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Locally Dually Flatness and Locally Projectively Flatness of Matsumoto change with m-th root Finsler metrics

Manoj Kumar[a*]

[a]Department of Mathematics and Stataistics
Faculty of Science
Dr. Rammanohar Lohia Avadh University
Ayodhya-224001, India.

E-mail: mvermamath@gmail.com and mkvermabhu1986@gmail.com

Abstract. In this paper, we study the Matsumoto change of m-th root Finsler metric. We find the necessary and sufficient conditions under which the transformed metric be locally dually flat. Also, we prove that for Matsumoto change of m-th root metric is locally projectively flat if and only if it is locally Minkowskian.

Keywords: Finsler metric, m-th root metric, Matsumoto change, Locally projectively flat metric and Locally dually flat metric.

1. Introduction

The Matsumoto metric is an important and interesting Finsler metric, which is realization of Finsler's idea of a slope measure of a mountain with respect to a time measure ([7] and [11]). If $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric on Earth's surface and $\beta = b_iy^i$ is a one form, depends on Earth's gravity then Matsumoto metric F is defined by

$$F(x,y) = \frac{\alpha^2(x,y)}{\alpha(x,y) - \beta(x,y)}.$$

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*Corresponding Author

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A β-change $\bar{F}$ of Finsler metric F, is defined as $\bar{F}(x,y) = f(F,\beta)$, where $f(F,\beta)$ is a positively homogeneous function. We discuss a special $\beta-$ change, as

$$\bar{F} = \frac{F^2}{F-\beta},$$

known Matsumoto change. In particular, if F is a Riemannian metric, then $\bar{F}$ becomes to Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$.

The theory of m-th root metrics has been developed by H. Shimada ([10]) and applied to Biology as an ecological metric and studied by many authors ([9], [12], [13] and [14]). It is regarded as a direct generalization of the theory of Riemannian metric in the sense that the second root metric is a Riemannian metric. The third and fourth root metrics are called the cubic metric and quartic metric, respectively. Recently studies show that the theory of m-th root Finsler metrics play a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory.

Suppose $F = \sqrt[m]{A}$ be m-th root metric, as A is given by

$$A := a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$$

with $a_{i_1 \ldots i_m}$ symmetric in every indices. Let us put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, A_{0l} = A_{x^i y^l} y^i, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j},$$

and

$$B_i = \frac{\partial B}{\partial y^i}, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i} y^i, \quad B_{0l} = B_{x^i y^l} y^i.$$

In information geometry on Riemannian manifolds, Amari and Nagaoka [1] proposed concept of locally dually flat Riemannian metrics. In [9], Shen enhanced concept of locally dually flatness. In [12], Tayebi-Najafi proved for locally dually flat and Antonelli m-th root metrics. Nowdays, A. Tayebi et.al. [14], studied Kropina change for locally dually flat.

In this paper, we prove the following.

Theorem 1.1. *Let $F = \sqrt[m]{A}$ be an m-th root metric on open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F^2/(F-\beta)$ is the Matsumoto change of F. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x) y^l$ such that the following hold*

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 4\theta A_l\Big], \tag{1.1}$$

$$4A\beta_l A_0 - \Big(\frac{9}{m} - 1\Big) A^{\frac{1}{m}} A_l A_0 - A^{\frac{1}{m}+1} A_{0l} + 4AA_l\beta_0 + mA^2\beta_{0l} + 2A^{\frac{1}{m}+1} A_{x^l} = 2mA^2 \beta_{x^l}, \tag{1.2}$$

$$\frac{1}{m} A^{\frac{2}{m}} A_0 A_l - \beta_l A^{\frac{1}{m}+1} A_0 - A^{\frac{1}{m}+1} A_l \beta_0 + mA^2 \beta_0 \beta_l = 0, \tag{1.3}$$

where

$$\beta_0=\beta_{x^i}y^i,\quad \beta_{0l}=\beta_{x^k y^l}y^k,\quad \beta_{x^l}=(b_i)_{x^l}y^i.$$

Distance functions induced by a Finsler metrics are regarded as *smooth* ones. The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Such Finsler metrics are called *projectively flat Finsler metrics* or briefly *projective Finsler metrics*. G. Hamel first characterizes projective Finsler metrics by a system of PDE's [4]. Later on, A. Rapcsák extends Hamel's result to projectively equivalent Finsler metrics [8]. In this paper, w we consider the Matsumoto change of an m-th root metric such that the transformed metric is locally projectively flat. Then we prove the following.

Theorem 1.2. *Let $F=\sqrt[m]{A}$ be an m-th root metric on open subset $U\subset\mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F}=F^2/(F-\beta)$ is the Matsumoto change of F. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.*

2. Preliminaries

Let M is n-dimensional C^∞-manifold. The tangent space at $x\in M$ are given by T_xM and tangent bundle of M denoted by $TM:=\bigcup_{x\in M}T_xM$. Every element of TM is of the form (x,y), where $x\in M$ and $y\in T_xM$. Let $TM_0=TM\setminus\{0\}$.

Definition: A metric is a function $F:TM\to[0,\infty)$ on M with following properties:
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on TM and
(iii) the Hessian of $F^2/2$ with components

$$g_{ij}=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j}$$

is positive definite on TM_0. The pair $F^n=(M,F)$ is said to be a Finsler space of dimension n. F is said fundamental function and tensor g with components g_{ij} is said fundamental tensor of Finsler space F^n.

The normalized element l_i and angular metric tensor h_{ij} are defined, respectively as:

$$l_i=\frac{\partial F}{\partial y^i},\quad h_{ij}=F\frac{\partial^2F}{\partial y^i\partial y^j}.$$

Locally, geodesics of a metric are given by

$$\frac{d^2x^i}{dt^2}+2G^i(x,\frac{dx^i}{dt})=0,$$

where

$$G^i = \frac{1}{4} g^{il} \Big\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \Big\}$$

are called spray coefficient.

For a non-zero vector $y \in T_x M_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_x M \to T_x M$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y) u^k \frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

For a flag $P := \text{span}\{y, u\} \subset T_x M$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

The flag curvature $\mathbf{K}(x, y, P)$ is a function of tangent planes $P = \text{span}\{y, v\} \subset T_x M$. This quantity tells us how curved the space is at a point. If F is a Riemannian metric, $\mathbf{K}(x, y, P) = \mathbf{K}(x, P)$ is independent of $y \in P \setminus \{0\}$. Thus the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry.

A metric F is said to be locally dually flat if,

$$F^2_{x^k y^l} y^k = 2F^2_{x^l}.$$

A metric is called Berwald metric, if spray coefficients G^i are quadratic. A metric $F(x, y)$ is called locally projectively flat if its geodesic coefficients G^i given as $G^i(x, y) = P(x, y) y^i$, where $P : TU = U \times \mathbb{R}^n \to \mathbb{R}$ is homogeneous (positively) of degree one in y, that is $P(x, \lambda y) = \lambda P(x, y), \lambda > 0$ [5]. Here P is projective factor.

A metric is called locally projectively flat if, the geodesics are straight lines. A metric $F = F(x, y)$ is projectively flat on $U \subset \mathbb{R}^n$ if and only if

$$F_{x^t y^l} y^t - F_{x^l} = 0.$$

3. Proof of Theorem 1.1

For proving theorem, we need the following lemma:

Lemma 3.1. [14] *Suppose that the following equation holds*

$$\Omega \Big\{ A^{\frac{1}{m}} - \beta \Big\}^3 + \Phi \Big\{ A^{\frac{1}{m}} - \beta \Big\}^2 + \zeta \left\{ A^{\frac{1}{m}} - \beta \right\} + \Delta = 0,$$

where Ω, Φ, ζ, Δ are polynomials in y and $m > 2$. Then

$$\Omega = \Phi = \zeta = \Delta = 0.$$

Proof. Suppose $\bar{F}$ is locally dually flat. The following holds

$$\bar{F}^2 = \frac{F^4}{(F-\beta)^2},$$

Thus we get

$$(\bar{F}^2)_{x^k} = \left[\frac{A^{\frac{4}{m}}}{(A^{\frac{1}{m}}-\beta)^2}\right]_{x^k} = -\frac{1}{(A^{\frac{1}{m}}-\beta)^3}\left[\frac{2}{m}A^{\frac{5}{m}-1}A_{x^k} - 2A^{\frac{4}{m}}\beta_{x^k}\right] + \frac{4}{m}\frac{A^{\frac{4}{m}-1}A_{x^k}}{(A^{\frac{1}{m}}-\beta)^2}$$

Also, we have

$$\begin{aligned}(\bar{F}^2)_{x^k y^l}y^k \quad = \quad & \frac{4}{m(A^{\frac{1}{m}}-\beta)^2}\left[\Big(\frac{4}{m}-1\Big)A^{\frac{4}{m}-2}A_lA_0 + A^{\frac{4}{m}-1}A_{0l}\right] \\ & -\frac{8}{m(A^{\frac{1}{m}}-\beta)^3}\left[\frac{1}{m}A^{\frac{5}{m}-2}A_0A_l - A^{\frac{4}{m}-1}A_0\beta_l\right] \\ & -\frac{2}{(A^{\frac{1}{m}}-\beta)^3}\Bigg[\frac{1}{m}\Big(\frac{5}{m}-1\Big)A^{\frac{5}{m}-2}A_lA_0 + \frac{1}{m}A^{\frac{5}{m}-1}A_{0l} \\ & -\frac{4}{m}A^{\frac{4}{m}-1}A_l\beta_0 - A^{\frac{4}{m}}\beta_{0l}\Bigg] + \frac{6}{(A^{\frac{1}{m}}-\beta)^5}\Bigg[\frac{1}{m^2}A^{\frac{6}{m}-2}A_0A_l \\ & -\frac{1}{m}A^{\frac{5}{m}-1}\Big(A_0\beta_l + A_l\beta_0\Big) + A^{\frac{4}{m}}\beta_0\beta_l\Bigg].\end{aligned}$$

Therefore, we obtain the following

$$\begin{aligned}(\bar{F}^2)_{x^k y^l} y^k - 2(\bar{F}^2)_{x^l} &= \frac{4}{m(A^{\frac{1}{m}} - \beta)^2}\left[(\frac{4}{m} - 1)A^{\frac{4}{m}-2}A_l A_0 + A^{\frac{4}{m}-1}A_{0l}\right] \\ &-\frac{8}{m(A^{\frac{1}{m}} - \beta)^3}\left[\frac{1}{m}A^{\frac{5}{m}-2}A_0 A_l - A^{\frac{4}{m}-1}A_0\beta_l\right] \\ &-\frac{1}{(A^{\frac{1}{m}} - \beta)^3}\left[\frac{2}{m}(\frac{5}{m} - 1)A^{\frac{5}{m}-2}A_l A_0 + \frac{2}{m}A^{\frac{5}{m}-1}A_{0l}\right. \\ &\left.-\frac{8}{m}A^{\frac{4}{m}-1}A_l\beta_0 - 2A^{\frac{4}{m}}\beta_{0l}\right] \\ &+\frac{6}{(A^{\frac{1}{m}} - \beta)^5}\left[\frac{1}{m^2}A^{\frac{6}{m}-2}A_0 A_l + A^{\frac{4}{m}}\beta_0\beta_l\right. \\ &\left.-\frac{1}{m}A^{\frac{5}{m}-1}(A_0\beta_l + A_l\beta_0)\right] - \frac{8}{m}\frac{A^{\frac{4}{m}-1}A_{x^l}}{(A^{\frac{1}{m}} - \beta)^2} \\ &+\frac{4}{(A^{\frac{1}{m}} - \beta)^3}\left(\frac{1}{m}A^{\frac{5}{m}-1}A_{x^l} - A^{\frac{4}{m}}\beta_{x^l}\right).\end{aligned}$$

Thus $(\bar{F}^2)_{x^k y^l} y^k - 2(\bar{F}^2)_{x^l} = 0$ implies that

$$\begin{aligned}&(A^{\frac{1}{m}} - \beta)^3\left[\frac{4}{m}(\frac{4}{m} - 1)A^{\frac{4}{m}-2}A_l A_0 + \frac{4}{m}A^{\frac{4}{m}-1}A_{0l} - \frac{8}{m}A^{\frac{4}{m}-1}A_{x^l}\right] \\ &+(A^{\frac{1}{m}} - \beta)^2\left[-\frac{8}{m^2}A^{\frac{5}{m}-2}A_l A_0 + \frac{8}{m}A^{\frac{4}{m}-1}\beta_l A_0 - \frac{2}{m}(\frac{5}{m} - 1)A^{\frac{5}{m}-2}A_l A_0\right. \\ &\left.-\frac{2}{m}A^{\frac{5}{m}-1}A_{0l} + \frac{8}{m}A^{\frac{4}{m}-1}A_l\beta_0 + 2A^{\frac{4}{m}}\beta_{0l} + \frac{4}{m}A^{\frac{5}{m}-1}A_{x^l} - 4A^{\frac{4}{m}}\beta_{x^l}\right] \\ &+\frac{1}{m^2}A^{\frac{6}{m}-2}A_l A_0 - \frac{1}{m}A^{\frac{5}{m}-1}A_0\beta_l - \frac{1}{m}A^{\frac{5}{m}-1}A_l\beta_0 + A^{\frac{4}{m}}\beta_l\beta_0 = 0. \qquad (3.1)\end{aligned}$$

By Lemma 3.1, the relation (3.1) implies that

$$\left(\frac{4}{m} - 1\right)A_l A_0 + AA_{0l} - 2AA_{x^l} = 0, \qquad (3.2)$$

$$\begin{aligned}&4A\beta_l A_0 - \left(\frac{9}{m} - 1\right)A^{\frac{1}{m}}A_l A_0 - A^{\frac{1}{m}+1}A_{0l} + 4AA_l\beta_0 + mA^2\beta_{0l} \\ &\qquad +2A^{\frac{1}{m}+1}A_{x^l} - 2mA^2\beta_{x^l} = 0, \qquad (3.3)\end{aligned}$$

$$\frac{1}{m}A^{\frac{2}{m}}A_0 A_l - \beta_l A^{\frac{1}{m}+1}A_0 - A^{\frac{1}{m}+1}A_l\beta_0 + mA^2\beta_0\beta_l = 0. \qquad (3.4)$$

(3.2) can be rewrite as follows

$$A(2A_{x^l} - A_{0l}) = \left(\frac{4}{m} - 1\right)A_0 A_l. \qquad (3.5)$$

Since $Deg(A_l) = m-1$, then by irreducibility of A, there exist a 1-form $\theta = \theta_l y^l$ such that

$$A_0 = \theta A. \tag{3.6}$$

From definition of A_0, we write

$$A_{x^i} y^i = \theta A. \tag{3.7}$$

Differentiating (3.7) with respect to l implies that

$$A_{x^i y^l} y^i + A_{x^i} \delta^i_l = \theta_l A + A_l \theta$$

or equivalently

$$A_{0l} = \theta_l A + A_l \theta - A_{x^l}. \tag{3.8}$$

Further, substituting (3.6) and (3.8) in (3.5) yields (1.1). The converse is obvious. This completes the proof. □

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. In order to prove Theorem 1.2, we need the following.

Proposition 4.1. *Let $F = \sqrt[m]{A}$ be an m-th root metric on open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F^2/(F-\beta)$ is the Matsumoto change of F. If $\bar{F}$ is a projectively flat metric, then it is reduced to a Berwald metric.*

Proof. Suppose $\bar{F}$ is a projectively flat metric. We have

$$\bar{F} = \frac{F^2}{F-\beta}.$$

Thus

$$(\bar{F})_{x^k} = \left[\frac{A^{\frac{2}{m}}}{(A^{\frac{1}{m}} - \beta)}\right]_{x^k} = \frac{2}{m}\frac{A^{\frac{2}{m}-1}A_{x^k}}{(A^{\frac{1}{m}} - \beta)} - \frac{1}{(A^{\frac{1}{m}} - \beta)^2}\left[\frac{1}{m}A^{\frac{3}{m}-1}A_{x^k} - A^{\frac{2}{m}}\beta_{x^k}\right].$$

We have

$$\begin{aligned}
(\bar{F})_{x^k y^l} y^k = {} & \frac{2}{m(A^{\frac{1}{m}} - \beta)}\left[(\frac{2}{m}-1)A^{\frac{2}{m}-2}A_l A_0 + A^{\frac{2}{m}-1}A_{0l}\right] \\
& - \frac{2}{m(A^{\frac{1}{m}} - \beta)^2}\left[\frac{1}{m}A^{\frac{3}{m}-2}A_0 A_l - A^{\frac{2}{m}-1}A_0\beta_l\right] \\
& - \frac{1}{(A^{\frac{1}{m}} - \beta)^2}\left[\frac{1}{m}(\frac{3}{m}-1)A^{\frac{3}{m}-2}A_l A_0 + \frac{1}{m}A^{\frac{3}{m}-1}A_{0l} - \frac{2}{m}A^{\frac{2}{m}-1}A_l\beta_0 - A^{\frac{2}{m}}\beta_{0l}\right] \\
& + \frac{2}{(A^{\frac{1}{m}} - \beta)^3}\left[\frac{1}{m^2}A^{\frac{4}{m}-2}A_0 A_l - \frac{1}{m}A^{\frac{3}{m}-1}\left(A_0\beta_l + A_l\beta_0\right) + A^{\frac{2}{m}}\beta_0\beta_l\right].
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(\bar{F})_{x^k y^l} y^k - (\bar{F})_{x^l} &= \frac{2}{m(A^{\frac{1}{m}} - \beta)} \left[(\frac{2}{m} - 1) A^{\frac{2}{m}-2} A_l A_0 + A^{\frac{2}{m}-1} A_{0l} \right] \\ &\quad - \frac{2}{m(A^{\frac{1}{m}} - \beta)^2} \left[\frac{1}{m} A^{\frac{3}{m}-2} A_0 A_l - A^{\frac{2}{m}-1} A_0 \beta_l \right] \\ - \frac{1}{(A^{\frac{1}{m}} - \beta)^2} &\left[\frac{1}{m} (\frac{3}{m} - 1) A^{\frac{3}{m}-2} A_l A_0 + \frac{1}{m} A^{\frac{3}{m}-1} A_{0l} - \frac{2}{m} A^{\frac{2}{m}-1} A_l \beta_0 - A^{\frac{2}{m}} \beta_{0l} \right] \\ &+ \frac{2}{(A^{\frac{1}{m}} - \beta)^3} \left[\frac{1}{m^2} A^{\frac{4}{m}-2} A_0 A_l - \frac{1}{m} A^{\frac{3}{m}-1} (A_0 \beta_l + A_l \beta_0) + A^{\frac{2}{m}} \beta_0 \beta_l \right] \\ &\quad - \frac{2}{m} \frac{A^{\frac{2}{m}-1} A_{x^l}}{(A^{\frac{1}{m}} - \beta)} + \frac{1}{(A^{\frac{1}{m}} - \beta)^2} \left(\frac{1}{m} A^{\frac{3}{m}-1} A_{x^l} - A^{\frac{2}{m}} \beta_{x^l} \right).\end{aligned}$$

Then $(\bar{F})_{x^k y^l} y^k - (\bar{F})_{x^l} = 0$ implies that

$$\begin{aligned}&(A^{\frac{1}{m}} - \beta)^2 \left[(\frac{2}{m} - 1) \frac{2}{m} A^{\frac{2}{m}-2} A_l A_0 + \frac{2}{m} A^{\frac{2}{m}-1} A_{0l} - \frac{2}{m} A^{\frac{2}{m}-1} A_{x^l} \right] \\ &+ (A^{\frac{1}{m}} - \beta) \left[-\frac{2}{m^2} A^{\frac{3}{m}-2} A_l A_0 + \frac{2}{m} A^{\frac{2}{m}-1} \beta_l A_0 - \frac{1}{m} (\frac{3}{m} - 1) A^{\frac{3}{m}-2} A_l A_0 \right. \\ &\left. - \frac{1}{m} A^{\frac{3}{m}-1} A_{0l} + \frac{2}{m} A^{\frac{2}{m}-1} A_l \beta_0 + A^{\frac{2}{m}} \beta_{0l} + \frac{1}{m} A^{\frac{3}{m}-1} A_{x^l} - A^{\frac{2}{m}} \beta_{x^l} \right] \\ &+ \frac{1}{m^2} A^{\frac{4}{m}-2} A_l A_0 - \frac{1}{m} A^{\frac{3}{m}-1} A_0 \beta_l - \frac{1}{m} A^{\frac{3}{m}-1} A_l \beta_0 + A^{\frac{2}{m}} \beta_l \beta_0 = 0. \quad (4.1)\end{aligned}$$

By Lemma 3.1, the relation (4.1) yields

$$\left(\frac{2}{m} - 1 \right) A_l A_0 + A A_{0l} - A A_{x^l} = 0, \quad (4.2)$$

$$\begin{aligned}2A\beta_l A_0 - \left(\frac{5}{m} - 1 \right) A^{\frac{1}{m}} A_l A_0 - A^{\frac{1}{m}+1} A_{0l} + 2A A_l \beta_0 + m A^2 \beta_{0l} \\ + A^{\frac{1}{m}+1} A_{x^l} = m A^2 \beta_{x^l}, \quad (4.3)\end{aligned}$$

$$\frac{1}{m} A^{\frac{2}{m}} A_0 A_l - \beta_l A^{\frac{1}{m}+1} A_0 - A^{\frac{1}{m}+1} A_l \beta_0 + m A^2 \beta_0 \beta_l = 0. \quad (4.4)$$

We have $Deg(A_l) = m-1 < deg(A)$. From (4.2), the irreducibility of A implies that A divides A_0. Therefore there exists a 1-form $\theta = \theta_l y^l$ such that

$$A_0 = 2mA\theta.$$

A simple fact is that a Finsler metric $F = F(x,y)$ on an open subset $U \subset R^n$ is projectively flat if and only if the spray coefficients are in the form $G^i = Py^i$. It is equivalent to the following Hamel equation $F_{x^m} y^k y^m = F_{x^k}$. In this case, we have

$$P = \frac{F_{x^m} y^m}{2F}.$$

Thus

$$P = \frac{2mA\theta}{2A} = \theta.$$

Then

$$G^i = Py^i = \theta y^i$$

which means that metric F is a Berwald metric. □

Now, we are going to prove Theorem 1.2. For this aim, we need the following.

Lemma 4.2. [3] *Every Berwald metric with vanishing flag curvature* $\boldsymbol{K} = 0$ *is a locally Minkowskian metric.*

Proof of Theorem 1.2: By Proposition 4.1, if the Finsler metric $F = F(x, y)$ is projectively flat, then it becomes Berwald metric. Let $\mathbf{K} \neq 0$. By Numata theorem, every Berwald metric with non zero scalar flag curvature $\mathbf{K}$ should be Riemannian. This contradicts with our assumption. Therefore, $\mathbf{K} = 0$. By Lemma 4.2, the Finsler metric F reduces to a locally Minkowskian metric. This is the proof of Theorem 1.2. □

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