Journal of Finsler Geometry and its Applications
Vol. 1, No. 1 (2020), pp 96-102
DOI: 10.22098/jfga.2020.1016

Characterization of 3-Dimensional Left-Invariant Locally Projectively Flat Randers Metrics

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Abstract. In this paper, we characterize locally projectively flat left-invariant Randers metrics on simply connected three dimensional Lie groups.

Keywords: Randers metric, locally projectively flat metric, left-invariant metric.

1. Introduction

Locally projectively flat Finsler metrics are interesting Finsler metrics that have the regular solution to Hilbert's fourth problem on a convex domain in $\mathbb{R}^n$[8]. Recently, many mathematicians (geometers) have studied the characterization and construction of locally projectively flat Finsler metrics [3, 9, 10, 12, 15]. For example, Z. Shen investigated the necessary and sufficient for (α, β)-metrics to be projectively flat apart from some special cases. In [5], it is shown that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed. According to Beltrami's Theorem, every Riemannian metrics of constant curvature is locally projectively flat and converse. But in Finsler metric every locally projectively flat Finsler metric is of scalar flag curvature and the converse is not true. For example, Randers metrics with constant flag curvature 1 are non-locally projectively flat [3].

A Randers metric $F(x, y) = \alpha(x, y) + \beta(x, y)$ is a Finsler metric which have defined as the sum of a Riemannian metric $\alpha(x, y) := \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta(x, y) := b_i(x)y^i$ such that the Riemannian metric controls the related form by $||\beta||_\alpha < 1$. The history of Randers metrics goes back to G. Randers's research

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AMS 2020 Mathematics Subject Classification: 22E60, 53C60, 53C50.

on general relativity of 4-dimensional Riemannian manifolds. He regarded these metrics not as Finsler metrics, but as affinely connected Riemannian metrics. This non-Riemannian metric was first recognized as a kind of Finsler metric by Ingarden, who first named it Randers metric. Since then it has been widely applied in many areas, including electron optics and biology [4]. In Finsler geometry, the class of Randers metrics is computable and this may lead to a better understanding of non-Riemannian curvature properties of Finsler metrics.

A natural problem is to study and characterize locally projectively flat Randers metrics of constant flag curvature. In [1], Bácsó-Matsumoto proved that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed. According to the Beltrami theorem in Riemann geometry, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Then a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is of constant sectional curvature and β is closed.

In this paper, we study locally projectively flat left-invariant Randers metrics on simply connected three dimensional Lie group. More precisely, we prove the following.

Theorem 1.1. *Let G be a simply connected 3-dimensional Lie group. Then G admits a left-invariant locally projectively flat Randers metric defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U, $F(x,y) = \sqrt{\alpha_x(y,y)} + \alpha_x(U,y)$, if and only if it is one of the following cases*

1) Abelian group: $[X,Y] = [Y,Z] = [Z,X] = 0$ with the left-invariant Riemannian metric

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) The solvable Lie group $\tilde{E}_0(2)$: $[X,Y] = Z$, $[Z,X] = Y$, $[Z,Y] = -X$ with the left invariant Riemannian metric

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

where $\mu = 1$ and $\nu > 0$;

3) The non-unimodular Lie group G_I: $[X,Y] = 0$, $[Z,X] = X$, $[Z,Y] = Y$ with the left invariant Riemannian metric

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

where $\nu > 0$.

2. Preliminaries

Let M be an n-dimensional smooth manifold and TM be its tangent bundle. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is smooth on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is positive definite

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \Big[F^2(y + su + tv) \Big]_{s,t=0}, \quad u, v \in T_xM.$$

Given a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i(x, y) := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big\}.$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F).

A distance function on a set U is a function $d : U \times U \to \mathbb{R}$ with the following properties

(a) $d(p, q) \geq 0$ and equality holds if and only if $p = q$;
(b) $d(p, q) \leq d(p, r) + d(r, q)$.

A distance function on a convex domain $U \subset \mathbb{R}^n$ is said to be *projective* (or *rectilinear*) if straight lines are the shortest paths. Hilbert's Fourth Problem is to characterize projective distance functions.

A distance function d on a manifold M is said to be *smooth* if it is induced by a Finsler metric F on M,

$$d(p, q) := \inf_c \int_0^1 F(\dot{c}(t)) dt,$$

where the infimum is taken over all curves $c(t)$, $0 \leq t \leq 1$, joining $p = c(0)$ to $q = c(1)$.

Now we start to discuss smooth projective distance functions or projective Finsler metrics on an open domain $U \subset \mathbb{R}^n$. First, let us use the following notations. The local coordinates of a tangent vector $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_p \in T_xU$ will be denoted by (x, y). Hence all quantities are functions of $(x, y) \in U \times \mathbb{R}^n$. It is known that a Finsler metric $F = F(x, y)$ on U is projective if and only if its geodesic coefficients G^i are in the form

$$G^i(x, y) = P(x, y)y^i,$$

where $P : TU = U \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. We call $P(x, y)$ the projective factor of $F(x, y)$.

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and vice versa.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y)u^iv^jw^k \frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[G^i - \frac{2}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right]. \tag{2.1}$$

The quantity $\mathbf{D}$ is called the Douglas curvature of F. Then F is called a Douglas metric if it satisfies $\mathbf{D} = 0$.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we remark a theorem of Ha-Lee. In [7] Ha-Lee studied 3-dimensional Lie algebra. They classified the Left-invariant Riemannian metrics on simply connected 3-dimensional Lie groups in [7]. If $\{X, Y, Z\}$ is a basis for a Lie algebra then it is isomorphic to one of the following Lie algebras

1) Abelian group:

$$[X, Y] = [Y, Z] = [Z, X] = 0.$$

2) Nilpotent group:

$$[X, Y] = Z, [Z, X] = [Z, Y] = 0.$$

3) Unimodular solvable group:

$$(a)\quad [X, Y] = 0,\quad [Z, X] = X,\quad [Z, Y] = Y,$$

or

$$(b)\quad [X, Y] = 0,\quad [Z, X] = Y,\quad [Z, Y] = X.$$

4) Simple group:

$$(a)\quad [X, Y] = 2Z,\quad [Z, X] = 2Y,\quad [Z, Y] = 2X,$$

or

$$(b)\quad [X, Y] = Z,\quad [Z, X] = Y,\quad [Z, Y] = -X.$$

5) Non-unimodular solvable group:

$$(a)\quad [X, Y] = 0,\quad [Z, X] = X,\quad [Z, Y] = Y,$$

or

$$(b)\quad [X,Y]=0,\quad [Z,X]=Y,\quad [Z,Y]=-cX+2Y,$$

where $c\in\mathbb{R}$. Every left-invariant Riemannian metric on a simply connected 3-dimensional Lie groups is affine equivalent to one of the left-invariant Riemannian metric belong to Lie algebras $(1-5)$ introduced in [7].

Let g be a Riemannian metric on a connected Lie group G. Suppose that $\varphi\in Aut(\mathbf{g})$ with $[\tilde{g}]=[\varphi]^t[g][\varphi]$. Let ∇ and$\tilde{\nabla}$ be the Levi-Civita connections determined by the left-invariant metrics g and $\tilde{g}$, respectively, on G. Then

$$\varphi\tilde{\nabla}_x y=\nabla_{\varphi x}\varphi y$$

Therefore, applying an automorphism on the left-invariant metric given each case establishes a new left-invariant metric which is affinely related to the first one.

Let G be a smooth n-dimensional connected Lie group endowed with a Riemannian metric $\alpha=a_{ij}dx^i\otimes dx^j$. We denote the inverse of (a_{ij}) by (a^{ij}). We know that α induces the musical bijection between 1-forms and vector fields on G, which is denoted by $\flat: T_xG\longrightarrow T_x^*G$ and given by $y\longrightarrow\alpha_x(y,-)$. The inverse of $\flat$ is denoted by $\sharp: T_x^*G\longrightarrow T_xG$. Suppose $\beta=b_idx^i$ is a 1-form on G, in which we have used Einstein's convention for summation. Then

$$(\beta^\sharp=)U=b^i\frac{\partial}{\partial x^i},$$

where $b^i=a^{ij}b_j$. Consider β such that $\|\beta\|_\alpha:=\sqrt{a_{ij}b^ib^j}<1$. A Randers metric F on G is defined as follows

$$F(x,y)=\sqrt{\alpha_x(y,y)}+\alpha_x(U,y)\qquad \forall x\in M,\ \forall y\in T_xM\ .$$

$$\alpha(x,y)=\sqrt{a_{ij}y^iy^j},\qquad \beta(x,y)=\alpha_x(U,y),$$

In order to prove Theorem 1.1, we remark a key Lemma of Yibing-Yaoyongy in [14].

Lemma 3.1. *Let $F=\alpha+\beta$ and $\bar{F}=\bar{\alpha}+\bar{\beta}$ be two Randers metrics, $\boldsymbol{D}$ and $\bar{\boldsymbol{D}}$ be the Douglas tensors of F and$\bar{F}$, respectively. Then $\boldsymbol{D}=\bar{\boldsymbol{D}}$ if and only if the following conditions are satisfied.*
(1) $\alpha\neq\lambda(x)\bar{\alpha}$, $s_{ij}=\bar{s}_{ij}=0$, that is, β and $\bar{\beta}$ are closed;

(2) $\alpha=\lambda(x)\bar{\alpha}$, $s_{ij}=\lambda(x)\bar{s}_{ij}$.

A Finsler metric F on a Lie group G is called left-invariant if for all $a\in G$ and $Y\in T_aG$

$$F(a,Y)=F(e,(L_{a^{-1}})_{*a}Y).\tag{3.1}$$

One of the main quantities in Finsler geometry is the flag curvature which is defined as follows:

$$\mathbf{K}(P,Y) = \frac{\mathbf{g}_Y(\mathbf{R}(U,Y)Y,U)}{\mathbf{g}_Y(Y,Y).\mathbf{g}_Y(U,U) - \mathbf{g}_Y^2(Y,U)}, \tag{3.2}$$

where

$$\mathbf{R}(U,Y)Y = \nabla_U\nabla_Y Y - \nabla_Y\nabla_U Y - \nabla_{[U,Y]}Y,$$

$P = span\{U,Y\}$ is a 2-plane in T_xM and ∇ is the Chern connection induced by F (for more details, see [2, 13]).

Proof of Theorem 1.1: It is known that F is locally projectively flat if and only if α is locally projectively flat and β is closed [5]. On the other hand, we have $\beta(x,y) := \alpha_x(U,y)$ is closed if and only if $\langle U, [\mathbf{g},\mathbf{g}]\rangle = 0$. Therefore by using Lema 3.1, it is enough to prove that α is locally projectively flat.

Case 1: We can consider $U = ax + by + cz$ with $\sqrt{(U,U)} < 1$. In this case, β is closed and α is locally projectively flat metric.

Case 2: Let $U = az$. Then β is closed. On the other hand, the sectional curvature with $\mu = 1$ on $\tilde{E}_0(2)$ are zero. Then

$$k(x,y) = k(y,z) = k(z,x) = 0.$$

Hence F is a locally projectively flat in this case F is not a non-Riemannian Randers metric.

Case 3: Like the previous one, if we assume that $U = az$ with $\sqrt{(U,U)} < 1$. This demonstrates β is closed. In addition the sectional curvature are

$$k(x,y) = k(y,z) = k(z,x) = -\frac{1}{v}.$$

In other cases, since sectional curvature is not constant according to Beltrami's Theorem, then α is not locally projectively flat. Thus, F is not locally projectively flat. This complete the proof. $\square$

In [6], Deng-Hu give the following formula for the flag curvature of left-invariant Randers metrics of Douglas type

$$\mathbf{K}(P,y) = \frac{\alpha^2}{F^2}\tilde{\mathbf{K}}(P) + \frac{1}{4F^4}\Big(3\langle U(y,y),u\rangle^2 - 4F\langle U(y,U(y,y)),u\rangle\Big), \tag{3.3}$$

where $U : \mathbf{g}\times\mathbf{g}\to\mathbf{g}$ be the bilinear symmetric map defined by

$$2\langle U(X,Y),Z\rangle = \langle [Z,X],Y\rangle + \langle [Z,Y],Z\rangle, \quad \forall Z\in\mathbf{g},$$

u is the vector in m corresponding to the 1-form and $\tilde{\mathbf{K}}$ is the sectional curvature of α.

Corollary 3.1. *By using* (3.3)*, one can see that there are some cases in Theorem 1.1 which are not constant flag curvature although the sectional curvature are constant. Therefore, Beltrami's Theorem dose not hold for the simply connected* 3*-dimensional Lie groups with left-invariant Randers metric.*

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Received: 28.01.2020
Accepted: 29.05.2020