Journal of Finsler Geometry and its Applications
Vol. 1, No. 1 (2020), pp 45-53
DOI: 10.22098/jfga.2020.1009

On generalized symmetric Finsler spaces with some special $(\alpha, \beta)-$metrics

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Abstract. In this paper, we study generalized symmetric $(\alpha, \beta)-$spaces. We prove that generalized symmetric $(\alpha, \beta)-$spaces with Matsumoto metric, infinite series metric and exponential metric are Riemannian.

Keywords: $(\alpha, \beta)-$metric, generalized symmetric space, Matsumoto metric, infinite series metric, exponential metric.

1. Introduction

The notion of symmetric spaces is due to Cartan. Later, Kowalski [6] defined generalized symmetric spaces or regular $s-$spaces following the introduction of $s-$manifolds in [8, 9]. Generalized symmetric Finsler spaces are a natural generalization of generalized symmetric spaces and they keep many of their properties [5, 10] . Let (M, F) be a connected Finsler manifold. A symmetry at $x \in M$ is an isometry of (M, F) for which x is an isolated fixed point. A $s-$structure on (M, F) is a family $\{s_x\}_{x\in M}$ such that s_x is a symmetry at $x \in M$, for each $x \in M$. An $s-$structure is called regular if for any two points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y).$$

An $s-$structure $\{s_x\}_{x\in M}$ is called of order k if $(s_x)^k = id_M$ for all $x \in M$ and k is the minimal number with this property. It is well known that if (M, F) admits an $s-$structure, then it always admits an $s-$structure of finite order.

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AMS 2020 Mathematics Subject Classification: 53C60, 53C35.

In particular if (M, F) admits an $s-$structure of order two then it is a usual symmetric Finsler space.

An $(\alpha,\beta)-$metric is a Finsler metric of the form $F=\alpha\phi(s),\ s=\frac{\beta}{\alpha}$ where $\alpha=\sqrt{\tilde{a}_{ij}(x)y^iy^j}$ is induced by a Riemannian metric $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$ on a connected smooth $n-$dimensional manifold M and $\beta=b_i(x)y^i$ is a $1-$form on M. Some important classes of $(\alpha,\beta)-$metrics are Randers metric $F=\alpha+\beta$, Matsumoto metric $F=\frac{\alpha^2}{(\alpha-\beta)}$, infinite series metic $F=\frac{\beta^2}{\beta-\alpha}$ and exponential metric $F=\alpha\exp(\frac{\beta}{\alpha})$

In this paper, we study generalized symmetric Finsler spaces with Matsumoto metric, infinite series metric and exponential metric.

2. Preliminaries

Let M be a smooth $n-$dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F: TM\longrightarrow R$ with the following properties [2]:

(1) F is smooth on the slit tangent bundle $TM^0:=TM\setminus\{0\}$.
(2) $F(x,\lambda y)=\lambda F(x,y)$ for any $x\in M$, $y\in T_xM$ and $\lambda>0$.
(3) The following bilinear symmetric form $g_y: T_xM\times T_xM\longrightarrow R$ is positive definite

$$g_y(u,v)=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2(x,y+su+tv)|_{s=t=0}.$$

Definition 2.1. Let $\alpha=\sqrt{\tilde{a}_{ij}(x)y^iy^j}$ be a norm iduced ba a Riemannian metric $\tilde{a}$ and $\beta(x,y)=b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold M. Let

$$\|\beta(x)\|_\alpha:=\sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{2.1}$$

Now, let the function F is defined as follows

$$F:=\alpha\phi(s)\quad,\quad s=\frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi=\phi(s)$ is a positive C^∞ function on $(-b_0,b_0)$ satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0\quad,\quad |s|\leq b<b_0. \tag{2.3}$$

Then by lemma 1.1.2 of [3], F is a Finsler metric if $\|\beta(x)\|_\alpha<b_0$ for any $x\in M$. A Finsler metric in the form (2.2) is called an $(\alpha,\beta)-$metric [1, 3].

A Finsler space having the Finsler function:

$$F(x,y)=\frac{\alpha^2(x,y)}{\alpha(x,y)-\beta(x,y)} \tag{2.4}$$

is called a Matsumoto space.

A Finsler space having the Finsler function:

$$F(x,y)=\frac{\beta^2(x,y)}{\beta(x,y)-\alpha(x,y)} \tag{2.5}$$

is called a Finsler space with an infinite series $(\alpha, \beta)-$metric.

A Finsler space having the Finsler function:

$$F(x,y) = \alpha(x,y)\exp(\frac{\beta(x,y)}{\alpha(x,y)}) \tag{2.6}$$

is called a Finsler space with an exponential metric $(\alpha, \beta)-$metric.

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T_x^*M such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T_x^*M induces a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x,y). \tag{2.7}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write $(\alpha, \beta)-$metrics as follows:

$$F(x,y) = \alpha(x,y)\phi(\frac{\tilde{a}(\tilde{X}(x), y)}{\alpha(x,y)}), \tag{2.8}$$

where for any $x \in M$, $\sqrt{\tilde{a}(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < b_0$. Symmetric Finsler spaces form a natural extension to the symmetric spaces of Cartan. A symmetric Finsler spaces is a Finsler space (M, F) such that for all $p \in M$ there exist an involutive isometry $s_p \in M$ such that p is an isolated fixed point of s_p [4, 7]. Generalized symmetric Finsler spaces were introduced as generalization of generalized symmetric spaces [5]. A Finsler space (M, F) is said to be symmetric space if for any point $p \in M$ there exist an involutive isometry s_p of (M, F) such that p is an isolated fixed point of (M, F) . Let (M, F) be a connected Finsler space. An isometry s_x of (M, F) for which $x \in M$ is an isolated fixed point will be called a symmetry of M at x.
An $s-$structure on (M, F) is a family $\{s_x | x \in M\}$ of symmetries of (M, F). The corresponding tensor field S of type (1,1) defined by $S_x = (s_x)_x$ for each $x \in M$ is called the symmetry tensor field of $s-$structure [6, 5].

Definition 2.2. An $s-$structure $\{s_x | x \in M\}$ on a Finsler space (M, F) is said to be regular if it satisfies the rule

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y)$$

for every two points $x, y \in M$.

3. Generalized symmetric (α, β) spaces

Lemma 3.1. *Let (M, F) be a generalized symmetric Matsumoto space with F defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then the regular $s-$structure $\{s_x\}$ of (M, F) is also a regular $s-$structure of the Riemannian manifold $(M, \tilde{a})$.*

Proof. Let s_x be a symmetry of (M,F) at x and $p\in M$. Then for any $Y\in T_pM$ we have

$$\begin{aligned} F(p,Y) &= F(s_x(p), ds_x(Y)) \\ \frac{\tilde a(Y,Y)}{\sqrt{\tilde a(Y,Y)}-\tilde a(X_p,Y)} &= \frac{\tilde a(ds_xY, ds_xY)}{\sqrt{\tilde a(ds_xY,ds_xY)}-\tilde a(X_{s_x(p)},ds_xY)}. \end{aligned}$$

Applying the above equation to $-Y$, we get

$$\frac{\tilde a(Y,Y)}{\sqrt{\tilde a(Y,Y)}+\tilde a(X_p,Y)} = \frac{\tilde a(ds_xY, ds_xY)}{\sqrt{\tilde a(ds_xY,ds_xY)}+\tilde a(X_{s_x(p)},ds_xY)}.$$

Combining the above two equations, we get

$$\begin{aligned} \tilde a(Y,Y) &= \tilde a(ds_xY, ds_xY) \\ \tilde a(X_p,Y) &= \tilde a(X_{s_x(p)}, ds_xY). \end{aligned}$$

Thus s_x is a symmetry with respect to the Riemannian metric $\tilde a$. □

Lemma 3.2. *Let $(M,\tilde a)$ be a generalized symmetric Riemannian space. Also suppose that F is a Matsumoto metric introduced by $\tilde a$ and a vector field X. Then the regular $s-$structure $\{s_x\}$ of $(M,\tilde a)$ is also a regular $s-$structure of (M,F) if and only if X is s_x-invariant for all $x\in M$.*

Proof. Let X be s_x-invariant. Therefore for any $p\in M$, we have $X_{s_x(p)} = ds_xX_p$. Then for any $y\in T_pM$ we have

$$\begin{aligned} F(s_x(p), ds_xy_p) &= \frac{\tilde a(ds_xy_p, ds_xy_p)}{\sqrt{\tilde a(ds_xy,ds_xy)}-\tilde a(X_{s_x(p)},ds_xy)} \\ &= \frac{\tilde a(y,y)}{\sqrt{\tilde a(y,y)}-\tilde a(ds_xX_p,ds_xy)} \\ &= \frac{\tilde a(y,y)}{\sqrt{\tilde a(y,y)}-\tilde a(X_p,y)}. \\ &= F(p,y) \end{aligned}$$

Conversely, let s_x be a symmetry of (M,F) at x. Then for any $p\in M$ and $y\in T_pM$ we have

$$\begin{aligned} F(p,y) &= F(s_x(p), ds_xy) \\ \frac{\tilde a(y,y)}{\sqrt{\tilde a(y,y)}-\tilde a(X_p,y)} &= \frac{\tilde a(ds_xy_p, ds_xy_p)}{\sqrt{\tilde a(ds_xy,ds_xy)}-\tilde a(X_{s_x(p)},ds_xy)}. \end{aligned}$$

So we have

$$\tilde a(ds_xX_p - X_{s_x(p)}, ds_xy_p) = 0.$$

Therefore $ds_xX_p = X_{s_x(p)}$. □

Theorem 3.3. *A generalized symmetric Matsumoto space must be Riemannian.*

Proof. Let (M, F) be a generalized symmetric Matsumoto space with F defined by the Riemannian metric $\tilde{a}$ and the vector field X, and let $\{s_x\}$ be the regular $s-$structure of (M, F). Let s_x be a symmetry of (M, F). Then by lemma 3.1, s_x is also a symmetry of $(M, \tilde{a})$. Thus we have

$$\begin{aligned} F(x, ds_x(y)) &= \frac{\tilde{a}(ds_x y, ds_x y)}{\sqrt{\tilde{a}(ds_x y, ds_x y)} - \tilde{a}(X_x, ds_x y)} \\ &= \frac{\tilde{a}(y, y)}{\sqrt{\tilde{a}(y, y)} - \tilde{a}(X_x, ds_x y)} \\ &= F(x, y). \end{aligned}$$

Therefore $\tilde{a}(X_x, ds_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_x M$. Since x is an isolated fixed point of the symmetry s_x, the tangent map $S_x = (ds_x)_x$ is an orthogonal transformation of $T_x M$ having no nonzero fixed vectors. So we have $\tilde{a}(X_x, (S - id)_x(y)) = 0$, $\forall y \in T_x M$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

Lemma 3.4. *Let (M, F) be a generalized symmetric Finsler space with infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then the regular $s-$structure $\{s_x\}$ of (M, F) is also a regular $s-$structure of the Riemannian manifold $(M, \tilde{a})$.*

Proof. Let s_x be a symmetry of (M, F) at x and let $p \in M$. Then for any $Y \in T_p M$ we have

$$F(p, Y) = F(s_x(p), ds_x(Y)).$$

Applying equation (2.5) we get

$$\frac{\tilde{a}(X_p, Y)^2}{\tilde{a}(X_p, Y) - \sqrt{\tilde{a}(Y, Y)}} = \frac{\tilde{a}(X_{s_x(p)}, ds_x Y)^2}{\tilde{a}(X_{s_x(p)}, ds_x Y) - \sqrt{\tilde{a}(ds_x Y, ds_x Y)}},$$

which implies

$$\begin{aligned} &\tilde{a}(X_p, Y)^2 \tilde{a}(X_{s_x(p)}, ds_x Y) - \tilde{a}(X_p, Y)^2 \sqrt{\tilde{a}(ds_x Y, ds_x Y)} \\ = \ &\tilde{a}(X_{s_x(p)}, ds_x Y)^2 \tilde{a}(X_p, Y) - \tilde{a}(X_{s_x(p)}, ds_x Y)^2 \sqrt{\tilde{a}(Y, Y)}. \end{aligned} \tag{3.1}$$

Applying the above equation to $-Y$, we get

$$\begin{aligned} &\tilde{a}(X_p, Y)^2 \tilde{a}(X_{s_x(p)}, ds_x Y) + \tilde{a}(X_p, Y)^2 \sqrt{\tilde{a}(ds_x Y, ds_x Y)} \\ = \ &\tilde{a}(X_{s_x(p)}, ds_x Y)^2 \tilde{a}(X_p, Y) + \tilde{a}(X_{s_x(p)}, ds_x Y)^2 \sqrt{\tilde{a}(Y, Y)}. \end{aligned} \tag{3.2}$$

Adding equations (3.1) and (3.2), we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{s_x(p)}, ds_x Y). \tag{3.3}$$

Subtracting equation (3.2) from equation (3.1) and using equation (3.3), we get

$$\tilde{a}(Y,Y) = \tilde{a}(ds_xY, ds_xY).$$

Therefore s_x is a symmetry with respect to the Riemannian metric $\tilde{a}$. □

Lemma 3.5. *Let $(M,\tilde{a})$ be a generalized symmetric Riemannian space. Also suppose that F is an infinite series metric defined by $\tilde{a}$ and a vector field X. Then the regular $s-$structure $\{s_x\}$ of $(M,\tilde{a})$ is also a regular $s-$structure of (M,F) if and only if X is s_x-invariant for all $x \in M$.*

Proof. Let X be s_x-invariant. Therefore for any $p \in M$, we have $X_{s_x(p)} = ds_xX_p$. Then for any $y \in T_pM$ we have

$$\begin{aligned} F(s_x(p), ds_xY_p) &= \frac{\tilde{a}(X_{s_x(p)}, ds_xY)^2}{\tilde{a}(X_{s_x(p)}, ds_xY) - \sqrt{\tilde{a}(ds_xY, ds_xY)}} \\ &= \frac{\tilde{a}(ds_xX_p, ds_xY)^2}{\tilde{a}(ds_xX_p, ds_xY) - \sqrt{\tilde{a}(ds_xY, ds_xY)}} \\ &= \frac{\tilde{a}(X_p, Y)^2}{\tilde{a}(X_p, Y) - \sqrt{\tilde{a}(Y,Y)}} \\ &= F(p, Y). \end{aligned}$$

Conversely, let s_x be a symmetry of (M,F) at x. Then for any $p \in M$ and $y \in T_pM$ we have

$$\begin{aligned} F(p,Y) &= F(s_x(p), ds_xY). \\ \frac{\tilde{a}(X_p, Y)^2}{\tilde{a}(X_p, Y) - \sqrt{\tilde{a}(Y,Y)}} &= \frac{\tilde{a}(X_{s_x(p)}, ds_xY)^2}{\tilde{a}(X_{s_x(p)}, ds_xY) - \sqrt{\tilde{a}(ds_xY, ds_xY)}}, \end{aligned}$$

which implies

$$\begin{aligned} & \tilde{a}(X_p,Y)^2\tilde{a}(X_{s_x(p)}, ds_xY) - \tilde{a}(X_p,Y)^2\sqrt{\tilde{a}(Y,Y)} \\ = \; & \tilde{a}(X_{s_x(p)}, ds_xY)^2\tilde{a}(X_p,Y) - \tilde{a}(X_{s_x(p)}, ds_xY)^2\sqrt{\tilde{a}(Y,Y)}. \end{aligned} \tag{3.4}$$

Replacing Y by $-Y$ in equation (3.4), we get

$$\begin{aligned} & \tilde{a}(X_p,Y)^2\tilde{a}(X_{s_x(p)}, ds_xY) + \tilde{a}(X_p,Y)^2\sqrt{\tilde{a}(Y,Y)} \\ = \; & \tilde{a}(X_{s_x(p)}, ds_xY)^2\tilde{a}(X_p,Y) + \tilde{a}(X_{s_x(p)}, ds_xY)^2\sqrt{\tilde{a}(Y,Y)}. \end{aligned} \tag{3.5}$$

Subtracting equation (3.5) from equation (3.4) we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{s_x(p)}, ds_xY).$$

Therefore $(ds_x)_pX_p = X_{s_x(p)}$. □

Theorem 3.6. *A generalized symmetric infinite series $(\alpha,\beta)-$space must be Riemannian.*

Proof. Let (M, F) be a generalized symmetric Finsler space with infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X and let $\{s_x\}$ be the regular $s-$structure of (M, F). Let s_x be a symmetry of (M, F). Then by lemma 3.4, s_x is also a symmetry of $(M, \tilde{a})$. Thus we have

$$\begin{aligned} F(x, ds_x(y)) &= \frac{\tilde{a}(X_x, ds_x(y))^2}{\tilde{a}(X_x, ds_x(y)) - \sqrt{\tilde{a}(ds_x(y), ds_x(y))}} \\ &= \frac{\tilde{a}(X_x, ds_x(y))^2}{\tilde{a}(X_x, ds_x(y)) - \sqrt{\tilde{a}(y, y)}} \\ &= F(x, y). \end{aligned}$$

Therefore $\tilde{a}(X_x, ds_x(y)) = \tilde{a}(X_x, y)$, $\forall y \in T_xM$. Since x is an isolated fixed point of the symmetry s_x, the tangent map $S_x = (ds_x)_x$ is an orthogonal transformation of T_xM having no nonzero fixed vectors. So we have

$$\tilde{a}(X_x, (S - id)_x(y)) = 0, \quad \forall y \in T_xM.$$

Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

Lemma 3.7. *Let (M, F) be a generalized symmetric Finsler space with exponential metric $F = \alpha \exp(\frac{\beta}{\alpha})$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then the regular $s-$structure $\{s_x\}$ of (M, F) is also a regular $s-$structure of the Riemannian space $(M, \tilde{a})$.*

Proof. Let s_x be a symmetry of (M, F) and let $p \in M$. Therefore for every $Y \in T_pM$ we have $F(p, Y) = F(s_x(p), ds_xY)$. Applying equation (2.6) we get

$$\sqrt{\tilde{a}(Y, Y)} \exp(\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y, Y)}}) = \sqrt{\tilde{a}(ds_xY, ds_xY)} \exp(\frac{\tilde{a}(X_{s_x(p)}, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}). \quad (3.6)$$

Replacing Y by $-Y$ in equation 3.6 we get

$$\sqrt{\tilde{a}(Y, Y)} \exp(\frac{-\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y, Y)}}) = \sqrt{\tilde{a}(ds_xY, ds_xY)} \exp(\frac{-\tilde{a}(X_{s_x(p)}, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}). \quad (3.7)$$

Combining the above equations (3.6) and (3.7) we have

$$\exp(\frac{2\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y, Y)}}) = \exp(\frac{2\tilde{a}(X_{s_x(p)}, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}),$$

which implies

$$\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y, Y)}} = \frac{\tilde{a}(X_{s_x(p)}, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}. \quad (3.8)$$

From equation (3.6) and (3.8), we have

$$\tilde{a}(Y, Y) = \tilde{a}(ds_xY, ds_xY).$$

Therefore s_x is a symmetry with respect to the Riemannian metric $\tilde{a}$. □

Lemma 3.8. *Let $(M,\tilde{a})$ be a generalized symmetric Riemannian space. Let F be a exponential metric defined by $\tilde{a}$ and a vector field X. Then the regular $s-$structure $\{s_x\}$ of $(M,\tilde{a})$ is also a regular $s-$structure of (M,F) if and only if X is s_x-invariant for all $x \in M$.*

Proof. Let X be s_x-invariant. Therefore for any $p \in M$, we have $X_{s_x(p)} = ds_x X_p$. Then for any $Y \in T_pM$ we have

$$\begin{aligned} F(s_x(p), ds_xY_p) &= \sqrt{\tilde{a}(ds_xY, ds_xY)}\exp(\frac{\tilde{a}(X_{s_x(p)}, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}) \\ &= \sqrt{\tilde{a}(ds_xY, ds_xY)}\exp(\frac{\tilde{a}(ds_xY, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}) \\ &= \sqrt{\tilde{a}(Y,Y)}\exp(\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}}) \\ &= F(p,Y). \end{aligned}$$

Conversely, let s_x be a symmetry of (M,F) at x. Then for any $p \in M$ and $y \in T_pM$ we have $F(p,Y) = F(s_x(p), ds_xY)$. Applying the theorem 3.7 we get

$$\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}} = \frac{\tilde{a}(X_{s_x(p)}, ds_xY)}{\sqrt{\tilde{a}(ds_xY, ds_xY)}}, \tag{3.9}$$

which implies

$$\tilde{a}(Y,Y) = \tilde{a}(ds_xY, ds_xY). \tag{3.10}$$

From equation (3.9) and (3.10), we have

$$\tilde{a}(X_x, Y) = \tilde{a}(X_{s_x(p)}, ds_xY).$$

Therefore $(ds_x)_pX_p = X_{s_x(p)}$. □

Theorem 3.9. *A generalized symmetric exponential metric space must be Riemannian.*

Proof. The proof is similar to the above cases. □

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Received: 23.01.2020
Accepted: 29.05.2020