Journal of Finsler Geometry and its Applications
Vol. 1, No. 1 (2020), pp 84-88
DOI: 10.22098/jfga.2020.1014

Characterization of the Killing and homothetic vector fields on Lorentzian pr-waves three-manifolds with Recurrent Curvature

P. Atashpeykar[a*] and A. Haji-Badali[b]

[a,b]Department of Mathematics, Basic Sciences Faculty, University of Bonab, Bonab 5551761167, Iran.

E-mail: p.atashpeykar@ubonab.ac.ir
E-mail: haji.badali@ubonab.ac.ir

Abstract. We consider the Lorentzian pr-waves three-manifolds with recurrent curvature. We obtain a full classification of the Killing and homothetic vector fields of these spaces.

Keywords: Pr-waves manifolds, Killing vector fields, Homothetic vector fields, Lorentzian.

1. Introduction

A Lorentzian manifold with a parallel light-like vector field is called Brinkmann-wave, due to [1]. A Brinkmann-wave manifold (M, g) is called pp-wave if its curvature tensor R satisfies the trace condition $tr_{(3,5)(4,6)}(R \otimes R) = 0$. In [2], Schimming proved that an $(n + 2)$-dimensional pp-wave manifold admits coordinates $(x, y_1, \ldots, y_n, z)$ such that g has the form

$$g = 2dxdz + \sum_{k=1,\ldots,n} (dy_k)^2 + f(dz)^2, \ with \ \partial_x f = 0. \qquad (1.1)$$

In [3], Leistner gave another equivalence for pp-wave manifold. More precisely, he proved that a Brinkmann-wave manifolds (M, g) with parallel light-like vector field X and induced parallel distributions Ξ and $\Xi^{\perp}$ is a pp-wave if and

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53C60, 53C30.

only if its curvature tensor satisfies

$$R(U,V) : \Xi^{\perp} \to \Xi, \ for \ all \ U,V \in TM, \tag{1.2}$$

or equivalently $R(Y_1, Y_2) = 0 \ for \ all \ Y_1, Y_2 \in \Xi^{\perp}$. From this description, it follows that a pp-wave manifold is Ricci-isotropic, which means that the image of the Ricci operator is totally light-like, and has vanishing scalar curvature [3]. Furthermore, Leistner introduced a new class of non-irreducible Lorentzian manifolds satisfying (1.2) but only for a recurrent vector field X, that is,$\nabla X = \omega \otimes X$ where ω is a one-form on M. Following [3], such manifolds are called pr-waves. Moreover, a description in terms of local coordinates similar to the one for pp-waves manifolds was given in [3]: a Lorentzian manifold (M, g) of dimension $n+2 > 2$ is a pr-wave if and only if around any point $o \in M$ exist coordinates $(x, y_1, \ldots, y_n, z)$ in which the metric g has the following form:

$$g = 2dxdz + \sum_{k=1,\ldots,n} (dy_k)^2 + f(dz)^2,$$

where f is a real valued smooth function on (M, g).

In this paper, we shall investigate killing and homothetic vector fields on the Lorentzian pr-waves three-manifolds with recurrect curvature. If (M, g) denotes a Lorentzian manifold and T a tensor on (M, g), codifying some either mathematical or physical quantity, a symmetry of T is a one-parameter group of diffeomorphisms of (M, g), leaving T invariant. As such, it corresponds to a vector field X satisfying $\mathcal{L}_X T = 0$, where $\mathcal{L}$ denotes the Lie derivative. Isometries are a well known example of symmetries, for which $T = g$ is the metric tensor. The corresponding vector field X is then a Killing vector field. Homotheties and conformal motions on (M, g) are again examples of symmetries. (see, for example, [[4], [5], [6], [7], [8], [9]] and references therein).

2. Killing and homothetic vector fields of pr-wave three-manifold

We first classify Killing and homothetic and affine vector fields of (M, g). The classifications we obtain are summarized in the following theorem. Put $f_x := \partial_x f, \ f_y := \partial_y f$ and $f_z := \partial_z f$.

Theorem 1. *Let $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary vector field on the three-dimensional pr-wave manifold (M, g). Then*

(i) *X is a Killing vector field if and only if*

$$X^1 = -f_1^{'}(z)y - f_2^{'}(z)x + f_3(z), \ X^2 = f_1(z), \ X^3 = f_2(z). \tag{2.1}$$

where $f_i(z)$ are arbitrary smooth functions on M, satisfying

$$\begin{aligned} 2f_2^{'}(z)f - 2f_1^{''}(z)y - 2f_2^{''}(z)x + 2f_3^{'}(z) + (f_3(z) - f_1^{'}(z)y \\ - f_2^{'}(z)x)f_x + f_3(z)f_y + f_2(z)f_z = 0. \end{aligned} \tag{2.2}$$

(ii) *X is a homothetic, non-Killing vector field if and only if*

$$X^1 = -f_1^{'}(z)y + (\eta - f_2^{'}(z))x + f_3(z),\ X^2 = \frac{1}{2}\eta y + f_1(z),\ X^3 = f_2(z).$$

where $\eta \neq 0$ *is a real constant and*

$$-\eta f + 2f_2^{'}(z)f - 2f_1^{''}(z)y - 2f_2^{''}(z)x + 2f_3^{'}(z) + (f_3(z) - f_1^{'}(z)y + (\eta - f_2^{'}(z))x)f_x$$
$$+ (\frac{1}{2}\eta y + f_3(z))f_y + f_2(z)f_z = 0.$$

Proof. We start from an arbitrary smooth vector field $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ on the three-dimensional pr-wave manifold (M, g), and calculate $\mathcal{L}_X g$. we assume $\partial_x = \partial_1, \partial_y = \partial_2, \partial_z = \partial_3$. With regard to

$$(\mathcal{L}_X g)_{\mu\nu} = X^i \partial_i g_{\mu\nu} + g_{i\nu}\partial_\mu X^i + g_{\mu i}\partial_\nu X^i,$$

We have

$$\begin{aligned}(\mathcal{L}_X g)_{11} &= \Sigma_{i=1}^3 (X^i\partial_i g_{11} + g_{i1}\partial_1 X^i + g_{1i}\partial_1 X^i)\\ &= X^1\partial_1 g_{11} + g_{11}\partial_1 X^1 + g_{11}\partial_1 X^1 + X^2\partial_2 g_{11} + g_{21}\partial_1 X^2 + g_{12}\partial_1 X^2\\ &\quad + X^3\partial_3 g_{11} + g_{31}\partial_1 X^3 + g_{13}\partial_1 X^3\\ &= 2\partial_1 X^3,\end{aligned}$$

$$\begin{aligned}(\mathcal{L}_X g)_{12} &= \Sigma_{i=1}^3 (X^i\partial_i g_{12} + g_{i2}\partial_1 X^i + g_{1i}\partial_2 X^i)\\ &= X^1\partial_1 g_{12} + g_{12}\partial_1 X^1 + g_{11}\partial_2 X^1 + X^2\partial_2 g_{12} + g_{22}\partial_1 X^2 + g_{12}\partial_2 X^2\\ &\quad + X^3\partial_3 g_{12} + g_{32}\partial_1 X^3 + g_{13}\partial_2 X^3\\ &= \partial_1 X^2 + \partial_2 X^3,\end{aligned}$$

$$\begin{aligned}(\mathcal{L}_X g)_{13} &= \Sigma_{i=1}^3 (X^i\partial_i g_{13} + g_{i3}\partial_1 X^i + g_{1i}\partial_3 X^i)\\ &= X^1\partial_1 g_{13} + g_{13}\partial_1 X^1 + g_{11}\partial_3 X^1 + X^2\partial_2 g_{13} + g_{23}\partial_1 X^2 + g_{12}\partial_3 X^2\\ &\quad + X^3\partial_3 g_{13} + g_{33}\partial_1 X^3 + g_{13}\partial_3 X^3\\ &= \partial_1 X^1 + f\partial_1 X^3 + \partial_3 X^3,\end{aligned}$$

By following this process we get

$$\begin{aligned}\mathcal{L}_X g &= 2\partial_1 X^3 dxdx + 2(\partial_1 X^2 + \partial_2 X^3)dxdy + 2(\partial_1 X^1 + f\partial_1 X^3 + \partial_3 X^3)dxdz + 2\partial_2 X^2 dydy\\ &\quad + 2(\partial_2 X^1 + \partial_3 X^2 + f\partial_2 X^3)dydz + (X^1\partial_1 f + 2\partial_3 X^1 + X^2\partial_2 f + X^3\partial_3 f + 2f\partial_3 X^3)dzdz,\end{aligned}$$

Then, X satisfies $\mathcal{L}_X g = \eta g$ for some real constant η if and only if the following system of partial differential equations is satisfied:

$$\partial_1 X^3 = 0,\ \partial_2 X^2 = \frac{\eta}{2},\ \partial_1 X^2 + \partial_2 X^3 = 0,\ \partial_1 X^1 + f\partial_1 X^3 + \partial_3 X^3 = \eta, \quad (2.3)$$

$$\partial_2 X^1 + \partial_3 X^2 + f\partial_2 X^3 = 0,\ X^1\partial_1 f + 2\partial_3 X^1 + X^2\partial_2 f + X^3\partial_3 f + 2f\partial_3 X^3 = \eta f.$$

We then proceed to integrate (2.3). From the first three equations in (2.3) we get

$$X^2 = \frac{\eta}{2}y - f_1(z)x + f_3(z), \quad X^3 = f_1(z)y + f_2(z).$$

Then, the fourth equation in 2.3 yields

$$\begin{aligned} X^1 &= f_5^{'}(z)xy + f_6^{'}(z)x + f_4(x,y), \\ f_1(z) &= -f_5(z) + c_1, \\ f_2(z) &= -f_6(z) + \eta z + c_2. \end{aligned}$$

Where c_1 and c_2 are real constants. substituting this into the fifth equation, we have

$$(-f_5(z) + c_1)f + 2f_5^{'}(z)x + f_3^{'}(z) + \partial_y f_4(x,y) = 0.$$

Then, we have

$$\begin{aligned} f_3(z) &= -f_6(z)y + c_1, \\ f_4(x,y) &= f_6^{'}(z)y + f_7(z), \\ f_5(z) &= c_1 \end{aligned}$$

Now, the last equation in (2.3) gives

$$\begin{aligned} &-\eta f + 2f_2^{'}(z)f - 2f_1^{''}(z)y - 2f_2^{''}(z)x + 2f_3^{'}(z) + (-f_1^{'}(z)y + (\eta - f_2^{'}(z))x + f_3(z))f_x \\ &+ (\frac{1}{2}\eta y + f_3(z))f_y + f_2(z)f_z = 0. \end{aligned}$$

So, we have

$$\begin{aligned} X^1 &= -f_1^{'}(z)y + (\eta - f_2^{'}(z))x + f_3(z), \\ X^2 &= \frac{1}{2}\eta y + f_1(z), \\ X^3 &= f_2(z). \end{aligned}$$

This proves the statement i) in the case $\eta = 0$ and the statement ii) if we assume $\eta \neq 0$. □

Example 2. *The functions in equation 2.2 for the killing vector fields on the three-dimensional pr-wave manifold produce a various family of killing vector fields on the three-dimensional pr-wave manifold. for example, let $f(x,y,z) = x$, we have*

$$f_2^{'}(z)x - 2f_1^{''}(z)y - 2f_2^{''}(z)x + 2f_3^{'}(z) - f_1^{'}(z)y + f_3(z) = 0.$$

Therefore,

$$f_3(z) = \left(\int (\frac{1}{2}f_2^{'}(z)x + f_1^{''}(z)y + f_2^{''}(z)x - \frac{1}{2}f_1^{'}(z)y)e^{\frac{1}{2}z}dz + c_1\right)e^{-\frac{1}{2}z}.$$

where c_1 and c_2 are real constants.

Now, with the arbitrary selection for function $f_1(z)$ and $f_2(z)$, killing vector fields are generated, which is a special example as follows:

$$f_1(z) = f_2(z) = 2e^{-\frac{1}{2}z}.$$

So, we have

$$f_3(z) = (yz + c_1)e^{-\frac{1}{2}z}.$$

In a special case, it can be assumed $c_1 = 0$. Hence,

$$f_3(z) = e^{-\frac{1}{2}z}yz.$$

Therefore,

$$X^1 = -2e^{-\frac{1}{2}z}y - 2e^{-\frac{1}{2}z}x + e^{-\frac{1}{2}z}yz,$$
$$X^2 = X^3 = 2e^{-\frac{1}{2}z}.$$

References

1. Brinkmann H.W., *Einstein spaces which are mapped conformally on each other,* Math. Ann. 94 (1925) 119145.
2. Schimming R., *Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie,* Math. Nachr. 59 (1974) 128162.
3. Leistner T., *Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds,* Differential Geom. Appl. 24 (2006) 458478.
4. Aichelburg P.C., *Curvature collineations for gravitational pp- waves,*J. Math. Phys. 11 (1970), 2458-2462.
5. Calvaruso G., Zaeim A., *Invariant symmetries on non-reductive homogeneous pseudo-Riemannian four manifolds,*Rev. Mat. Complut. 28 (2015), 599-622.
6. Calvaruso G., Zaeim A., *Geometric structures over four-dimensional generalized symmetric spaces,*Collect. Math., to appear.
7. Calvino-Louzao E., Seoane-Bascoy J., Vsazquez-Abal M.E., Vsazquez-Lorenzo R., *Invariant Ricci collineations on three-dimensional Lie groups,*J. Geom. Phys. 96 (2015), 59-71.
8. Hall G.S., *Symmetries and curvature structure in general relativity,*World Scientific Lecture Notes in Physics, Vol. 46, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
9. Hall G.S., Capocci M.S., *Classification and conformal symmetry in three-dimensional space-times,*J. Math. Phys. 40 (1999), 1466-1478.

Received: 19.01.2020
Accepted: 20.06.2020