Journal of Finsler Geometry and its Applications
Vol. 1, No. 2 (2020), pp 83-93
DOI: 10.22098/jfga.2020.1242

Projective vector fields on special (α, β)-metrics

Saeedeh Masoumi

Department of Mathematics, Faculty of Science
Urmia University, Urmia, Iran.

E-mail: s.masoumi94@gmail.com

Abstract. In this paper, we study the projective vector fields on two special (α, β)-metrics, namely Kropina and Matsumoto metrics. First, we consider the Kropina metrics, and show that if a Kropina metric $F = \alpha^2/\beta$ admits a projective vector field, then this is a conformal vector field with respect to Riemannian metric α or F has vanishing S-curvature. Then we study the Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ and prove that if the Matsumoto metric $F = \alpha^2/\beta$ admits a projective vector field, then this is a conformal vector field with respect to Riemannian metric α or F has vanishing S-curvature.

Keywords: Projective vector field, Kropina metric, Matsumoto metric, S-curvature.

1. Introduction

The projective Finsler metrics are smooth solutions to the historic Hilberts fourth problem. Unlike the Riemannian metrics, a non-projective Finsler metric may be of constant flag curvature in Finsler geometry [2]. A good way to characterizing the projective metrics is the projective vector fields. A vector field V is called projective if its flow takes (unparameterized) geodesics to geodesics. The collection of all projective vector fields on a Finsler space (M, F) is a finite dimensional Lie algebra with respect to the usual Lie bracket, called the projective algebra and denoted by $p(M, F)$. Searching about projective vector fields and determining the dimension of this algebra is of interest in physical and geometrical discussions.

AMS 2020 Mathematics Subject Classification: 53C60, 53C25

In this paper, we study a class of Finsler metric called general (α,β)-metrics. An (α,β)-metric is a scalar function F on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on an open interval $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. The Randers metric $F = \alpha + \beta$, the Kropina metric $F = \frac{\alpha^2}{\beta}$, the generalized Kropina metric $F = \alpha^{1-m}\beta^m$ and Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ are special (α,β)-metrics with $\phi(s) = 1 + s$, $\phi(s) = 1/s$, $\phi(s) = s^m$ and $\phi = 1/(1 - s)$, respectively. The class of Randers metrics are popular Finsler metrics appearing in many physical and geometric studies. In [10], M. Rafie-Rad and B. Rezaei studied the projective vector fields on Randers metrics. They proved that if (M, F) be an n-dimensional $(n \geq 3)$ equipped with a Randers metric of constant flag curvature and M be compact, then the dimension of the projective algebra $p(M, F)$ is either $n(n + 2)$ or at most equals $n(n + 1)/2$. Moreover, they showed that a vector field V on Randers space (M, F) is projective vector field if and only if V is projective vector field on (M, α) and

$$\ell_{\hat{V}}({s^i}_0) = 0.$$

In [9], Rafie-Rad studied the projective vector fields on the class of Randers metrics. He introduced Lie sub-algebra of projective vector fields of a Finsler metric and proved that a Randers metric of non-zero constant S-curvature is projective if and only if the dimension of this sub-algebra is $n(n + 1)/2$.

In this paper, we study the projective vector fields on two important subclass of (α,β)-metrics. First, we study the Kropina metrics. The Kropina metrics are closely related to physical theories. These metrics, was introduced by Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [8]. We prove the following.

Theorem 1.1. *Let $F = \alpha^2/\beta$ be a Kropina metric on manifold M. Suppose that F admits a projective vector field V. Then one of the following holds*

a) *V is a conformal vector field with respect to α;*
b) *F has vanishing S-curvature* $\mathbf{S} = 0$.

The Matsumoto metric was introduced by Matsumoto as a realization of Finsler's idea "a slope measure of a mountain with respect to a time measure" [12]. He gave an exact formulation of a Finsler surface to measure the time on the slope of a hill and introduced the Matsumoto metrics in [6]. Here we study the projective vector fields on Matsumoto metric and prove the following.

Theorem 1.2. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a Matsumoto metric on a manifold M. suppose that F admits a projective vector field V. Then one of the following holds*

a) *V is a conformal vector field with respect to α;*
b) *F has vanishing S-curvature* $\mathbf{S} = 0$.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM as the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ as the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM\backslash\{0\}$. The natural projection $\pi : TM \to M$ is given by

$$\pi(x, y) = x.$$

The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π_v^*TM at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \Big\{(x, y, v)|y \in T_xM_0, v \in T_xM\Big\}.$$

A Finsler metric on a manifold M is a function $F : TM \to [0,\infty)$ which has the following properties:

(1) F is C^∞ on TM_0;
(2) $F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0$;
(3) For any tangent vector $y \in T_xM$, the vertical Hessian of $F^2/2$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^iy^j}$$

is positive definite.

Every Finsler metric F induces a spray

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$$

by

$$G^i(x, y) := \frac{1}{4}g^{il}(x, y)\Big\{2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\Big\}y^jy^k. \tag{2.1}$$

The homogeneous scalar functions G^i are called the geodesic coefficients of F. The vector field $\mathbf{G}$ is called the associated spray to (M, F).

The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \wedge \cdots \wedge dx^n$ related to F is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \;\Big|\; F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}},$$

where $\mathbb{B}^n(1)$ denotes the unit ball in $\mathbb{R}^n$.

The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form on M, i.e., $dV_{BH} = \sigma(x)dx^1 \wedge dx^2 ... \wedge dx^n$, is defined by following

$$\tau(x, y) = \ln\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S-curvature is defined by

$$\mathbf{S}(x,y)=\frac{d}{dt}\Big[\tau\big(c(t),\dot{c}(t)\big)\Big]_{t=0},$$

where $c=c(t)$ is the geodesic with $c(0)=x$ and $\dot{c}(0)=y$. In a local coordinates, the S-curvature is given by

$$\mathbf{S}=\frac{\partial G^m}{\partial y^m}-y^m\frac{\partial(\ln\sigma)}{\partial x^m}.$$

A Finsler metric F has vanishing S-curvature if $\mathbf{S}=0$.

As we know, the geodesic coefficients G^i of F and geodesic coefficients G^i_α of α are related as follows [7]:

$$G^i=G^i_\alpha+\alpha Q s^i{}_\circ+\alpha^{-1}\Theta\{r_{00}-2\alpha Q s_\circ\}y^i+\Psi\{r_{00}-2\alpha Q s_\circ\}b^i, \tag{2.2}$$

where

$$\begin{aligned}Q&=\frac{\phi'}{\phi-s\phi'},\\ \Theta&=\frac{\phi\phi'-s(\phi\phi''-\phi'\phi')}{2\{(\phi-s\phi')+(b^2-s^2)\phi''\}},\\ \Psi&=\frac{\phi''}{2\{(\phi-s\phi')+(b^2-s^2)\phi''\}}.\end{aligned}$$

Denote the Levi-Civita connection of α by ∇ and define $b_{i|j}$ by $(b_{i|j})\theta^j:=db_i-b_j\theta_i{}^j$, where $\theta^i:=dx^i$ and $\theta_i{}^j:=\Gamma^j_{ik}dx^k$. For a generic (α,β)-metric, we use usually the following notations:

$$r_{ij}:=\frac{1}{2}\Big(b_{i|j}+b_{j|i}\Big),\qquad s_{ij}:=\frac{1}{2}\Big(b_{i|j}-b_{j|i}\Big).$$

Furthermore, we denote

$$\begin{gathered}r^i{}_j:=a^{ik}r_{kj},\quad r_{00}:=r_{ij}y^iy^j,\quad r_{i0}:=r_{ij}y^j,\quad r:=r_{ij}b^ib^j,\\ s^i{}_j:=a^{ik}s_{kj},\quad s_j:=b^is_{ij},\quad s_0:=s_iy^i,\quad s_{i0}:=s_{ij}y^j,\quad b^2:=b^ib_i.\end{gathered}$$

Let us define

$$\Delta:=1+sQ+(b^2-s^2)Q', \tag{2.3}$$

$$\Phi:=-(Q-sQ')(n\Delta+1+sQ)-(b^2-s^2)(1+sQ)Q''. \tag{2.4}$$

In [3], Cheng-Shen characterized (α,β)-metrics with isotropic S-curvature.

Theorem A. ([3]) Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an non-Riemannian (α,β)-metric on a manifold and $b:=\|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$, if and only if one of the following holds

(i) β satisfies

$$r_{ij}=\varepsilon(b^2a_{ij}-b_ib_j),\quad s_j=0, \tag{2.5}$$

where $\varepsilon=\varepsilon(x)$ is a scalar function, and $\phi=\phi(s)$ satisfies

$$\Phi=-2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}, \tag{2.6}$$

where k is a constant. In this case, $\mathbf{S}=(n+1)cF$ with $c=k\varepsilon$.

(ii) β satisfies

$$r_{ij}=0,\quad s_j=0. \tag{2.7}$$

In this case, $\mathbf{S}=0$, regardless of choices of a particular ϕ.

One of special type of the (α,β)-metrics that we are interested to study in this paper is Kropina metric. Let $F=\alpha^2/\beta$ be a Kropina metric on a manifold M. Then geodesic coefficients $G^i(x,y)$ are given by

$$G^i=G^i_\alpha-\frac{\alpha^2}{2\beta}s^i{}_0+\frac{1}{2b^2}\Big(\frac{\alpha^2}{\beta}s_0+r_{00}\Big)b^i-\frac{1}{b^2}\Big(s_0+\frac{\beta}{\alpha^2}r_{00}\Big)y^i. \tag{2.8}$$

For more details, see [15].

Another metric that we study in this paper is named Matsumoto metric $F=\alpha^2/\alpha-\beta$. In this case, by (2.2) the geodesic coefficients of F are as follows

$$G^i=G^i_\alpha-\frac{\alpha}{A_1}s^i{}_0+\frac{(2\alpha s_0+A_1r_{00})}{2\alpha A_1A_2}\Big[(2A_1+1)y^i-2\alpha b^i\Big], \tag{2.9}$$

where

$$A_1=A_1(s):=2s-1,$$
$$A_2=A_2(s):=3s-2b^2-1.$$

See [13].

Every vector field V on M induces naturally a transformation under the following infinitesimal coordinate transformations on TM, $(x^i,y^i)\longrightarrow(\bar{x}^i,\bar{y}^i)$ given by

$$\bar{x}^i=x^i+V^idt,$$
$$\bar{y}^i=y^i+y^k\frac{\partial V^i}{\partial x^k}dt.$$

This leads to the notion of the complete lift $\hat{V}$ (or traditionally denoted by V^C, see [14]) of V to a vector field on TM_0, given by

$$\hat{V} = V^i \frac{\partial}{\partial x^k} + y^k \frac{\partial V^i}{\partial x^k} \frac{\partial}{\partial y^i}. \tag{2.10}$$

Since almost geometric objects in Finsler geometry depends on the both points and velocities, the Lie derivatives of such geometric objects should be regarded with respect to $\hat{V}$ (Receives a family to the theory of Lie derivatives in Finsler geometry in [12]). It is a notable remark in the Lie derivative computations that $\ell_{\hat{V}} y^i = 0$ and the differential operators $\ell_{\hat{V}}$, $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ commute. A smooth vector field V on (M, F) is called projective if each local flow diffeomorphism associated with V maps geodesics onto geodesics. If V is projective and each such map preserves affine parameters, then V is called affine, otherwise it is said to be proper projective. It is easy to prove that a vector field V on the Finsler space (M, F) is a projective if and only if there is a function P defined on TM_0 such that

$$\ell_{\hat{V}} G^i = P y^i \tag{2.11}$$

and V is affine if and only if $P = 0$.

3. Proof of Main Theorems

Kropina metrics were first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and were investigated by Kropina [5]. This metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics, irreversible thermodynamics and general dynamical system represented by a Lagrangian function [1, 4, 11]. We consider that there is a projective vector field on Kropina space and prove it.

Proof of Theorem 1.1. A vector field V on (M, F) is projective if and only if there is a 1-form $P = P_i(x, y) y^i$ on M such that $\ell_{\hat{V}} G^i = P y^i$. In the case of Kropina metrics, by (2.8) we can write this equation as follows

$$\ell_{\hat{V}} \left(G^i_\alpha - \frac{\alpha^2}{2\beta} s^i{}_0 + \frac{1}{2b^2} \big(\frac{\alpha^2}{\beta} s_0 + r_{00} \big) b^i - \frac{1}{b^2} \big(s_0 + \frac{\beta}{\alpha^2} r_{00} \big) y^i \right) = P y^i.$$

Let $\ell_{\hat{V}} a_{ij} = t_{ij}$ where $t_{ij} = t_{ij}(x)$ is a scalar function on M, then equation mentioned above is equivalent to the following equality

$$\begin{aligned}
0 &= \ell_{\hat{v}} G^i_\alpha - P y^i - \Big(\frac{2\beta t_{00} - 2\alpha^2 \ell_{\hat{V}}\beta}{4\beta^2}\Big) s^i{}_{{}_0} - \frac{\alpha^2}{2\beta}\ell_{\hat{V}} s^i{}_0 \\
&\quad - \frac{1}{2b^4}\ell_{\hat{V}} b^2 b^i \Big(\frac{\alpha^2}{\beta}s_0 + r_{00}\Big) + \frac{1}{2b^2}\Big(\frac{\beta t_{00} - \alpha^2 \ell_{\hat{V}}\beta}{\beta^2}s_0 + \frac{\alpha^2}{\beta}\ell_{\hat{V}} s_0 + \ell_{\hat{V}} r_{00}\Big) b^i \\
&\quad + \frac{1}{2b^2}\Big(\frac{\alpha^2}{\beta}s_0 + r_{00}\Big)\ell_{\hat{V}} b^i + \frac{\ell_{\hat{V}} b^2}{b^4} y^i \Big(s_0 + \frac{\beta}{\alpha^2} r_{00}\Big) \\
&\quad - \frac{1}{b^2}\Big(\ell_{\hat{V}} s_0 + \frac{\alpha^2 \ell_{\hat{V}}\beta - \beta t_{00}}{\alpha^4} r_{00} + \frac{\beta}{\alpha^2}\ell_{\hat{V}} r_{00}\Big) y^i. \qquad (3.1)
\end{aligned}$$

Multipling both sides of this very equation by $2\alpha^4\beta^2 b^4$ to remove denominators and sorting by α, we can rewrite (3.1) as follows

$$0 = A^i_2 \alpha^6 + A^i_4 \alpha^4 + A^i_6 \alpha^2 + A^i_8, \qquad (3.2)$$

where

$$\begin{aligned}
A^i_2 &= b^4 \ell_{\hat{V}}\beta s^i{}_0 - \beta b^4 \ell_{\hat{V}} s^i{}_0 - \beta s_0 \ell_{\hat{V}} b^2 b^i + \beta b^2 b^i \ell_{\hat{V}} s_0 - \ell_{\hat{V}}\beta s_0 b^2 b^i \\
&\quad + \beta \ell_{\hat{V}} s_0 b^2 b^i, \\
A^i_4 &= 2\beta^2 b^4 \ell_{\hat{V}} G^i_\alpha - \beta b^4 t_{00} s^i{}_0 - \beta^2 r_{00} \ell_{\hat{V}} b^2 b^i + \beta b^2 t_{00} s^i{}_0 + \beta^2 b^2 \ell_{\hat{V}} r_{00} b^i \\
&\quad + \beta^2 b^2 r_{00} \ell_{\hat{V}} b^i + 2\beta^2 s_0 \ell_{\hat{V}} b^2 y^i - 2\beta^2 b^2 \ell_{\hat{V}} s_0 y^i - 2\beta^2 b^4 P y^i, \\
A^i_6 &= 2\beta^3 r_{00} \ell_{\hat{V}} b^2 y^i - 2\beta^2 b^2 r_{00} \ell_{\hat{V}}\beta y^i - 2\beta^3 b^2 \ell_{\hat{V}} r_{00} y^i, \\
A^i_8 &= -2\beta^3 b^2 t_{00} r_{00} y^i.
\end{aligned}$$

By (3.2) we can conclude that A^i_8 must be coefficient of α^2, i.e., there is scalar function $c(x)$ on M such that

$$r_{00} = c(x)\alpha^2$$

Then F must has vanishing S-curvature, or

$$t_{00} = c(x)\alpha^2.$$

Thus V is conformal projective vector field with respect to the Riemannian metric α. □

The Matsumoto metric was introduced by Matsumoto as a realization of Finsler's idea (a slope measure of a mountain with respect to a time measure) [12]. He gave an exact formulation of a Finsler surface to measure the time on the slope of a hill and introduced the Matsumoto metric [6, 13]. In this paper, we also study the projective vector field on Matsumoto space and get the following result:

Proof of Theorem 1.2: If a Matsumoto metric $F=\alpha^2/(\alpha-\beta)$ admits a projective vector field V, then by (2.9) and (2.11) we can say

$$\ell_{\hat{V}}\left(G^i_\alpha-\frac{\alpha}{2s-1}{s^i}_0+\frac{(2\alpha s_0+(2s-1)r_{00})}{2\alpha(2s-1)(3s-2b^2-1)}\Big[(2(2s-1)+1)y^i-2\alpha b^i\Big]\right)=Py^i. \tag{3.3}$$

We simplify the equation mentioned above by using Maple program and multiply this equation by $4\alpha^3(\alpha-2\beta)^2((1+2b^2)\alpha-3\beta)^2$ to remove denominators. Then we get the following

$$0=B^i_1\alpha^8+B^i_2\alpha^7+B^i_3\alpha^6+B^i_4\alpha^5+B^i_5\alpha^4+B^i_6\alpha^3+B^i_7\alpha^2+B^i_8\alpha+B^i_9, \tag{3.4}$$

where

$$\begin{aligned}
B^i_1 &= 16b^4\ell_{\hat{V}}{s^i}_0+16\ell_{\hat{V}}b^2b^is_0-16\ell_{\hat{V}}b^ib^2s_0-16\ell_{\hat{V}}s_0b^2b^i\\
&\quad+16b^2\ell_{\hat{V}}{s^i}_0-8\ell_{\hat{V}}b^is_0-8\ell_{\hat{V}}s_0b^i+8b^is_0+4\ell_{\hat{V}}{s^i}_0,
\end{aligned}$$

$$\begin{aligned}
B^i_2 &= -8\ell_{\hat{V}}b^2b^ir_{00}+8\ell_{\hat{V}}b^ib^2r_{00}+40\ell_{\hat{V}}b^i\beta s_0-16\beta b^is_0-16y^iPb^4\\
&\quad-8\ell_{\hat{V}}s_0y^ib^2-16y^iPb^2+40\ell_{\hat{V}}s_0\beta b^i+32\ell_{\hat{V}}\beta b^2{s^i}_0-32b^4\beta\ell_{\hat{V}}{s^i}_0\\
&\quad+8\ell_{\hat{V}}r_{00}b^2b^i-40\ell_{\hat{V}}\beta b^is_0+32\ell_{\hat{V}}\beta b^4{s^i}_0+8\ell_{\hat{V}}\beta{s^i}_0+4\ell_{\hat{V}}r_{00}b^i\\
&\quad+4\ell_{\hat{V}}b^ir_{00}-4b^ir_{00}+16\ell_{\hat{V}}{G^i}_\alpha b^4+16\ell_{\hat{V}}{G^i}_\alpha b^2-4\ell_{\hat{V}}s_0y^i+4y^is_0\\
&\quad-32\ell_{\hat{V}}\beta b^2b^is_0+32\ell_{\hat{V}}s_0b^2\beta b^i-32\ell_{\hat{V}}b^2\beta b^is_0+32\ell_{\hat{V}}b^ib^2\beta s_0\\
&\quad+8\ell_{\hat{V}}b^2y^is_0-32\beta\ell_{\hat{V}}{s^i}_0-80b^2\beta\ell_{\hat{V}}{s^i}_0+4\ell_{\hat{V}}{G^i}_\alpha-4Py^i,
\end{aligned}$$

$$\begin{aligned}
B^i_3 &= -64\ell_{\hat{V}}{G^i}_\alpha b^4\beta-112\ell_{\hat{V}}{G^i}_\alpha b^2\beta+36\ell_{\hat{V}}s_0y^i\beta+40y^iP\beta+2t_{00}{s^i}_0\\
&\quad-24y^i\beta s_0-48\ell_{\hat{V}}bi\beta^2s_0-48\ell_{\hat{V}}s_0\beta^2b^i-48\ell_{\hat{V}}\beta\beta{s^i}_0-28\ell_{\hat{V}}b^i\beta r_{00}\\
&\quad+12\ell_{\hat{V}}\beta b^ir_{00}-4\ell_{\hat{V}}b^2y^ir_{00}+4\ell_{\hat{V}}r_{00}y^ib^2-4\ell_{\hat{V}}\beta y^is_0+8b^2t_{00}{s^i}_0\\
&\quad-4b^is_0t_{00}+2\ell_{\hat{V}}r_{00}y^i-2y^ir_{00}-8b^2b^is_0r_{00}+16\ell_{\hat{V}}\beta y^ib^2s_0\\
&\quad+32\ell_{\hat{V}}b^2\beta b^ir_{00}-32\ell_{\hat{V}}b^ib^2\beta r_{00}-32\ell_{\hat{V}}r_{00}b^2\beta b^i+96\ell_{\hat{V}}\beta\beta b^is_0\\
&\quad-48\ell_{\hat{V}}b^2y^i\beta s_0+48\ell_{\hat{V}}s_0y^ib^2\beta+112y^iPb^2\beta+84\beta^2\ell_{\hat{V}}{s^i}_0\\
&\quad+96b^2\beta^2\ell_{\hat{V}}{s^i}_0-28\ell_{\hat{V}}r_{00}\beta b^i+16\beta b^ir_{00}+8b^4r_{00}{s^i}_0+64y^iPb^4\beta\\
&\quad-96\ell_{\hat{V}}\beta b^2\beta{s^i}_0-40\ell_{\hat{V}}{G^i}_\alpha\beta,
\end{aligned}$$

$$\begin{aligned}
B^i_4 &= 148\ell_{\hat{V}}G^i{}_\alpha\beta^2+32b^2\beta b^i s_0 t_{00}+72\ell_{\hat{V}}\beta\beta^2 s^i{}_0+64\ell_{\hat{V}}b^i\beta^2 r_{00}\\
&\quad +64\ell_{\hat{V}}r_{00}\beta^2 b^i-16\beta^2 b^i r_{00}-20\beta t_{00}s^i{}_0+64\ell_{\hat{V}}G^i{}_\alpha b^4\beta^2\\
&\quad +256\ell_{\hat{V}}G^i{}_\alpha b^2\beta^2-104\ell_{\hat{V}}s_0 y^i\beta^2-148y^i P\beta^2+32y^i\beta^2 s_0\\
&\quad -22\ell_{\hat{V}}r_{00}y^i\beta-72\ell_{\hat{V}}s^i{}_0\beta^3-2\ell_{\hat{V}}\beta y^i r_{00}-56b^2\beta t_{00}s^i{}_0\\
&\quad -32\ell_{\hat{V}}b^2\beta^2 b^i r_{00}+32\ell_{\hat{V}}b^i\beta^2 b^2 r_{00}+32\ell_{\hat{V}}r_{00}b^2\beta^2 b^i+16y^i\beta r_{00}\\
&\quad -48\ell_{\hat{V}}\beta\beta b^i r_{00}+40\beta b^i s_0 r_{00}-64y^i P b^4\beta^2+64\ell_{\hat{V}}b^2 y^i\beta^2 s_0\\
&\quad -64\ell_{\hat{V}}s_0 y^i b^2\beta^2-256y^i P b^2\beta^2+32\ell_{\hat{V}}b^2 y^i\beta r_{00}-32\ell_{\hat{V}}r_{00}y^i b^2\beta\\
&\quad +48\ell_{\hat{V}}\beta y^i\beta s_0-32b^4\beta t_{00}s^i{}_0-16\ell_{\hat{V}}\beta y^i b^2 r_{00},
\end{aligned}$$

$$\begin{aligned}
B^i_5 &= 192y^i P b^2\beta^3+64\ell_{\hat{V}}\beta y^i b^2\beta r_{00}-80\ell_{\hat{V}}b^2 y^i\beta^2 r_{00}-72\beta^2 b^i s_0 t_{00}\\
&\quad -192\ell_{\hat{V}}G^i_\alpha b^2\beta^3+96b^2\beta^2 t_{00}s^i{}_0-96\ell_{\hat{V}}betay^i\beta^2 s_0-y^i r_{00}t_{00}\\
&\quad -48\ell_{\hat{V}}r_{00}\beta^3 b^i+96\ell_{\hat{V}}s_0 y^i\beta^3-2y^i b^2 r_{00}t_{00}+80\ell_{\hat{V}}r_{00}y^i b^2\beta^2\\
&\quad +8\ell_{\hat{V}}\beta y^i\beta r_{00}+88\ell_{\hat{V}}r_{00}y^i\beta^2-40y^i\beta^2 r_{00}-240\ell_{\hat{V}}G^i_\alpha\beta^3\\
&\quad +66\beta^2 t_{00}s^i{}_0-6\beta b^i r_{00}t_{00}+48\ell_{\hat{V}}\beta\beta^2 b^i r_{00}+2y^i\beta s_0 t_{00}\\
&\quad -8y^i b^2\beta s_0 t_{00}-48\ell_{\hat{V}}b^i\beta^3 r_{00}+240y^i P\beta^3,
\end{aligned}$$

$$\begin{aligned}
B^i_6 &= -4\beta(16\ell_{\hat{V}}\beta y^i b^2\beta r_{00}-16\ell_{\hat{V}}b^2 y^i\beta^2 r_{00}+16\ell_{\hat{V}}r_{00}y^i b^2\beta^2\\
&\quad -6y^i b^2 r_{00}t_{00}+2\ell_{\hat{V}}\beta y^i\beta r_{00}+38\ell_{\hat{V}}r_{00}y^i\beta^2-8y^i\beta^2 r_{00}\\
&\quad -36\ell_{\hat{V}}G^i_\alpha\beta^3+18\beta^2 t_{00}s^i{}_0-6\beta b^i r_{00}t_{00}-3y^i r_{00}t_{00}\\
&\quad +6y^i\beta s_0 t_{00}+36y^i P\beta^3),
\end{aligned}$$

$$\begin{aligned}
B^i_7 &= 24\beta^2(-3y^i b^2 r_{00}t_{00}+4\ell_{\hat{V}}r_{00}y^i\beta^2+2y^i\beta s_0 t_{00}\\
&\quad -\beta b^i r_{00}t_{00}-2y^i r_{00}t_{00}),
\end{aligned}$$

$$\begin{aligned}
B^i_8 &= 16y^i\beta^3 r_{00}t_{00}(4b^2+5),\\
B^i_9 &= -48y^i\beta^4 r_{00}t_{00}.
\end{aligned}$$

From equation (3.4), we can get two fundamental equations

$$0=B^i_1\alpha^8+B^i_3\alpha^6+B^i_5\alpha^4+B^i_7\alpha^2+B^i_9, \tag{3.5}$$

$$0=B^i_2\alpha^6+B^i_4\alpha^4+B^i_6\alpha^2+B^i_8. \tag{3.6}$$

From these equations we can conclude that α^2 divides B^i_8 and B^i_9, in this way we have the following cases

Case 1: α^2 divides t_{00}, therefore there is scalar function $c=c(x)$ on M such that

$$t_{00}=\ell_{\hat{V}}\alpha^2=c(x)\alpha^2.$$

Then V is a conformal vector field respect on α.

Case 2: α^2 divides r_{00}, therefore there is scalar function $c = c(x)$ on M such that

$$r_{00} = c(x)\alpha^2.$$

Replacing this quantity into (3.3) and sorting again by α, we can get the following equation

$$0 = \bar{B}^i_0\alpha^7 + \bar{B}^i_1\alpha^6 + \bar{B}^i_2\alpha^5 + \bar{B}^i_3\alpha^4 + \bar{B}^i_4\alpha^3 + \bar{B}^i_5\alpha^2 + \bar{B}^i_6\alpha + \bar{B}^i_7, \tag{3.7}$$

where

$$\bar{B}^i_7 = 48y^i\beta^3 t_{00}(\beta c + s_0). \tag{3.8}$$

From (3.7) we have this fundamental equation

$$0 = \bar{B}^i_1\alpha^6 + \bar{B}^i_3\alpha^4 + \bar{B}^i_5\alpha^2 + \bar{B}^i_7. \tag{3.9}$$

By the equation mentioned above we can conclude that $\bar{B}^i_7$ must be divided by α^2, if α^2 divide t_{00}, then the equality and the reduce to the case 1, otherwise $(\beta c + s_0)$ must be remove. So, we have $s_i = -b_i c$. By contracting it with b^i we can obtain $c(x) = 0$. Then $s_0 = r_{00} = 0$. It means that Matsumoto metric has vanishing S-curvature. □

References

1. G.S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, D. Reidel Publishing Company, Dordrecht, Holland (1985).
2. D. Bao and Z. Shen, *Finsler metrics of constant positive curvature on the Lie group S_3*, J. Lond. Math. Soc., **66** (2002), 453-467.
3. X. Cheng and Z. Shen, *A class of Finsler metrics with isotropic S-curvature*, Israel J. Math. **169**(2009), 317-340.
4. R.S. Ingarden, *Geometry of thermodynamics*, Diff. Geom. Methods in Theor. Phys, XV Intern. Conf. Clausthal 1986, World Scientific, Singapore (1987).
5. V.K. Kropina, *On projective two-dimensional Finsler spaces with a special metric*, Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 277-292.
6. M. Matsumoto, *A slope of a mountain is a Finsler surface with respect to a time measure*, J. Math. Kyoto Univ. **29**(1) (1989), 17-25.
7. M. Matsumoto, *Finsler spaces with (α, β)-metric of Douglas type*, Tensor, N.S. **60** (1998), 123-134.
8. M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys. **31**(1992), 43-84.
9. M. Rafie-Rad, *Some new characterizations of projective Randers metrics with constant S-curvature*, J. Geom. Phys. **6**(2) (2012), 272-278.
10. M. Rafie-Rad and B. Rezaei, *On the projective algebra of Randers metrics of constant flag curvature*, SIGMA, **085**(7), 12 (2011)
11. P. Stavrinos and F. Diakogiannni, *Finslerian structure of anisotropic gravitational field*, Gravit. Cosmol. **10**(4) (2004), 1-11.
12. H. Shimada and S. Sabau, *Introduction to Matsumoto metric, Nonlinear Anal*, Theory Methods Appl., Ser. A, Theory Methods. **63** (2005), 165-168.

13. A. Tayebi, T. Tabatabaeifar, and E.Peyghan, *On the second approximate Matsumoto metric*, Bull. Korean Math. Soc. **51**(1) (2014), 115-128.
14. K. Yano, *The theory of Lie derivative and its applications*, North Holland , Amsterdam (1957).
15. X. Zhang and Y.B. Shen, *On Einstein Kropina metrics*, Differ. Geom. Appl. **31**(1) (2013), 80-92.

Received: 13.05.2020
Accepted: 09.10.2020