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On weakly Landsberg 3-dimensional Finsler Spaces

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Abstract. In this paper, we study the class of 3-dimensional Finsler manifolds. We find the necessary and sufficient condition under which a 3-dimensional weakly Landsberg metric reduces to a Landsberg metric.

Keywords: 3-dimensional Finsler space, weakly Landsberg metric.

1. Introduction

Let (M, F) be a Finsler manifold and $c : [a, b] \to M$ be a piecewise C^∞ curve from $c(a) = p$ to $c(b) = q$. For every $u \in T_pM$, let us define $P_c : T_pM \to T_qM$ by $P_c(u) := U(b)$, where $U = U(t)$ is the parallel vector field along c such that $U(a) = u$. P_c is called the parallel translation along c. In [2], Ichijyō showed that if F is a Berwald metric, then all tangent spaces (T_xM, F_x) are linearly isometric to each other. Let us consider the Riemannian metric $\hat{g}_x$ on $T_xM_0 := T_xM - \{0\}$ which is defined by

$$\hat{g}_x := g_{ij}(x, y)\delta y^i \otimes \delta y^j,$$

where $g_{ij} := 1/2[F^2]_{y^iy^j}$ is the fundamental tensor of F and $\{\delta y^i := dy^i + N^i_j dx^j\}$ is the natural coframe on T_xM associated with the natural basis $\{\partial/\partial x^i|_x\}$ for T_xM. If F is a Landsberg metric, then for any C^∞ curve c, P_c preserves the induced Riemannian metrics on the tangent spaces, i.e.,

$$P_c : (T_pM, \hat{g}_p) \to (T_qM, \hat{g}_q)$$

is an isometry.

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On a Landsberg manifold, the volume function $Vol = Vol(x)$ of the unit tangent sphere $S_xM \subset (T_xM, \hat{g}_x)$ is a constant. The constancy of $Vol(x)$ is required to establish a Gauss-Bonnet theorem for Finsler manifolds. The volume function is closely related to a weaker non-Riemannian quantity, mean Landsberg curvature $\mathbf{J} = J_k dx^k$, where

$$J_k := g^{ij} L_{ijk}.$$

Finsler metrics with $\mathbf{J} = 0$ are called weakly Landsberg metrics. By definition, every Landsberg metric is a Landsberg metric, but the converse may not hold. Thus, we get the following

$$\{\text{Landsberg metrics}\} \subseteq \{\text{Weakly Landsberg metrics}\}.$$

In dimension two, any weakly Landsberg metric must be a Landsberg metric. It has been shown that on a weakly Landsberg manifold, the volume function is a constant. Some rigidity problems also lead to weakly Landsberg manifolds. For example, for a closed Finsler manifold of non-positive flag curvature, if the S-curvature is a constant, then it is weakly Landsbergian. Apparently, weakly Landsberg manifolds deserve further investigation. For the first step, it is an interesting problem to consider the class of 3-dimensional Finsler manifolds with vanishing mean Landsberg curvature [1].

In [8], Moór introduced a special orthonormal frame field (ℓ^i, m^i, n^i) in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element, the second is the normalized mean Cartan torsion vector, and third is the unit vector orthogonal to them. Let (M, F) be a 3-dimensional Finsler manifold. Suppose that $\ell_i := F_{y^i}$ is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e., $m_i := I_i/||\mathbf{I}||$, where $||\mathbf{I}|| := \sqrt{I_i I^i}$ and n_i is a unit vector orthogonal to the vectors ℓ_i and m_i. Then the triple (ℓ_i, m_i, n_i) is called the Moór frame.

An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M (for more details, see [11], [15], [16] and [17]). If $\phi = 1 + s$, then we get the Randers metric. In [14], Tayebi-Najafi classified the class of 3-dimensional (α, β)-metrics with vanishing Landsberg curvature. They showed that these metrics belong to one of the following main classes: Berwald metrics which contain the Randers or Kropina metrics, or satisfy an ODE. More precisely, they proved the following.

Theorem A. Every 3-dimensional non-Riemannian almost regular Landsberg (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ belongs to the one of the following three classes of Finsler metrics:

(i) F is a Berwald metric. In this case, F is a Randers metric or a Kropina metric;

(ii) $\phi(t)$ is given by the ODE

$$\phi(t)^{4-4c}\big(\phi(t)-t\phi'(t)\big)^{4-c}\Big[\phi(t)-t\phi'(t)+(b^2-t^2)\phi''(t)\Big]^{-c}=e^{k_0}, \qquad (1.1)$$

where c is a nonzero real constant, k_0 is a real number and $b:=||\beta||_\alpha$. In this case, F is a Berwald metric (regular case) or an almost regular unicorn.

It is well known that every Berwald space is a Landsberg space. However, it has been one of the longest standing problem in Finsler geometry whether there exists a Landsberg space which is not a Berwald space. In 1996, Matsumoto found a list of rigidity results which almost suggest that such metric does not exist [4]. In 2003, Matsumoto emphasized this problem again and looked on it as the most important open problem in Finsler geometry. Recently, Bao called such spaces unicorns in Finsler geometry. For the unicorn problem, one can see [12], [13] and [18].

Let (M,F) be a 3-dimensional non-Riemannian Finsler manifold. Let $\mathcal{H}=\mathcal{H}(x,y)$, $\mathcal{I}=\mathcal{I}(x,y)$ and $\mathcal{J}=\mathcal{J}(x,y)$ are the main scalars of F. It is well-known that F is semi-C-reducible if and only if $\mathcal{J}=0$ (see [6]). It is proved that F is a C-reducible metric if and only if its main scalars satisfy the following

$$\mathcal{H}=3\mathcal{I}, \quad \mathcal{J}=0.$$

In this case, $\mathcal{I}=F\|\mathbf{I}\,\|/4$. Also, it is showed that a 3-dimensional Finsler metric F is a Berwald metric if and only if it has horizontally constant main scalars with vanishing h-connection vectors. Then it is proved F is a Landsberg metric if and only if it has constant main scalars along Finslerian geodesics and $h_0=0$, where $h_0:=h_iy^i$. Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian (α,β)-metric on a 3-dimensional manifold M. In [14], Tayebi-Najafi proved that F has bounded Cartan torsion if and only if the following holds

$$3\mathcal{I}^2+\mathcal{H}^2<\infty.$$

For a 3-dimensional non-Riemannian Finsler manifold (M,F), let us define the function $b_i=b_i(x,y)$ as follows

$$b_i:=\frac{F}{3(\mathcal{H}+\mathcal{I})^2}\Big[(\mathcal{H}-3\mathcal{I})m_i-4\mathcal{J}n_i\Big]. \qquad (1.2)$$

Theorem 1.1. *Let (M,F) be a non-Riemannian 3-dimensional weakly Landsberg manifold. Then F is a Landsberg metric if and only if the quantity $b_i=b_i(x,y)$ is horizontally constant along Finsler geodesics.*

2. Preliminaries

Let (M, F) be an n-dimensional Finsler manifold. The fundamental tensor $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ of F is defined by following

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. It is well known that, $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. By Diecke Theorem, a positive-definite metric F is Riemannian if and only if $\mathbf{I}_y = 0$.

At any point $x \in M$, the norm of $\mathbf{I}$ is defined as follows

$$\begin{aligned} ||\mathbf{I}|| &= \sup_{y,u \in T_xM_0} \frac{F(y)|\mathbf{I}_y(u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}} \\ &= \sup_{y,u \in I_xM} \frac{|\mathbf{I}_y(u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}}, \end{aligned} \tag{2.1}$$

where I_xM is the indicatrix of F at point $x \in M$ (see [9]).

For a vector $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) := \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\},$$

where

$$\mathbf{h}_y(u, v) := \mathbf{g}_y(u, v) - F^{-2}(y)\mathbf{g}_y(y, u)\mathbf{g}_y(y, v)$$

is the angular metric [7]. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. In local coordinate, it is written as follows:

$$C_{ijk} = \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}.$$

In [3], Matsumoto-Hōjō proved the following.

Lemma 2.1. ([3]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, and in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are scalar functions on TM_0 given by

$$G^i := \frac{1}{4} g^{ij}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^j} y^k - \frac{\partial[F^2]}{\partial x^j}\right\}, \quad y \in T_xM.$$

The vector field $\mathbf{G}$ is called the spray associated with (M, F).

The Landsberg curvature of F is defined by following

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t),V(t),W(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U = U(t)$, $V = V(t)$ and $W = W(t)$ are linearly parallel vector fields along σ with $U(0) = u$, $V(0) = v$ and $W(0) = w$, respectively. Then the Landsberg curvature $\mathbf{L}_y$ is the rate of change of $\mathbf{C}_y$ along geodesics for any $y \in T_xM_0$. F is called a Landsberg metric if it satisfies $\mathbf{L} = 0$ (see [10]).

For a non-zero vector $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$\mathbf{J}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{L}_y(u, \partial_i, \partial_j).$$

$\mathbf{J}$ is called the mean Landsberg curvature of F. Then F is called a weakly Landsberg metric if $\mathbf{J} = 0$. Also, the mean Landsberg curvature of F is defined by following

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$. Mean Landsberg curvature can also be defined as following

$$J_i := y^m \frac{\partial I_i}{\partial x^m} - I_m \frac{\partial G^m}{\partial y^i} - 2G^m \frac{\partial I_i}{\partial y^m}.$$

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we prove the following.

Lemma 3.1. *Let (M, F) be a non-Riemannian 3-dimensional weakly Landsberg manifold. Suppose that the quantity $b_i = b_i(x, y)$ is horizontally constant along Finsler geodesics. Then F is a Landsberg metric.*

Proof. For 3-dimensional Finsler manifolds, we have

$$g_{ij} = \ell_i \ell_j + m_i m_j + n_i n_j.$$

Thus

$$g^{ij} = \ell^i \ell^j + m^i m^j + n^i n^j.$$

Then the angular metric is given by

$$h_{ij} = m_i m_j + n_i n_j. \tag{3.1}$$

In [5], Matsumoto showed that the Cartan torsion of F is written as follows

$$\begin{aligned} FC_{ijk} = \mathcal{H} m_i m_j m_k - \mathcal{J}\Big\{ m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k \Big\} \\ + \mathcal{I}\Big\{ n_i n_j m_k + n_j n_k m_i + n_i n_k m_j \Big\}, \end{aligned} \tag{3.2}$$

where $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are called the main scalars of F. Contracting (3.2) with g^{ij} implies that

$$FI_k = (\mathcal{H} + \mathcal{I}) m_k. \tag{3.3}$$

Multiplying (3.3) with g^{mk} yields

$$FI^k = (\mathcal{H} + \mathcal{I}) m^k. \tag{3.4}$$

By (3.3) and (3.4) we get

$$\mathcal{H} + \mathcal{I} = F||\mathbf{I}||. \tag{3.5}$$

From (3.3) and (3.5), we have

$$I_k = ||\mathbf{I}|| m_k. \tag{3.6}$$

Comparing (3.5) and (3.6) imply that F is a Riemannian if and only if $\mathcal{H}+\mathcal{I} = 0$. Throughout this paper, we assume that $\mathcal{H} + \mathcal{I} \neq 0$.

Now, by considering (3.1) and (3.3), one can rewrite (3.2) as

$$C_{ijk} = \Big\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \Big\} + \Big\{ b_i I_j I_k + b_j I_i I_k + b_k I_i I_j \Big\}, \tag{3.7}$$

where

$$a_i := \frac{1}{3F}\Big[3\mathcal{I} m_i + \mathcal{J} n_i \Big], \qquad b_i := \frac{F}{3(\mathcal{H}+\mathcal{I})^2}\Big[(\mathcal{H} - 3\mathcal{I}) m_i - 4\mathcal{J} n_i \Big]. \tag{3.8}$$

It is easy to see that $a_i y^i = 0$ and $b_i y^i = 0$.

The horizontal derivation of the members of the Moór frame with respect to the Berwald connection are given by following

$$\ell_{i|j} = 0,$$
$$m_{i|j} = h_j n_i - m_r L^r{}_{ij},$$
$$n_{i|j} = -h_j m_i - n_r L^r{}_{ij}.$$

Multiplying (3.7) with g^{ij} implies that

$$a_i = \frac{1}{4}\Big\{(1 - 2I^m b_m)I_i - ||\mathbf{I}||^2 b_i\Big\}, \tag{3.9}$$

where $||\mathbf{I}||^2 := I^m I_m$. By plugging (3.9) in (3.7), we get

$$C_{ijk} - \frac{1}{4}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} = -\frac{2b_m I^m}{4}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}$$
$$-\frac{||\mathbf{I}||^2}{4}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} + \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}, \tag{3.10}$$

which yields

$$M_{ijk} = \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\} - \frac{2b_m I^m}{4}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}$$
$$-\frac{||\mathbf{I}||^2}{n+1}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\}. \tag{3.11}$$

Let us put

$$M'_{ijk} := M_{ijk|s} y^s = L_{ijk} - \frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}.$$

By taking a horizontal derivation of (3.11), we have

$$\begin{aligned} M'_{ijk} \;=\; & \Big\{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i I_j + b_k I_i J_j\Big\} \\ & -\frac{1}{2}(b_m J^m + b'_m I^m)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ & -\frac{b_m I^m}{2}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} \\ & -\frac{1}{2} I_m J^m \Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ & -\frac{||\mathbf{I}||^2}{4}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} \\ & +\Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\}. \end{aligned} \tag{3.12}$$

where $b'_i = b_{i|s} y^s$. By assumption F is a weakly Landsberg metric and then (3.12) reduces to following

$$M'_{ijk} = \Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\} - \frac{1}{2} b'_m I^m \Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}$$
$$-\frac{||\mathbf{I}||^2}{4}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\}. \tag{3.13}$$

By assumption, b_i is constant along geodesics, i.e., $b'_i = 0$, then (3.13) reduces to following

$$M'_{ijk} = 0 \tag{3.14}$$

or equivalently

$$L_{ijk} - \frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} = 0. \tag{3.15}$$

Since F is a weakly Landsberg metric, then (3.15) implies that

$$L_{ijk} = 0. \tag{3.16}$$

This means that F is a Landsberg metric. □

Proposition 3.2. *Let (M, F) be a non-Riemannian 3-dimensional weakly Landsberg manifold. Then $b_i = b_i(x, y)$ is constant along Finslerian geodesics if and only if the following hold*

$$(\mathcal{H}' - 3\mathcal{I}') + 4\mathcal{J}h_0 = 0, \tag{3.17}$$

$$(\mathcal{H} - 3\mathcal{I})h_0 - 4\mathcal{J}' = 0, \tag{3.18}$$

where

$$\mathcal{H}' := \mathcal{H}_{|i}y^i, \quad \mathcal{I}' := \mathcal{I}_{|i}y^i, \quad \textit{and} \quad \mathcal{J}' := \mathcal{J}_{|i}y^i$$

denote the horizontal derivation of main scalars along Finslerian geodesics.

Proof. The following holds

$$b'_i = \frac{1}{3F||\mathbf{I}||^4}\Big\{\big[(\mathcal{H}' - 3\mathcal{I}')m_i + (\mathcal{H} - 3\mathcal{I})h_0 n_i - 4(\mathcal{J}'n_i - \mathcal{J}h_0 m_i)\big]||\mathbf{I}||^2$$
$$-2I_m J^m\big[(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J}n_i\big]\Big\}. \tag{3.19}$$

Then $b'_i = 0$ if and only if the following holds

$$\Big[2I_m J^m(\mathcal{H} - 3\mathcal{I}) - (\mathcal{H}' - 3\mathcal{I}' + 4\mathcal{J}h_0)||\mathbf{I}||^2\Big]m_i$$
$$= \Big[(\mathcal{H}h_0 - 3\mathcal{I}h_0 - 4\mathcal{J}')||\mathbf{I}||^2 + 8I_m J^m \mathcal{J}\Big]n_i. \tag{3.20}$$

Multiplying (3.20) with m^i and n^i yields

$$\Big[(\mathcal{H}' - 3\mathcal{I}') + 4\mathcal{J}h_0\Big]||\mathbf{I}||^2 = 2I_m J^m\Big[\mathcal{H} - 3\mathcal{I}\Big], \tag{3.21}$$

$$\Big[(\mathcal{H} - 3\mathcal{I})h_0 - 4\mathcal{J}'\Big]||\mathbf{I}||^2 = -8I_m J^m \mathcal{J}, \tag{3.22}$$

By assumption F is a weakly Landsberg metric, namely it satisfies

$$\mathbf{J} = 0.$$

Putting it in (3.21) and (3.22) yield (3.17) and (3.18), respectively. □

Corollary 3.3. *Let $F = \alpha + \beta$ be a 3-dimensional non-Riemannian Randers metric on a manifold M. Then $b_i = b_i(x, y)$ is constant along Finslerian geodesics.*

Proof. It is proved that F is a C-reducible metric if and only if its main scalars satisfy the following

$$\mathcal{H} = 3\mathcal{I}, \quad \mathcal{J} = 0. \tag{3.23}$$

In this case,

$$\mathcal{I} = \frac{F\|\mathbf{I}\|}{4}.$$

Since F is positive-definite Finsler metric, then by Matsumoto-Hōjō's lemma, F is a Randers metric. By considering (3.23), it is easy to see that F satisfies (3.21) and (3.22). □

By (3.21) and (3.22), we conclude the following.

Corollary 3.4. *Let (M, F) be a non-Riemannian 3-dimensional Finsler manifold. Suppose that F satisfies following*

$$\mathbf{g}(\mathbf{I}, \mathbf{J}) = 0.$$

Then $b_i = b_i(x, y)$ is constant along Finslerian geodesics if and only if (3.17) and (3.18) hold.

Also, the Landsberg metrics satisfy following.

Lemma 3.5. *Let (M, F) be a non-Riemannian 3-dimensional Landsberg manifold. Then the quantity $b_i = b_i(x, y)$ is horizontally constant along Finsler geodesics.*

Proof. Let F be a Landsberg metric. Then (3.12) reduces to following

$$2b'_m I^m\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - 4\Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\} + ||\mathbf{I}||^2\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} = 0. \tag{3.24}$$

Contracting (3.24) with $I^i I^j$ yields

$$b'_k ||\mathbf{I}||^4 = 0.$$

Since $||\mathbf{I}|| \neq 0$, then we get $b'_k = 0$. □

Proof of Theorem 1.1: By Lemmas 3.1 and 3.5, we get the proof. □

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