Journal of Finsler Geometry and its Applications
Vol. 5, No. 1 (2024), pp 115-126.
https://doi.org/10.22098/jfga.2024.14815.1122

On projective Riemann quadratic (PR-quadratic) Finsler metrics

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Abstract. This paper focuses on the projective Riemann quadratic (PR-quadratic) Finsler metrics, which are variant of the Finsler metrics in Finsler geometry. The paper introduces a special class of PR-quadratic Finsler metrics, called SPR-quadratic Finsler metrics, which is closed under projective changes with respect to a fixed volume form on M. This class contains the class of Douglas-Weyl metrics and is a subset of the class of Weyl metrics. The paper shows that any SPR-quadratic Finsler metric has a scalar flag curvature and a PR-quadratic Finsler metric has a scalar curvature if and only if it is of SPR-quadratic type. The results presented in this paper contribute to a deeper understanding of the behavior of PR-quadratic Finsler metrics and provide insights into the geometric properties of these metrics.

Keywords: Projective Ricci curvature, PR-quadratic Finsler metrics, SPR-quadratic, DW-metrics.

1. Introduction

Two regular metrics on a manifold M are considered projectively related if they share the same geodesics as the point sets. Geodesics represent the equations of motion that describe the behavior of space, making them significant in Physics. Utilizing the characteristics of regular metrics on a manifold M and applying them to introduce new projectively invariant quantities is a classical

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AMS 2020 Mathematics Subject Classification: 53B40, 53C30

approach in projective geometry. In the realm of Finsler metrics, there exist well-known projective invariants, such as the Douglas curvature, Weyl curvature, and generalized Douglas-Weyl curvature [2], [3]. These invariants play a crucial role in understanding the geometric properties of Finsler spaces. The concept of projective invariants is important in Finsler geometry. The tensors that contain both Ricci curvature $Ric = Ric(x, y)$ and S-curvature $S = S(x, y)$ are more applicable in this field [17][18][20]. Z. Shen introduced the concept of Projective Ricci curvature $PRic$ for a Finsler metric F [8], which is defined as follows

$$PRic = Ric + (n-1)\Big(\frac{\mathbf{S}_{|m} y^m}{n+1} + \frac{\mathbf{S}^2}{(n+1)^2}\Big).$$

The Projective Ricci curvature of Finsler metrics on a manifold M is projective invariant with respect to a fixed volume form on M. Here, the Ricci curvature is defined as the trace of the Riemann curvature. The Ricci curvature, which is defined as the trace of the Riemann curvature, plays a significant role in Finsler geometry. The Riemann curvature is a fundamental quantity in this field and is represented by a family of linear transformations

$$\mathbf{R}_y : T_xM \to T_xM,$$

where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, for every $\lambda > 0$.

A Finsler metric (M, F) is considered R-quadratic if its Riemann curvature $\mathbf{R}y$ is quadratic in $y \in TxM$. R-quadratic Finsler spaces are a rich class of Finsler spaces.

In Finsler geometry, the Riemann curvature of a projective spray in a Finsler metric (M, F) is referred to as the Projective Riemann curvature. Similarly, the Projective Ricci curvature is defined as the Ricci curvature of the projective spray.

In this paper, the focus is on studying Projective Riemann quadratic (PR-quadratic) Finsler metrics. In particular, a special class of PR-quadratic Finsler metrics, called SPR-quadratic Finsler metrics, is considered. It is proved that, this class of Finsler metrics contains the class of Douglas-Weyl metrics (DW-metrics). The class of DW-metrics is closed under projective changes and is equal to the intersection of two classes of Douglas metrics and Weyl metrics. In special case, it is proved that,

$$\{DW - metrics\} \subsetneq \{SPR - quadratic\ metrics\} \subsetneq \{Weyl\ metrics\} \quad (1.1)$$

The Weyl and Douglas curvatures of PR-quadratic Finsler metrics are also considered. It is shown that a PR-quadratic Finsler metrics is of scalar curvature if and only if it is of SPR-quadrartic. Moreover, It is proved that Every PR-quadratic Finsler metric is a GDW-metric.

In the paper, the vertical and horizontal derivatives with respect to the Berwald connection are denoted by "." and "|", respectively.

2. Preliminaries

A Finsler metric on a manifold M is a non-negative function F on TM having the following properties

(a) F is C^∞ on $TM \setminus \{0\}$,
(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$,
(c) For each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big]\Big|_{s,t=0}, \qquad u,v \in T_xM. \tag{2.1}$$

To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]\Big|_{t=0}, \qquad u,v,w \in T_xM. \tag{2.2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM\setminus\{0\}}$ is called the *Cartan torsion.*

A curve $c = c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(c,\dot{c}) = 0, \tag{2.3}$$

where $\dot{c} = \frac{dc}{dt}$ and $G^i = G^i(x,y)$ are local functions on TM given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}, \quad y \in T_xM. \tag{2.4}$$

and called the spray coefficients of F. The Riemann curvature $R_y = R^i{}_k \frac{\partial}{\partial x^i} \bigotimes dx^k$ of F is given by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

For the Riemann curvature of Finsler metric F one has [8]

$$R^i{}_{kl} = \frac{1}{3}(R^i{}_{k.l} - R^i{}_{l.k}), \quad \text{and} \quad R_j{}^i{}_{kl} = R^i{}_{kl.j}. \tag{2.5}$$

A Finsler metric F on a manifold M is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for all $x \in M$. For $y \in T_xM_0$, define

$$B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$$

$$B_y(u,v,w) = B_j{}^i{}_{kl}u^jv^kw^l\frac{\partial}{\partial x^i},$$

where $B^i_{j\,kl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$. Put

$$E_y : T_xM \otimes T_xM \longrightarrow \mathbb{R}$$

$$E_y(u,v) = E_{jk}u^jv^k,$$

where $E_{jk} = \frac{1}{2}B_j{}^m{}_{km}$, $u = u^i\frac{\partial}{\partial x^i}$, $v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$. B and E are called the Berwald curvature and mean Berwald curvature, respectively. F is called a Berwald metric and Weakly Berwald (WB) metric if $B = 0$ and $E = 0$, respectively [9].

The S-curvature $\mathbf{S} = \mathbf{S}(x, y)$ was introduced as follows [9]

$$\mathbf{S}(x, y) = \frac{d}{dt}[\tau\big(\gamma(t), \gamma'(t)\big)]_{|t=0},$$

where $\tau(x, y)$ is the distortion of the metric F and $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\gamma'(0) = y$ on M. It is considerable that [8]

$$E_{ij} = \frac{1}{2}\mathbf{S}_{.i.j}. \tag{2.6}$$

The non-Riemannian quantity χ-curvature is denoted by $\chi = \chi_j dx^j$ and defined as

$$\chi_j = \mathbf{S}_{.j|m}y^m - \mathbf{S}_{|j} = -\frac{1}{3}(2R^m{}_{k.m} - R^m{}_{m.k}). \tag{2.7}$$

Let

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{1}{n+1}\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}(\frac{\partial G^m}{\partial y^m}y^i).$$

It is easy to verify that $D := D_j{}^i{}_{kl}dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. It is known as the Douglas tensor, referred to as D. The Douglas tensor D is a non-Riemannian projective invariant, namely, If we consider two projectively equivalent Finsler metrics F and $\bar{F}$, with a positively y-homogeneous projective factor $P = P(x, y)$ of degree one, it follows that the expression

$$G^i = \bar{G}^i + Py^i,$$

holds true. Moreover, this implies that the Douglas tensor associated with F is equal to the Douglas tensor associated with $\bar{F}$ [4], [8]. One could easily show that

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jkl}y^i\Big\}. \tag{2.8}$$

Douglas curvature, $D_j{}^i{}_{kl}$, is a projective invariant constructed from the Berwald curvature. Finsler metrics with $D_j{}^i{}_{kl} = 0$ are called Douglas metrics. The metrics with the following condition are called GDW metric which are projective invariant.

$$D_j{}^i{}_{kl|m}y^m = T_{jkl}y^i,$$

for some tensors T_{jkl}.

Lemma 2.1. [8] *Let F and $\bar{F}$ be two projectively equivalent Finsler metrics on a manifold M. Then, their Riemann curvatures are related by*

$$\bar{R}^i{}_k = R^i{}_k + E\delta^i{}_k + \tau_k y^i, \tag{2.9}$$

where

$$E = P^2 - P_{|m}y^m, \qquad \tau_k = 3(P_{|k} - PP_{.k}) + E_{.k}.$$

Here $P_{|k}$ denotes the covariant derivative of projective factor P with respect to $\bar{F}$.

For a spray G on an n-dimensional manifold M and given a volume form dV on M, we can construct a new spray by [8]

$$\tilde{G} := G + \frac{2\mathbf{S}}{n+1} Y.$$

The spray $\tilde{G}$ is called the projective spray of (G, dV). In local coordinates,

$$\widetilde{G}^i = G^i - \frac{\mathbf{S}}{n+1} y^i. \tag{2.10}$$

The projective Ricci curvature of (G, dV) is defined as the Ricci curvature of $\tilde{G}$, namely,

$$PRic_{(G,dV)} := Ric_{\tilde{G}}.$$

Then by a simple computation one has

$$PRic_{(G,dV)} = Ric + (n-1)\Big\{\frac{\mathbf{S}_{|0}}{n+1} + [\frac{\mathbf{S}}{n+1}]^2\Big\}, \tag{2.11}$$

where $Ric = Ric_G$ is the Ricci curvature of the spray G, $\mathbf{S} = \mathbf{S}_{(G,dV)}$ is the S-curvature of (G, dV) and $S_{|0}$ is the covariant derivative of S along a geodesic of G. It is known that $\tilde{G}$ remains unchanged under a projective change of G with dV fixed, thus $PRic_{(G,dV)} = Ric_{\tilde{G}}$ is a projective invariant of (G, dV). For a Finsler metric (M, F), the Riemann curvature of a projective spray is called projective Riemann curvature,

$$PR^i{}_{k(G,dV)} = R^i{}_{k\tilde{G}}.$$

A Finsler metric (M, F) is called PR-quadratic Finsler metric if

$$PR_j{}^i{}_{kl.m} = 0.$$

3. **Finsler metrics of PR-quadratic type**

In this section, we dive into the fascinating world of PR-quadratic Finsler metrics. These metrics are a special type of Finsler metrics that possess certain unique properties. By carefully considering these metrics, we can gain valuable insights and further our understanding of the mathematical principles behind them.

Lemma 3.1. *Let Finsler metric (M, F) is of PR-quadratic type. Then*

$$PR_j{}^i{}_{kl} = \mu_{jk}\delta^i{}_l - \mu_{jl}\delta^i{}_k + (\mu_{lk} - \mu_{kl})\delta^i{}_j + t_j{}^i{}_{kl}. \tag{3.1}$$

μ_{kl} and ${t_j}^i{}_{kl}$ on M can be expressed as $\mu_{kl} = \mu_{kl}(x)$ and ${t_j}^i{}_{kl} = {t_j}^i{}_{kl}(x)$. Moreover, ${t_j}^i{}_{kl}$ do not include any terms involving $\delta^i{}_j$, $\delta^i{}_k$, $\delta^i{}_l$, or y^i and one has

$$ {t_j}^i{}_{kl} = -{t_j}^i{}_{lk}, \quad {t_j}^i{}_{kl} + {t_k}^i{}_{lj} + {t_l}^i{}_{jk} = 0. \tag{3.2} $$

Proof. By using the assumption that $\widetilde{G}^i = G^i - \mathbf{S}/(n+1)y^i$ and referring to (2.5) as well as (2.9), it becomes possible to establish the Projective Riemann curvature of F as follows.

$$ PR^i{}_{kl} = R^i{}_{kl} + \frac{1}{3}(\tau_k - E_{.k})\delta^i{}_l - \frac{1}{3}(\tau_l - E_{.l})\delta^i{}_k + \frac{1}{3}(\tau_{k.l} - \tau_{l.k})y^i, \tag{3.3} $$

where

$$ E = \frac{\mathbf{S}^2}{(n+1)^2} + \frac{\mathbf{S}_{|0}}{n+1}, \quad \tau_k = -3\frac{\mathbf{S}\mathbf{S}_{.k}}{(n+1)^2} - 3\frac{\mathbf{S}_{|k}}{n+1} + E_{.k}. $$

It is known that

$$ E_{.k} = \frac{1}{n+1}\Big[\frac{(\mathbf{S}^2)_{.k}}{n+1} + (\mathbf{S}_{|k} + \mathbf{S}_{|m.k}y^m)\Big]. \tag{3.4} $$

On the other hand,

$$ \tau_{k.l} = -\frac{3\mathbf{S}_{|k.l}}{n+1} - \frac{3(\mathbf{S}^2)_{.k.l}}{2(n+1)^2} + E_{.k.l}. \tag{3.5} $$

It should be noted that one might discover that

$$ \tau_{k.l} - \tau_{l.k} = -\frac{3}{n+1}\Big(\mathbf{S}_{|k.l} - \mathbf{S}_{|l.k}\Big), \tag{3.6} $$

$$ \tau_k - E_{.k} = -\frac{3}{n+1}(\mathbf{S}_{|k} + \frac{\mathbf{S}\mathbf{S}_{.k}}{n+1}), \tag{3.7} $$

and

$$ \tau_k + \frac{E_{.k}}{2} = \frac{3}{2(n+1)}\chi_k, \tag{3.8} $$

where χ_k is the χ-curvature of F defined as (2.7). Now put

$$ R^i{}_{kl} = a_k\delta^i{}_l - a_l\delta^i{}_k + b_{kl}y^i + t^i{}_{kl}, $$

where the factors $\delta^i{}_l$, $\delta^i{}_k$ and y^i are not present in $t^i{}_{kl}$. After placing the equation mentioned above in (3.3), the result is obtained.

$$ \begin{aligned} P{R_j}^i{}_{kl} &= (a_k + \frac{1}{3}(\tau_k - E_{.k})_{.j}\delta^i{}_l - (a_l + \frac{1}{3}(\tau_l - E_{.l})_{.j}\delta^i{}_k \\ &+ (b_{kl} + \frac{1}{3}(\tau_{k.l} - \tau_{l.k}))\delta^i_j + (b_{kl} + \frac{1}{3}(\tau_{k.l} - \tau_{l.k}))_{.j}y^i + t^i{}_{kl.j}. \end{aligned} \tag{3.9} $$

Upon discovering that F is of PR-quadratic type, one can deduce that

$$ (a_k + \frac{1}{3}(\tau_k - E_{.k}))_{.j} = \mu_{jk}, \quad b_{kl} = -\frac{1}{3}(\tau_{k.l} - \tau_{l.k}) + \sigma_{kl}(x), \quad t^i{}_{kl.j} = {t_j}^i{}_{kl}(x), $$

where $\mu_{jk} = \mu_{jk}(x)$, $\sigma_{kl} = \sigma_{kl}(x)$ and $t_j{}^i{}_{kl} = t_j{}^i{}_{kl}(x)$ exhibit dependence on M, where σ_{kl} satisfies the condition $\sigma_{kl} = -\sigma_{lk}$ and $t_j{}^i{}_{kl}$ satisfies the condition $t_j{}^i{}_{kl} = -t_j{}^i{}_{lk}$. Following that, we obtain [8]

$$PR_j{}^i{}_{kl} = \mu_{jk}\delta^i{}_l - \mu_{jl}\delta^i{}_k + \sigma_{kl}\delta^i{}_j + t_j{}^i{}_{kl}. \tag{3.10}$$

The following identities hold for the Riemann curvature of $\widetilde{G}^i = G^i - \mathbf{S}/(n+1)y^i$,

$$PR_j{}^i{}_{kl} + PR_k{}^i{}_{lj} + PR_l{}^i{}_{jk} = 0, \quad PR_j{}^i{}_{kl} = -PR_j{}^i{}_{lk}.$$

After considering the previous identity and referring to equation (3.10), it becomes apparent that we can readily deduce the result referenced in equation (3.2). The previous identity and (3.10) produce

$$\sigma_{kl} = -\sigma_{lk} = \mu_{lk} - \mu_{kl},$$

which (3.1) is determined. □

An interesting topic to delve into next is the relationship between PR-quadratic and GDW-metrics. The remarkable finding is that every PR-quadratic Finsler metric can be regarded as a GDW-metric.

Proposition 3.2. *The Douglas curvature of PR-quadratic Finsler metric satisfies the following equation*

$$D_j{}^i{}_{kl|0} = \frac{1}{n+1}\mathbf{S}_{.r}D_j{}^r{}_{kl}y^i,$$

where $D_j{}^i{}_{kl|0} = D_j{}^i{}_{kl|m}y^m$ and $\mathbf{S} = \mathbf{S}(x,y)$ denotes the S-curvature of F.

Proof. By employing the assumption, which states that F is PR-quadratic, as well as the subsequent Ricci identity [8]

$$PB_j{}^i{}_{kl||m} - PB_j{}^i{}_{km||l} = PR_j{}^i{}_{ml.k},$$

we conclude

$$PB_j{}^i{}_{kl||m} - PB_j{}^i{}_{km||l} = 0,$$

where "$||$" denotes the horizontal derivative with respect to Berwald connection of $\widetilde{G}^i$, while $PB_j{}^i{}_{kl}$ denotes the Berwald tensor of $\widetilde{G}^i$. By referring to (2.10) and (2.8), it becomes evident that

$$PB_j{}^i{}_{kl} = D_j{}^i{}_{kl}.$$

It indicates that

$$D_j{}^i{}_{kl||m} - D_j{}^i{}_{km||l} = 0. \tag{3.11}$$

Nevertheless, based on (2.10), we have

$$D_j{}^i{}_{kl||m} = D_j{}^i{}_{kl|m} + \frac{1}{n+1}\{\mathbf{S}_{.j}D_m{}^i{}_{kl} + \mathbf{S}_{.k}D_j{}^i{}_{ml} + \mathbf{S}_{.l}D_j{}^i{}_{km}$$
$$+\mathbf{S}_{.m}D_j{}^i{}_{kl} + \mathbf{S}D_j{}^i{}_{kl.m} - \mathbf{S}_{.r}D_j{}^r{}_{kl}\delta^i_m - \mathbf{S}_{.r.m}D_j{}^r{}_{kl}y^i\}.$$

It can be derived from the two equations above that

$$D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = \frac{1}{n+1}\Big\{\mathbf{S}_{.r} D_j{}^r{}_{kl}\delta^i_m - \mathbf{S}_{.r} D_j{}^r{}_{km}\delta^i_l \\ +(\mathbf{S}_{.r.m} D_j{}^r{}_{kl} - \mathbf{S}_{.r.l} D_j{}^r{}_{km})y^i\Big\}.$$

Now, contracting the aforementioned equation by y^m leads to the desired outcome. □

3.1. **Douglas-Weyl (DW)-metric.** DW-metrics possess certain unique properties that make them particularly interesting to study. By being the intersection of Douglas and Weyl metrics, they inherit some characteristics from both classes. To further understand these DW-metrics, we investigate their relationship with the Projective Riemann curvature tensor. In particular, we observe that these metrics fall within the realm of PR-quadratic Finsler metrics. This class encompasses a wide range of Finsler metrics that exhibit certain quadratic properties with respect to projective transformations. Within the PR-quadratic Finsler metric class, we identify a specific subset known as SPR-quadratic metrics. These metrics possess additional special properties that make them particularly interesting and worthy of investigation. In particular, we have

Theorem 3.3. *Let F be a Finsler metric on a connected manifold M $(n>2)$. If F is a DW-metric then it is PR-quadratic with*

$$PR^i{}_k = \theta_{pq} y^q(\delta^i{}_k y^p - \delta^p{}_k y^i), \tag{3.12}$$

where $\theta_{pq} = \theta_{qp} = \theta_{pq}(x)$.

Finsler metrics characterized by the projective Riemann curvature in the expression (3.12) *are referred to as Special PR-quadratic (SPR-quadratic) Finsler metrics.*

Proof. Let F be a Finsler metric of DW type. Then the vanishing of the Weyl curvature implies that F is of scalar flag curvature.

$$R^i{}_k = \lambda(F^2\delta^i{}_k - y_k y^i),$$

where $\lambda = \lambda(x,y)$ is a function on TM.

Now, by referring to (2.5), one can deduce the validity of the subsequent identity

$$R_j{}^i{}_{ml} = \Big(\frac{F^2}{3}\lambda_{.l} + \lambda y_l\Big)_{.j}\delta^i{}_m - \Big(\frac{F^2}{3}\lambda_{.m} + \lambda y_m\Big)_{.j}\delta^i{}_l \tag{3.13}$$

$$+\frac{1}{3}\Big(\lambda_{.m} y_l - \lambda_{.l} y_m\Big)\delta^i{}_j + \frac{1}{3}\Big(\lambda_{.m} y_l - \lambda_{.l} y_m\Big)_{.j} y^i.$$

In [8], we can find the following equation.

$$R_j{}^i{}_{ml.k} = B_j{}^i{}_{kl|m} - B_j{}^i{}_{km|l}. \tag{3.14}$$

However, as assumed, F belongs to Douglas classification. Then with reference to (2.8), it can be concluded

$$B_j{}^i{}_{kl} = \frac{2}{n+1}\Big(E_{jk}\delta^i{}_l + E_{jl}\delta^i{}_k + E_{kl}\delta^i{}_j + E_{jkl}y^i\Big). \tag{3.15}$$

(3.13), (3.14) and (3.15) yield

(3.16)

(1) $(\frac{F^2}{3}\lambda_{.l} + \lambda y_l)_{.j.k} = -\frac{2}{n+1}E_{jk|l}$,
(2) $\frac{1}{3}(\lambda_{.m}y_l - \lambda_{.l}y_m)_{.k} = -\frac{2}{n+1}(E_{km|l} - E_{kl|m})$,
(3) $\frac{1}{3}(\lambda_{.m}y_l - \lambda_{.l}y_m)_{.j.k} = -\frac{2}{n+1}(E_{jkm|l} - E_{jkl|m})$.

According to the reference (2.6) and the initial equation in (3.16), one discovers

$$\begin{aligned} -\frac{2}{n+1}E_{jk|l} = -\frac{1}{n+1}S_{.j.k|l} &= -\frac{1}{n+1}(S_{|l.j.k} + S_{.r}B_j{}^r{}_{kl}) \\ &= \Big(\frac{F^2}{3}\lambda_{.l} + \lambda y_l\Big)_{.j.k}. \end{aligned} \tag{3.17}$$

Then

$$-\frac{1}{n+1}S_{.r}B_j{}^r{}_{kl} = \Big(\frac{F^2}{3}\lambda_{.l} + \lambda y_l + \frac{S_{|l}}{n+1}\Big)_{.j.k}.$$

Noting (3.15), one gets $-\frac{1}{n+1}S_{.r}B_j{}^r{}_{kl} = -(\frac{S^2}{2(n+1)^2})_{.j.k.l}$ and then

$$\Big(\frac{F^2}{3}\lambda_{.l} + \lambda y_l + \frac{S_{|l}}{n+1} + \frac{SS_{.l}}{(n+1)^2}\Big)_{.j.k} = 0.$$

Then there exists a function $\theta_{kl} = \theta_{kl}(x)$ on M such that

$$\theta_{kl} = \Big(\frac{F^2}{3}\lambda_{.l} + \lambda y_l + \frac{S_{|l}}{n+1} + \frac{SS_{.l}}{(n+1)^2}\Big)_{.k}. \tag{3.18}$$

Thus

$$\lambda F^2 + \frac{S_{|0}}{n+1} + \frac{S^2}{(n+1)^2} = \theta_{pq}y^py^q,$$

which based on (2.9), it is equivalent to

$$E + \lambda F^2 = \theta_{pq}y^py^q. \tag{3.19}$$

Now we show that $\theta_{kl} = \theta_{lk}$. In the beginning, it is worth mentioning that according to (3.18), one obtains

$$\theta_{kl} - \theta_{lk} = \frac{1}{n+1}(S_{|l.k} - S_{|k.l}) - \frac{1}{3}(\lambda_{.l}y_k - \lambda_{.k}y_l). \tag{3.20}$$

The contraction of the above equation by y^k, with reference to (2.7), produces

$$-(\theta_{kl} - \theta_{lk})y^k = \frac{1}{n+1}\chi_l + \frac{F^2}{3}\lambda_{.l}.$$

Assuming that λ is the scalar curvature of F, utilizing (2.7) will produce

$$\chi_l = -\frac{n+1}{3}F^2\lambda_{.l}. \tag{3.21}$$

The two equations above demonstrate that

$$\theta_{kl} = \theta_{kl}. \tag{3.22}$$

Now, noting (3.8) and (3.21), we have

$$\tau_k - \lambda y_k = -\frac{1}{2}(E + \lambda F^2)_{.k},$$

which after observing (3.22) and (3.19), one reaches the following.

$$\tau_k - \lambda y_k = -\theta_{pk} y^p. \tag{3.23}$$

By its contradiction by y^k, one has

$$\lambda F^2 + E = \theta_{pq} y^p y^q.$$

By putting (3.19) and (3.23) in (2.9), one gets $PR^i{}_k = (\lambda F^2 + E)\delta^i{}_k + (\tau_k - \lambda y_k)y^i = \theta_{pq}y^p y^q - \theta_{pk}y^p y^i$. This expression, based on (2.5), allows us to find

$$PR_j{}^i{}_{kl} = \theta_{jl}\delta^i_k - \theta_{jk}\delta^i_l.$$

□

According to the theorem mentioned above, the class of DW-metrics is a subset of the class of SPR-quadratic Finsler metrics that have been specified in (3.12). It can be demonstrated below that each Finsler metric which is SPR-quadratic has scalar flag curvature. To put it in another way, the class of SPR-quadratic Finsler metrics is closed under projective transformations with a fixed volume form dV, and one must fulfill

$$\{DW - metrics\} \subseteq \{SPR - quadratic\, metrics\} \subseteq \{Weyl\ metrics\}.$$

Theorem 3.4. *Let F be a SPR-quadratic Finsler metric on a manifold M. Then it is of scalar flag curvature.*

Proof. Provided that F is classified as SPR-quadratic, a function $\theta_{pq} = \theta_{qp}$ will exist on M such that

$$PR^i_k = \theta_{pq}y^p y^q \delta^i_k - \theta_{kp}y^p y^i. \tag{3.24}$$

By considering the definition provided by [8] for Weyl curvature and acknowledging its characteristic as a projective invariant, one obtains

$$W^i_k = PW^i_k = PR^i_k - \frac{1}{n-1}PR^m_m\delta^i_k - \frac{1}{n+1}\Big(PR^m_{k.m} - \frac{1}{n-1}PR^m_{m.k}\Big)y^i, \tag{3.25}$$

where PW^i_k and W^i_k are the Weyl tensor related to $\widetilde{G}^i$ and G^i, respectively. By substituting (3.24) in the equation above, we obtain $PW^i_k = W^i_k = 0$. The prior equation implies that F is of scalar flag curvature. □

Corollary 3.5. *A PR-quadratic Finsler metric is of scalar curvature if and only if it is of SPR-quadratic type.*

Proof. Let (M, F) be considered as PR-quadratic. Using reference (3.25) and Lemma (3.1), the calculation of the Weyl curvature becomes possible as follows.

$$W^i_k = PW^i_k = {t_0}^i{}_{k0} - \frac{1}{n-1} {t_0}^m{}_{m0} \delta^i_k + \Big(\frac{n-2}{n+1} {t_k}^m{}_{m0} - \frac{2n-1}{n-1} {t_0}^m{}_{mk}\Big) y^i.$$

As shown in the proof procedure of Lemma 3.1, $t^i{}_{kl} = {t_j}^i{}_{kl} y^j$ do not include any terms involving $\delta^i{}_j$, $\delta^i{}_k$, $\delta^i{}_l$ and y^i. Then the Weyl curvature vanishes if and only if ${t_0}^i{}_{k0} = 0$, and this point combined with Lemma (3.1) gives

$$PR^i{}_k = -\mu_{00} \delta^i{}_k + (2\mu_{0k} - \mu_{k0}) y^i,$$

where $\mu_{00} = \mu_{pq} y^p y^q$ and $\mu_{k0} = \mu_{kp} y^p$. By substituting $\theta_{pq} = \mu_{qp} - 2\mu_{pq}$, we arrive at the desired conclusion. $\square$

In conclusion, the examples provided demonstrate that DW-metrics are strictly contained within SPR-quadratic Finsler metrics, and these Finsler metrics are in turn strictly contained within Weyl metrics. In other words, (1.1) is proved. Presented here is the SPR quadratic Finsler metric, which does not fall under the category of DW-metric.

Example 3.6. *Put*

$$\Omega = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 < 1\}, \quad p = (x, y, z) \in \Omega, \quad y = (u, v, w) \in T_p\Omega.$$

Define the Randers metric $F = \alpha + \beta$ by

$$\alpha = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2}, \quad \beta = \frac{-yu + xv}{1 - x^2 - y^2}.$$

The above Randers metric has vanishing flag curvature $K = 0$ and S-curvature $\mathbf{S} = 0$. F has zero Weyl curvature then F is of GDW metric. But β is not closed then F is not of Douglas type.

The subsequent example provides a clear indication that the class of SPR-quadratic Finsler metrics is distinct from the class of Weyl metrics, with the former being a subset of the latter.

Example 3.7. [8] *The family of Randers metrics on $\mathbb{S}^3$ constructed by Bao-Shen are weakly Berwald which are not Berwaldian. Denote generic tangent vectors on S^3 as*

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

The Finsler function for Bao-Shen's Randers space is given by

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w),$$

with

$$\alpha = \frac{\sqrt{\lambda(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\pm\sqrt{\lambda - 1}(cu - zv + yw)}{1 + x^2 + y^2 + z^2},$$

where $\lambda > 1$ is a real constant. The above Randers metric has vanishing S-curvature and with positive constant flag curvature 1. Due to vanishing S-curvature, $\widetilde{G}^i$ equals G^i, followed by

$$PR^i{}_k = R^i{}_k = F^2\delta^i{}_k - y_k y^i,$$

which does not possess SPR-quadratic property, unless F is Riemannian.

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