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Complete Ricci-Bourguignon solitons on Finsler manifolds

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Abstract. In this paper, we study Ricci-Bourguignon soliton on Finsler manifolds and prove any forward complete shrinking Finslerian Ricci-Bourguignon soliton under some conditions on vector filed and scalar curvature is compact and its fundamental group is finite.

Keywords: Finsler metric, Ricci-Bourguignon soliton, fundamental group, Ricci flow.

1. Introduction

Over the last few years, geometric flows have been a topic of active research interest in both mathematics and physics. A geometric flow is an evolution of a geometric structure as metric under a differential equation related to a functional on a manifold, usually associated with some curvatures. Also, Ricci solitons and Yamabe solitons play an important role in geometric flow where they correspond to self-similar solutions of the flow. Hence, given a geometric flow it is natural to investigate the solitons associated to that flow. In 1982, R. S. Hamilton introduced the intrinsic Riemannian geometric flows on Riemannain manifolds, Ricci flow [16] as

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\mathbf{Ric}(\mathbf{g}(t)), \tag{1.1}$$

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and Yamabe flow [17] as

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -R_{\mathbf{g}(t)}\mathbf{g}(t), \tag{1.2}$$

which are evolution equations for Riemannian metrics and R is the scalar curvature. Then J. P. Bourguignon [12] introduced Ricci-Bourguignon flow on Riemannian manifolds $(M^n, \mathbf{g}(t))$ as

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2(\mathbf{Ric} - \rho R\mathbf{g}) \tag{1.3}$$

where ρ is a real constant. Short-time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ have been shown by Catino et al. [13] for

$$\rho < \frac{1}{2(n-1)}.$$

When $\rho = 0$, the Ricci-Bourguignon flow reduces to the Ricci flow.

On a Riemannain manifold $(M^n, \mathbf{g})$ and a non-vanishing vector field X is said to define a Ricci-Bourguignon soliton if there exists a real constant λ such that

$$\mathbf{Ric} + \frac{1}{2}\mathcal{L}_X\mathbf{g} = \lambda\mathbf{g} + \rho R\mathbf{g}, \tag{1.4}$$

where $\mathcal{L}_X\mathbf{g}$ denotes the Lie derivative of the metric $\mathbf{g}$ in the direction of the vector field X. If the vector field X is of gradient type, $X = \nabla f$, for a smooth function f on M, then the Ricci-Bourguignon soliton is called a gradient Ricci-Bourguignon soliton. The soliton is called expanding, steady, shrinking when λ is negative, zero and positive, respectively.

In recent years, many authors studied the Ricci-Bourguignon soliton on Riemannain manifolds. In [14], Catino et al. classified noncompact gradient shrinkers of gradient Ricci-Bourguignon soliton with bounded non-negative sectional curvature. In [11, 15] the authors have obtained some results on Ricci-Bourguignon solitons and almost Ricci-Bourguignon solitons on Riemannian manifolds.

On the other hand, the concept of the Ricci flow on Finsler manifolds is defined by Bao [5], choosing the Ricci tensor introduced by Akbar-Zadeh. The existence and uniqueness of solutions to the Ricci flow and Yamabe flow on Finsler manifolds are shown in [3, 4, 7]. In [8], Bidabad and Yar Ahmadi introduced Ricci solitons on Finsler manifolds as a generalization of Einstein space and shown that if there is a Ricci soliton on a compact Finsler manifold then there exists a solution to the Finsler-Ricci flow. Then, in [10, 19], they established a forward complete shrinking Finsler-Ricci soliton space is compact if and only if the corresponding vector field is bounded and a compact shrinking Finsler-Ricci soliton space has a finite fundemental group. Also, they obtained similar results for complete Finslerian Yamabe soliton [9].

Motivated by the above studies, in the present paper, we establish some properties of Ricci-Bourguignon solitons on Finsler manifolds. In fact, we prove the following theorems.

Theorem 1.1. *Let (M, F) be a forward geodesically complete Finsler manifold satisfying*

$$2Ric_{ij} + \mathcal{L}_{\hat{V}} g_{ij} \geq 2(\lambda + \rho H) g_{ij}, \tag{1.5}$$

where $\hat{V}$ denotes the complete lift of the vector field V on M. Suppose that the following holds

$$H \leq K_1, \quad and \quad \lambda + \rho H \geq 0$$

for some positive real constants ρ and K_1. Then M is compact if and only if $||V||$ is bounded on M by a constant K_2. Moreover, in this case we have

$$diam(M) \leq \frac{\pi}{\lambda + \rho K_1} \left(K_2 + \sqrt{K_2^2 + (n-1)(\lambda + \rho K_1)} \right).$$

Then, we prove the following.

Theorem 1.2. *Let (M, F) be a geodesically complete Finsler manifold satisfying (1.5), where $\hat{V}$ denotes the complete lift of the vector field V on M. Suppose that the following holds*

$$H \leq K_1, \quad and \quad \lambda + \rho H \geq 0$$

for some positive real constants ρ and K_1. Then, for any two points p, q in M we have

$$d(p,q) \leq \max \left\{ 1, \frac{1}{\lambda + \rho K_1} \Big(2(n-1) + \Lambda_p + \Lambda_q + ||V||_p + ||V||_q \Big) \right\}. \tag{1.6}$$

Finally, we show the following.

Theorem 1.3. *Let (M, F) be a complete connected Finsler manifold satisfying (1.5), where $\hat{V}$ denotes the complete lift of the vector field V on M. If $H \leq K_1$ and $\lambda + \rho H \geq 0$ for some positive real constants ρ and K_1., then the fundamental group $\pi_1(M)$ of M is finite.*

2. Preliminaries

In this section, we recall some basic concepts and facts in Finsler geometry from [1, 5, 18].

Let M^n be a smooth, connected differentiable manifold and TM be the tangent bundle. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is smooth function on $TM_0 := TM \setminus \{0\}$;

(ii) $F(x,\lambda y)=\lambda F(x,y)$ for all $(x,y)\in TM$ and all $\lambda>0$,
(iii) the $n\times n$ matrix

$$g_{ij}(y):=\frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}F^2(x,y)$$

is positive definite for every $(x,y)\in TM_0$.
Such a pair (M,F) is called a Finsler manifold and $g(x,y)=g_{ij}y^iy^j$ is called fundamental tensor of F, where $y\in T_xM$.

The natural projection map $\pi: TM_0\to M$ gives rise to the pull-back bundle π^*TM and its dual bundle π^*T^*M over TM_0. The pull-back bundle π^*TM admits a unique connection which is called the Chern connection. The Chern connection is determined by the following structure equations,

$$D^V_XY-D^V_YX=[X,Y],$$

and

$$Xg_V(Y,Z)=g_V(D^V_XY,Z)+g_V(Y,D^V_XZ)+2C_V(D^V_XV,Y,Z),$$

for $V\in T_xM\setminus\{0\}$, $X,Y,Z\in T_xM$, where

$$C_V(X,Y,Z):=C_{ijk}(V)X^iY^jZ^k=\frac{1}{4}\frac{\partial^3F^2}{\partial V^i\partial V^j\partial V^k}(V)X^iY^jZ^k$$

is the Cartan tensor of F and D^V_XY the covariant derivative with respect to vector $V\in T_xM\setminus\{0\}$. The coefficients of the Chern connection are

$$\Gamma^k_{ij}=\frac{1}{2}g^{il}\left(\frac{\delta g_{kl}}{\delta x^j}+\frac{\delta g_{jl}}{\delta x^k}-\frac{\delta g_{jk}}{\delta x^l}\right)$$

where

$$\begin{aligned}\gamma^i_{jk}&:=\frac{1}{2}g^{is}\Big(\frac{\partial g_{sj}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{ks}}{\partial x^j}\Big),\\ G^j&=\frac{1}{2}\gamma^i_{jk}y^iy^j,\\ N^j_i&=\frac{\partial G^j}{\partial y^i},\\ \frac{\delta}{\delta x^i}&=\frac{\partial}{\partial x^i}-N^j_i\frac{\partial}{\partial y^j},\end{aligned}$$

and the pair $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\}$ forms a horizontal and vertical frame for TTM.

The coefficients of the Riemann curvature $R_y=R^i_kdx^i\otimes\frac{\partial}{\partial x^i}$ are given by

$$R^i{}_k:=2\frac{\partial G^i}{\partial x^k}-\frac{\partial^2G^i}{\partial x^j\partial y^k}y^j+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k},\tag{2.1}$$

and the Ricci scalar function of F is given by

$$\mathcal{R}ic:=\frac{1}{F^2}R^i_i.$$

A companion of the Ricci scalar is the Akbar-Zadeh's Ricci tensor

$$Ric_{ij} := \left(\frac{1}{2}F^2\mathcal{R}ic\right)_{y^iy^j}.$$

Let $V = v^i\frac{\partial}{\partial x^i}$ be a smooth vector field on Finsler manifold M. The complete lift of V is a globally defined vector field on TM_0 given by

$$\hat{V} = v^i\frac{\partial}{\partial x^i} + y^j(\frac{\partial v^i}{\partial x^j})\frac{\partial}{\partial y^i}$$

and the Lie derivative of a Finsler metric tensor g_{jk} in direction $\hat{V}$ is given by

$$\mathcal{L}_{\hat{V}}g_{jk} = \nabla_j v_k + \nabla_k v_j + 2(\nabla_0 v^l)C_{ljk}, \tag{2.2}$$

where ∇ is the Cartan h-covariant derivative, $\nabla_0 := y^p\nabla_{\frac{\delta}{\delta x^p}}$. For any piecewise smooth curve $\gamma : [a, b] \to M$ on (M, F) with the velocity

$$\frac{d\gamma}{dt} = \frac{d\gamma^i}{dt}\frac{\partial}{\partial x^i} \in T_{\gamma(t)}M,$$

the integral length $L(\gamma)$ is given by

$$L(\gamma) = \int_a^b F\big(\gamma, \frac{d\gamma}{dt}\big)dt$$

and distance function $d : M \times M \to [0, \infty)$ defined by

$$d(p, q) = \inf\Big\{L(\gamma) : \gamma \in \Gamma(p, q)\Big\},$$

where $\Gamma(p, q)$ denotes the collection of all piecewise smooth curve $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$.

Now, suppose that $\gamma(s)$, $s \in [0, r]$, is a geodesic of Cartan connection parameterized by the arc length s with variation $\beta(s, t)$. Let

$$T = \frac{\partial\beta}{\partial s},$$
$$U = \frac{\partial\beta}{\partial t},$$
$$\hat{\beta} : \Big\{(s, t)|0 \le s \le r, -\epsilon \le t \le \epsilon\Big\} \to TM_0$$

defined by

$$\hat{\beta}(s, t) = \Big(\beta(s, t), T(s, t)\Big)$$

be canonical lift of β. Then, the second variation of arc length in Finsler geometry is given by

$$L''(0) = g(\nabla_{\hat{U}}U, T)\Big|_0^r + \int_0^r \left[g(\nabla_{\hat{T}}U, \nabla_{\hat{T}}U) - g(R(U, T)T, U) - \Big|\frac{\partial}{\partial s}g(U, T)\Big|^2\right] ds, \tag{2.3}$$

where

$$\hat{T} = \frac{\partial \hat{\beta}}{\partial s}, \quad \hat{U} = \frac{\partial \hat{\beta}}{\partial t}.$$

Let us denote by S_xM the set consisting of all rays $[y] := \{\lambda y | \lambda > 0\}$, where $y \in T_xM_0$. The sphere bundle of M, i.e. SM, is the union of S_xM's , $SM = \bigcup\limits_x S_xM$, and it has a natural $(2n-1)$-dimensional manifold structure.

Let $u : M \to SM$ be a unitary vector fields and $\omega = u_i dx^i$ the corresponding 1-form. We consider the volume form

$$\eta = \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \omega \wedge (d\omega)^{n-1}$$

on the sphere bundle SM. Let $\alpha = \alpha_i dx^i$ be a horizontal 1-form on SM, then the divergence of α with respect to the Cartan connection is defined as

$$div(\omega) = -\nabla_i \alpha^i + \alpha_i \nabla_0 C^i,$$

where C^i is the trace of Cartan tensor, and when manifold M is closed we get

$$\int_{SM} div(\alpha)\eta = -\int_{SM} (\nabla_i \alpha^i - \alpha_i \nabla_0 C^i)\eta = 0. \tag{2.4}$$

Also, given a vector field $V = v^i \frac{\partial}{\partial x^i}$ on M define

$$||V||_x = \max_{y \in S_xM} \sqrt{g_{ij} v^i v^j},$$

where $x \in M$.

3. Compact Ricci-Bourguignon soliton

Let (M, F) be a Finsler manifold and $V = v^i \frac{\partial}{\partial x^i}$ a vector field on M. Similar to Riemannian manifolds, a Finslerian Ricci-Bourguignon soliton is a Finsler manifold (M^n, F) endowed with a vector filed V on M such that the fundamental tensor g of F satisfies

$$2Ric_{ij} + \mathcal{L}_{\hat{V}} g_{ij} = 2\lambda g_{ij} + 2\rho H g_{ij}, \tag{3.1}$$

where $\hat{V}$ is the complete lift of V, $H = g^{ij} Ric_{ij}$ and λ is a real constant. Multiplying the both sides of (3.1) by $y^i y^j$, we obtain

$$2F^2 Ric + \mathcal{L}_{\hat{V}} F^2 = 2\lambda F^2 + 2\rho H F^2. \tag{3.2}$$

The Finslerian Ricci-Bourguignon soliton is called expanding, steady, shrinking when λ is negative, zero and positive, respectively. When manifold (M, F) is forward complete (res. compact) then Finslerian Ricci-Bourguignon soliton is called forward complete (res. compact).

Proof of Theorem 1.1: If M is compact manifold then $||V||$ will be bounded on M. Conversely, assume that $||V||$ is bounded on M by a constant K_2

and p, q be two points in M jointed by a minimal geodesic $\alpha : [0, \infty) \to M$ parameterized by the arc length t. According (2.2), along geodesic α we have

$$\alpha'^i\alpha'^j\mathcal{L}_{\hat{V}}g_{jk} = \alpha'^i\alpha'^j\Big(\nabla_j v_k + \nabla_k v_j + 2(\nabla_0 v^l)C_{ljk}\Big). \tag{3.3}$$

Since along the geodesic α, we have

$$\alpha'^i\alpha'^j\nabla_0 v^l C_{ljk}\Big(\alpha(t), \alpha'(t)\Big) = 0,$$

then (3.3) becomes

$$\alpha'^i\alpha'^j\mathcal{L}_{\hat{V}}g_{jk} = 2\alpha'^i\alpha'^j\nabla_j v_k. \tag{3.4}$$

Replacing

$$\alpha'^i\alpha'^j\nabla_j v_k = \frac{d}{dt}\Big(\alpha'^k v_k\Big)$$

in (3.4) we get

$$\alpha'^i\alpha'^j\mathcal{L}_{\hat{V}}g_{jk} = 2\frac{d}{dt}\Big(\alpha'^k v_k\Big). \tag{3.5}$$

Multiplying the both sides of (3.1) by $\alpha'^i\alpha'^j$ and using (3.5) we obtain

$$\alpha'^i\alpha'^j Ric_{ij} \geq \alpha'^i\alpha'^j(\lambda + \rho H)g_{ij} - \frac{d}{dt}\Big(\alpha'^k v_k\Big) \geq \lambda + \rho K_1 + \frac{d}{dt}(-\alpha'^k v_k). \tag{3.6}$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned} |-\alpha'^k v_k| &= |g_{kl}\Big(\alpha(t), \alpha'(t)\Big)\alpha'^k v_l| \leq |g_{kl}\Big(\alpha(t), \alpha'(t)\Big)v^k v^l|^{\frac{1}{2}} \\ &\leq \max_{y \in S_{\alpha(t)}M} |g_{kl}\Big(\alpha(t), \alpha'(t)\Big)v^k v^l|^{\frac{1}{2}} \\ &= ||V||_{\alpha(t)} \\ &\leq K_2. \end{aligned}$$

Now, the Lemma 1 of [2] implies that M is compact and

$$diam(M) \leq \frac{\pi}{\lambda + \rho K_1}\Big(K_2 + \sqrt{K_2^2 + (n-1)(\lambda + \rho K_1)}\Big).$$

This completes the proof. □

By Theorem 1.1, we get the following.

Corollary 3.1. *Let (M, F) be a forward complete shrinking Finslerian Ricci-Bourguignon soliton. If $H \leq K_1$ and $\lambda + \rho H \geq 0$ for some positive real constant ρ, then M is compact if and only if $||V||$ is bounded on M by a constant K_2 and moreover, in this case we have*

$$diam(M) \leq \frac{\pi}{\lambda + \rho K_1}\left(K_2 + \sqrt{K_2^2 + (n-1)(\lambda + \rho K_1)}\right).$$

Let (M, F) be a Finsler manifold and $p \in M$. Set

$$\Lambda_p := \sup_{x\in\mathcal{B}_p^+(1)\cup\mathcal{B}_p^-(1)} \max_{y\in S_xM} |Ric(x, y)|,$$

where

$$\mathcal{B}_p^+(1) := \Big\{x \in M|d(p, x) < 1\Big\}, \quad \mathcal{B}_p^-(1) := \Big\{x \in M|d(x, p) < 1\Big\}.$$

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2: Without loss of generality we assume that $d(p, q) > 1$. Let p, q be two points in M jointed by a minimal geodesic $\alpha : [0, \infty) \to M$ parameterized by the arc length t. Taking integral of both sides of (3.6) we get

$$\int_0^r Ric(\alpha, \alpha')dt \geq (\lambda + \rho K_1)r - \alpha'^k(r)v_k + \alpha'^k(0)v_k. \tag{3.7}$$

The Cauchy-Schwarz inequality implies that

$$|\alpha'^k(0)v_k| \leq ||V||_p$$

and

$$|\alpha'^k(r)v_k| \leq ||V||_q.$$

Hence, we can write (3.7) as

$$\int_0^r Ric(\alpha, \alpha')dt \geq (\lambda + \rho K_1)r - ||V||_p - ||V||_q. \tag{3.8}$$

On the other hand from Lemma 3.1 of [10], we have

$$\int_0^r Ric(\alpha, \alpha')dt \leq 2(n - 1) + \Lambda_p + \Lambda_q. \tag{3.9}$$

Substutiting (3.9) into (3.8), we conclude

$$2(n - 1) + \Lambda_p + \Lambda_q \geq (\lambda + \rho K_1)r - ||V||_p - ||V||_q, \tag{3.10}$$

which proves (1.6). □

Corollary 3.2. *Let (M, F) be a complete shrinking Finsler Ricci-Bourguignon soliton. If $H \leq K_1$ and $\lambda + \rho H \geq 0$ for some positive real constant ρ, then for any two points p, q in M we have (1.6).*

Proof of Theorem 1.3: Let $p : \tilde{M} \to M$ be the universal covering manifold of M, it is well known that the fundamental group is in one-to-one corresponding with discrete counterimage of a basepoint $x \in M$. The pullback of the complete lift $\hat{p} : T\tilde{M} \to TM$ given by

$$\hat{p}(\tilde{x}, \tilde{y}) = \Big(p(\tilde{x}),\ \tilde{y}^i \frac{\partial p}{\partial \tilde{x}^i} \frac{\partial}{\partial x^i}\Big).$$

It defines a Finsler structure on $\tilde{M}$ as

$$\tilde{F} = \hat{p}^* F := F \circ \hat{p} : T\tilde{M} \to [0, \infty).$$

Notice that $p : (\tilde{M}, \tilde{F}) \to (M, F)$ is a local isometry. We have

$$\begin{aligned} &\hat{p}^* g = \tilde{g}, \\ &\hat{p}^* Ric = \tilde{Ric}, \\ &\hat{p}^* \mathcal{L}_{\hat{V}} g = \mathcal{L}_{\hat{W}} \tilde{g}, \end{aligned}$$

where $W = p^* V$. Inequality (1.5) implies that

$$2\tilde{Ric}_{ij} + \mathcal{L}_{\hat{W}} \tilde{g}_{ij} \geq 2(\lambda + \rho \tilde{H}) \tilde{g}_{ij}. \tag{3.11}$$

Let h be a deck transformation on $\tilde{M}$ and $\tilde{x} \in \tilde{M}$. Since h is an isometry and $\tilde{H} = \hat{p}^* H$, we get

$$\tilde{H} \leq K_1.$$

By Theorem 1.2, we can write

$$\begin{aligned} d\Big(\tilde{x}, h(\tilde{x})\Big) &\leq \max \left\{ 1, \frac{1}{\lambda + \rho K_1} \Big(2(n-1) + \Lambda_{\tilde{x}} + \Lambda_{h(\tilde{x})} + ||W||_{\tilde{x}} + ||W||_{h(\tilde{x})} \Big) \right\} \\ &= \max \left\{ 1, \frac{2}{\lambda + \rho K_1} \Big(2(n-1) + \Lambda_{\tilde{x}} + ||W||_{\tilde{x}} \Big) \right\}. \end{aligned}$$

Let $x = p(\tilde{x})$. Then $p^{-1}(x)$ is forward bounded and the closed and forward bounded subset $p^{-1}(x)$ of $\tilde{M}$ is compact and being discrete. Since $\tilde{M}$ is a universal covering and $\pi_1(M, x)$ is in a bijective corresponding with $p^{-1}(x)$ we conclude $\pi_1(M, x)$ is finite. On the other hand M is connected, hence all of its fundamental group $\pi_1(M, x)$, $x \in M$ are isomorphic. Therefore $\pi_1(M)$ is finite. □

Then, we conclude the following.

Corollary 3.3. *Let (M, F) be a complete shrinking Finsler Ricci-Bourguinon soliton. If $H \leq K_1$ and $\lambda + \rho H \geq 0$ for some positive real constant ρ, then the fundamental group $\pi_1(M)$ of M is finite.*

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