatabaeifar, *Matsumoto metrics of reversible curvature*, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis., **32**(2016), 165-200.
13. X. Zhang and Y. Shen, *On Einstein Matsumoto metrics*, arXiv:math/1207.1944v1 [math.DG] 9 Jul 2012.

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