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On H-curvature of Finsler warped product metrics

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Abstract. In this paper, we study the H-curvature, an important non-Riemannian quantity, for a rich and important class of Finsler metrics called Finsler warped product metrics. We find an equation that characterizes the metrics of almost vanishing H-curvature. Further, we show that, if F is a Finsler warped product metric, then the H-curvature vanishes if and only if the χ-curvature vanishes.

Keywords: Finsler metrics, warped product metrics, H-curvature.

1. Introduction

The non-Riemannian quantity H was introduced by Akbar-Zadeh [1] and developed by some other Finslerian geometers [8, 11]. It is determined by the χ-curvature in the following way

$$H_{ij} = \frac{1}{4}(\chi_{i.j} + \chi_{j.i}), \tag{1.1}$$

where $\chi := \chi_B du^B$ and $H := H_{AB} du^A \otimes du^B$ denote the χ- and H-curvatures of F respectively, and " $\cdot$ " denote the vertical covariant derivatives with respect to the Chern connection [15]. See [9, 17, 18] for interesting results on

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Finsler metrics with the H-curvature. A Finsler metric F on an n-dimensional manifold M is called of almost vanishing H-curvature if

$$H_{AB} = \frac{n+1}{2}\theta F_{v^A v^B}, \tag{1.2}$$

where $\theta := \theta_A(u)v^A$ is a 1-form on M.

In [9], X. Mo gave a characterization of spherically symmetric Finsler metrics of almost vanishing H-curvature. Then, the equivalence property between $\chi = 0$ and $H = 0$ for the metrics proved in [14] and [9]. Recall that a Finsler metric F is said to be spherically symmetric if the orthogonal group $O(n)$ acts as isometries on F [13, 10].

In [3], Chen-Shen-Zhao introduced a new class of Finsler metrics using the concept of the warped product structure on an n-dimensional manifold $M := I \times \breve{M}$ where I is an interval of $\mathbb{R}$ and $\breve{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric. In fact, they consider metrics of the form:

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\Big), \tag{1.3}$$

which are called Finsler warped product metric, where $u = (u^1, \breve{u}), v = v^1\frac{\partial}{\partial u^1} + \breve{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$. This class of Finsler metrics includes spherically symmetric Finsler metrics. Furthermore, they obtained the formula of the flag curvature and Ricci curvature of Finsler warped product metrics and gave the characterization of such metrics to be Einstein. H. Liu and X. Mo obtained the differential equation that characterizes the metrics with vanishing Douglas curvature [7]. Then, Liu-Mo-Zhang found equations that characterize the metrics of constant flag curvature and constructed explicitly many new warped product Douglas metrics of constant Ricci [6].

Troughout this paper, our index conventions are as follows:

$$1 \le A \le B \le \ldots \le n, \qquad 2 \le i \le j \le \ldots \le n.$$

In this paper, we mainly study Finsler warped product metrics with the H-curvature. First, we give a characterization equation for such metrics to be of almost vanishing H-curvature (Theorem 1.1). Then we show the equivalence property between $\chi = 0$ and $H = 0$ for the metrics (Corollary 1.4).

In fact, we get the following main results.

Theorem 1.1. *Let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric on an n-dimensional manifold $M = I \times \breve{M}$, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Then F has almost vanishing H-curvature if and only if*

$$s\breve{\alpha}\Big\{\lambda_s - s\tau_s + \frac{1}{3}[(n-2)\,\mu_s + (2\,n-1)\,\tau]\Big\} = (n+1)\theta(\phi - s\phi_s), \tag{1.4}$$

where θ is a 1-form on M and λ, μ and τ are defined in (2.5), (2.6) and (2.7), respectively.

For $\theta = 0$, we get

Corollary 1.2. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Then F has vanishing H-curvature if and only if*

$$\lambda_s - s\tau_s + \frac{1}{3}[(n-2)\,\mu_s + (2\,n-1)\,\tau] = 0. \tag{1.5}$$

In [4], Liu-Mo-Zhang obtained the differential equation that characterizes Finsler warped product metrics with vanishing χ-curvature, and it is given at below.

Lemma 1.3. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Then F has vanishing χ-curvature if and only if*

$$3\lambda_s - 3s\tau_s + (2n-1)\tau + (n-2)\mu_s = 0. \tag{1.6}$$

Now, the following corollary easily obtain by Corollary 1.2 and Lemma 1.3.

Corollary 1.4. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Then $\chi = 0$ if and only if $H = 0$.*

A Finsler metric F on an n-dimensional manifold M is said to be R-quadratic if its Riemann curvature R_v is quadratic in $v \in T_uM$ [2]. Najafi-Bidabad-Tayebi and X. Mo showed that all of R-quadratic Finsler metrics have vanishing H-curvature in [12] and [8]. Together with Corollary 1.4, we have the following:

Corollary 1.5. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Suppose that F is R-quadratic, then F has vanishing χ-curvature.*

Finsler metrics with vanishing χ-curvature is closely related to Finsler metrics of constant curvature. A Finsler metric F is called to be of constant Ricci curvature, if

$$Ric = (n-1)\kappa F^2,$$

where the Ricci curvature Ric is defined as $Ric = R^B_B$.

Note that there is a notion of Ricci curvature tensor introduced in [5],

$$Ric_{AB} = \frac{1}{2}\Big\{R^{C}_{A\ CB} + R^{C}_{B\ CA}\Big\}, \tag{1.7}$$

where $Ric = Ric_{AB}v^Av^B$. By (1.7), one can easily see that $Ric_{AB} = (n-1)\kappa g_{AB}$ implies $Ric = (n-1)\kappa F^2$. It is an interesting problem to see the difference between $Ric = (n-1)\kappa F^2$ and $Ric_{AB} = (n-1)\kappa g_{AB}$ [14]. Hence, we discuss this interesting result via Finsler warped product metrics with the following theorem:

Theorem 1.6. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Then $Ric_{AB} = \kappa g_{AB}$, if and only if ϕ satisfies*

$$(n-1)\kappa\phi^2 = \breve{Ric} + \breve{\alpha}^2[\lambda + (n-1)\mu - \nu], \tag{1.8}$$

$$3\lambda_s + (2n-1)\tau - 3s\tau_s + (n-2)\mu_s = 0. \tag{1.9}$$

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and $G = v^A \frac{\partial}{\partial u^A} - 2G^A \frac{\partial}{\partial v^A}$ in local coordinate $u^1, \dots, u^n$ and $v = v^A \frac{\partial}{\partial v^A}$ be a spray induced by F. The spray coefficients G^A are defined by

$$G^A := \frac{1}{4} g^{AB} \{[F^2]_{u^C v^B} v^C - [F^2]_{u^B}\},$$

where $g_{AB}(u,v) = \left[\frac{1}{2}F^2\right]_{v^A v^B}$ and $(g^{AB}) = (g_{AB})^{-1}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(u,v) = \frac{1}{2}\Gamma^i_{jk}(u) v^j v^k$. The spray coefficients G^A of a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$ are given by [3]

$$G^1 = \Phi \breve{\alpha}^2, \qquad G^i = \breve{G}^i + \Psi \breve{\alpha}^2 \breve{l}^i, \tag{2.1}$$

where $\breve{l}^i = \frac{v^i}{\breve{\alpha}}$ and

$$\Phi = \frac{s^2(\omega_r \omega_{ss} - \omega_s \omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \quad \Psi = \frac{s(\omega_r \omega_{ss} - \omega_s \omega_{rs}) + \omega_s \omega_r}{2(2\omega\omega_{ss} - \omega_s^2)}, \tag{2.2}$$

where $\omega = \phi^2$.

The well-known non-Riemannian quantity, S-curvature, is given by

$$S(u,v) := \frac{d}{dt}[\tau(c(t), \dot{c}(t))] \mid_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = u$ and $\dot{c}(0) = v$, [16]. According to the given definition, S-curvature measures the rate of change of the distortion on $(T_u M, F_u)$ in the direction $v \in T_u M$. For Berwald metrics the S-curvature is zero, [19]. In local coordinates, the S-curvature is defined by

$$S = \frac{\partial G^C}{\partial v^C} - v^C \frac{\partial}{\partial u^C}\Big[\ln \sigma_{BH}\Big], \tag{2.3}$$

where $dV_F = \sigma_F(u) du^1 \wedge \cdots \wedge du^n$ is the Busemann-Hausdorff volume form.

The E-curvature $E = E_{AB} du^A \otimes du^B$ and χ-curvature $\chi = \chi_B du^B$ are defined by $E_{AB} := \frac{1}{2} S_{.A.B}$ and $\chi_A := S_{.A|C} v^C - S_{|A}$, and the quantity $H_v = H_{AB} du^A \otimes du^B$ is defined as the covariant derivative of E along geodesics, more precisely $H_{AB} := E_{AB|C} v^C$, where " $\cdot$ " and " $|$ " denote the vertical and horizontal covariant derivatives with respect to the Chern connection, respectively. The H-curvature can be also expressed in terms of χ_A, (1.1).

In section 3, we need the following lemma given by Liu-Mo-Zhang.

Lemma 2.1. [4] *For a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$, the χ-curvature is given by*

$$\begin{aligned}\chi &= \chi_A du^A \\ &= \chi_1 du^1 + \chi_j du^j \\ &= -\breve{\alpha}\Big\{\lambda_s - s\tau_s + \frac{1}{3}[(n-2)\,\mu_s + (2\,n-1)\,\tau]\Big\}(dr - s\breve{\omega}),\end{aligned} \tag{2.4}$$

where $\breve{\omega}$ is the Hilbert form of $\breve{\alpha}$ and

$$\lambda = (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Phi_s, \tag{2.5}$$

$$\mu = \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, \tag{2.6}$$

$$\tau = 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \tag{2.7}$$

$$\nu = s\tau + \mu. \tag{2.8}$$

In particular, F has vanishing χ-curvature if and only if

$$\lambda_s - s\tau_s + \frac{1}{3}[(n-2)\,\mu_s + (2\,n-1)\,\tau] = 0. \tag{2.9}$$

In [3], Chen-Shen-Zhao obtained a formula for the Ricci curvature Ric of a Finsler warped product metric, and it is given at below.

Lemma 2.2. *For a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$, the Ricci curvature Ric is given by*

$$Ric = \breve{Ric} + \breve{\alpha}^2[\lambda + (n-1)\mu - \nu]. \tag{2.10}$$

3. H-curvature

In this section, we first derive a formula for the H-curvature of a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$, then we find an equation that characterizes the Finsler metrics of almost vanishing H-curvature.

Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a Finsler warped product metric. It is easy to verify that

$$\breve{\alpha}_{v^1} = 0, \quad s_{v^1} = \frac{1}{\breve{\alpha}}, \quad s_{v^j} = -\frac{s\breve{l}_j}{\breve{\alpha}}, \quad \breve{\alpha}^2_{v^j} = 2\breve{\alpha}\breve{l}_j, \tag{3.1}$$

where $\breve{l}_j := \breve{\alpha}_{v^j}$.

By differentiating χ_A and using (2.4) we obtain

$$\begin{aligned}
H &= H_{AB}du^A \otimes du^B \\
&= H_{11}(dr)^2 + H_{1j}dr \otimes du^j + H_{i1}du^i \otimes dr + H_{ij}du^i \otimes du^j \\
&= -\frac{1}{6}[(n-2)(\mu_{ss} + 2\tau_s) - 3s\tau_{ss} + 3\lambda_{ss}](dr)^2 \\
&+ \frac{1}{6}s\breve{l}_j[(n-2)(\mu_{ss} + 2\tau_s) - 3s\tau_{ss} + 3\lambda_{ss}]dr \otimes du^j \\
&+ \frac{1}{6}s\breve{l}_i[(n-2)(\mu_{ss} + 2\tau_s) - 3s\tau_{ss} + 3\lambda_{ss}]du^i \otimes dr \\
&+ \frac{1}{6}s\Big\{[3\lambda_s + (2n-1)\tau - 3s\tau_s + (n-2)\mu_s]\breve{a}_{ij} \\
&+ \big[3s^2\tau_{ss} - s(n-2)(\mu_{ss} + 2\tau_s) + 3s\tau_s - 3s\lambda_{ss} - (2n-1)\tau \\
&\quad - (n-2)\mu_s - 3\lambda_s\big]\breve{l}_i\breve{l}_j\Big\}du^i \otimes du^j, \qquad (3.2)
\end{aligned}$$

where we have used (3.1).

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1

For a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$ a direct computation yields

$$F_{v^1} = \phi_s, \qquad (3.3)$$

$$F_{v^i} = (\phi - s\phi_s)\breve{l}_i. \qquad (3.4)$$

From the above equations we obtain

$$F_{v^1v^1} = \frac{1}{\breve{\alpha}}\phi_{ss}, \qquad (3.5)$$

$$F_{v^1v^i} = -\frac{s\phi_{ss}}{\breve{\alpha}}\breve{l}_i, \qquad (3.6)$$

$$F_{v^iv^j} = \frac{1}{\breve{\alpha}}\big[(\phi - s\phi_s)\breve{a}_{ij} - (\phi - s\phi_s - s^2\phi_{ss})\breve{l}_i\breve{l}_j\big]. \qquad (3.7)$$

Suppose that F has almost vanishing H-curvature, i.e.,

$$H_{AB} = \frac{n+1}{2}\theta F_{v^Av^B}, \qquad (3.8)$$

where $\theta := \theta_A(u)v^A$ is a 1-form on M. Thus, by (3.2) and (3.5)∼(3.7), (3.8) holds if and only if

$$-\frac{1}{6}\{(n-2)(\mu_{ss} + 2\tau_s) - 3s\tau_{ss} + 3\lambda_{ss}\} = \frac{n+1}{2\breve{\alpha}}\theta\phi_{ss}, \qquad (3.9)$$

$$\begin{aligned}
&\frac{1}{6}s\Big\{[3\lambda_s + (2n-1)\tau - 3s\tau_s + (n-2)\mu_s]\breve{a}_{ij} + \big[-s(n-2)(\mu_{ss} + 2\tau_s) \\
&3s^2\tau_{ss} + 3s\tau_s - 3s\lambda_{ss} - (2n-1)\tau - (n-2)\mu_s - 3\lambda_s\big]\breve{l}_i\breve{l}_j\Big\} \\
&\quad = \frac{n+1}{2\breve{\alpha}}\theta\big[(\phi - s\phi_s)\breve{a}_{ij} - (\phi - s\phi_s - s^2\phi_{ss})\breve{l}_i\breve{l}_j\big]. \qquad (3.10)
\end{aligned}$$

Substituting (3.9) into (3.10), we get

$$\frac{1}{6}s[3\,\lambda_s + (2\,n-1)\,\tau - 3\,s\tau_s + (n-2)\,\mu_s](\breve{a}_{ij} - \breve{l}_i\breve{l}_j)$$
$$= \frac{n+1}{2\breve{\alpha}}\theta(\phi - s\phi_s)(\breve{a}_{ij} - \breve{l}_i\breve{l}_j). \tag{3.11}$$

It follows from (3.11) that

$$s\breve{\alpha}\Big\{\lambda_s - s\tau_s + \frac{1}{3}[(n-2)\,\mu_s + (2\,n-1)\,\tau]\Big\} = (n+1)\theta(\phi - s\phi_s). \tag{3.12}$$

Conversely, assume that (3.12) holds. By differentiating (3.12) with respect to s, one obtains (3.9). By (3.12) and (3.9), (3.10) holds. Hence, we conclude that $F = \breve{\alpha}\phi(r,s)$ has almost vanishing H-curvature if and only if (3.12) holds. This completes the proof of Theorem 1.1. □

Proof of Corollary 1.4: Taking $\theta = 0$ in (3.12), we obtain

$$\lambda_s - s\tau_s + \frac{1}{3}[(m-2)\,\mu_s + (2\,m-1)\,\tau] = 0. \tag{3.13}$$

Then, the proof is easy to obtain by comparing (2.9) with (3.13). □

In [3, Lemma 3.1], Cheng-Shen-Zhao proved that a spherically symmetric metric is a Finsler warped product metric. Then, we have the following [14, 9].

Corollary 3.1. *Let $(\mathbb{B}^n(\upsilon), F)$ be a spherically symmetric Finsler manifold. Then the H-curvature vanishes if and only if the χ-curvature vanishes.*

Example 3.2. *Consider the following metric*

$$F = \breve{\alpha}_+ \frac{\big(\sqrt{s^2 + r^2(1-r^2)} + rs\big)^2}{(1-r^2)^2\sqrt{s^2 + r^2(1-r^2)}}.$$

It is warped product form of the Berwald's metric on $\mathbb{B}^n$ [3]. The flag curvature of F is $K = 0$, then it satisfies $H = 0$. Thus, the Finsler metric $F = \breve{\alpha}\phi(r,s)$ has vanishing χ-curvature.

Proof of Theorem 1.6: It is known that for any Finsler metric, $Ric_{AB} = (n-1)\kappa g_{AB}$ if and only if $Ric = (n-1)\kappa F^2$ and $H_{AB} = 0$. By Corollary 1.4, for any warped product metric, $H_{AB} = 0$ if and only if $\chi_A = 0$. Thus for a warped product metric $F = \breve{\alpha}\phi(r,s)$, $Ric_{AB} = (n-1)\kappa g_{AB}$ if and only if $Ric = (n-1)\kappa F^2$ and $\chi_A = 0$. By (2.9) and Lemma 2.2, we prove the theorem. □

References

1. H. Akbar-Zadeh, *Sur les espaces de Finsler à courbures sectionnelles constantes*. Acad. Roy. Belg. Bull. Cl. Sci. **74**(5)(1988), 281-322.
2. S.S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientic Publishers, 2004.
3. B. Chen, Z. Shen and L. Zhao, *Constructions of Einstein Finsler metrics by warped product*, preprint, (2016).
4. L. Huaifu, M. Xiaohuan and Z. Hongzhen, *Finsler warped product metrics with special Riemannian curvature properties*. Science China Math.
5. B. Li and Z. Shen, *On some non-Riemannian quantities in Finsler Geometry*. Canad. Math. Bull. **56**(2013), 184-293.
6. H. Liu, X. Mo and H. Zhang, *Finsler warped product metrics with special Riemannian curvature properties*, Science China Mathematics. 2019; 1-18.
7. H. Liu and X. Mo, *Finsler warped product metrics of Douglas type. Canadian Mathematical Bulletin, https://cms.math.ca/10.4153/CMB-2017-077-0.*
8. *X. Mo, On the non-Riemannian quantity H of a Finsler metric, Differ. Geom. Appl.* **27***(1)(2009), 7-14.*
9. *X. Mo, A class of Finsler metrics with almost vanishing H-curvature, Balkan. J. Geom. Appl.* **21***(2016), 58-66.*
10. *P. J. McCarthy and S. F. Rutz, The general general four-dimensional spherically symmetric Finsler space, Gen. Relativity Gravitation.* **25***(1993), 589-602.*
11. *B. Najafi, Z. Shen and A. Tayebi, Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geometriae Dedicata.* **131***(2008), 87-97.*
12. *B. Najafi, B. Bidabad and A. Tayebi, On R-quadratic Finsler metrics, Iran. J. Sci. Tech. Tran. A. Science.* **31***(2007), 439-443.*
13. *S. Rutz, Symmetry in Finsler spaces, Finsler geometry. Contemp. Math.* **196***(1996), 289-300.*
14. *E. S. Sevim, Z. Shen and S. Ulgen, Spherically symmetric Finsler metrics with constant Ricci and flag curvature, 2015.*
15. *Z. Shen, On some non-Riemannian quantities in Finsler geometry, Canad. Math. Bull.* **56***(2013), 184-193.*
16. *Z. Shen, Volume comparison and its applications in Riemannian-Finsler geometry, Advances in Math.* **128***(1997), 306-328.*
17. *D. Tang, On the non-Riemannian quantity H in Finsler geometry, Differ. Geom. Appl.* **29***(2011), 207-213.*
18. *A. Tayebi and M. Razgordani, On H-curvature of (α, β)-metrics, Turkish J. Math.* **44***(2020), 207-222.*
19. *A. Tayebi and H. Sadeghi, Generalized P-reducible (α, β)-metrics with vanishing S-curvature, Ann. Polon. Math.* **114***(2015), 67-79.*

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