

On Quintic (α, β) -Metrics in Finsler Geometry

Jila Majidi^{a*} and Ali Haji-Badali^b

^aDepartment of Mathematics, Basic Sciences Faculty, University of Bonab,
Bonab, Iran.

^bDepartment of Mathematics, Basic Sciences Faculty, University of Bonab,
Bonab, Iran.

E-mail: majidi.majidi.2020@gmail.com

E-mail: haji.badali@ubonab.ac.ir

Abstract. In this paper, we study the class of quintic (α, β) -metrics. We show that every weakly Landsberg 5-th root (α, β) -metrics has vanishing S -curvature. Using it, we prove that a quintic (α, β) -metric is a weakly Landsberg metric if and only if it is a Berwald metric. Then, we show that a quintic (α, β) -metric satisfies $\Xi = 0$ if and only if $\mathbf{S} = 0$.

Keywords: Weakly Landsberg metric, Landsberg metric, Berwald metric, (α, β) -metric, S -curvature, Ξ -curvature.

1. Introduction

Let $F = F(x, y)$ be a Finsler metric on tangent bundle TM defined as $F = \sqrt[m]{A}$, where $A := a_{i_1 \dots i_m}(x)y^{i_1}y^{i_2} \dots y^{i_m}$ and $a_{i_1 \dots i_m}$ are symmetric in all its indices. Then, F is called an m -th root Finsler metric on the manifold M . The class of m -th root Finsler metrics has been developed by Shimada in [10], and applied to biology as an ecological metric by Antonelli in [1]. The fifth root metrics $F = \sqrt[5]{a_{ijklp}(x)y^i y^j y^k y^l y^p}$ are called the quintic metrics.

In order to understand the structure of quintic root metrics, one can study the non-Riemannian curvatures of these metrics [11][12][13]. Among these quantities, the mean Landsberg curvature \mathbf{J} and the S -curvature \mathbf{S} have important and deep relation with each other. Let us give a brief explanation of

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C30

their relations. The distortion $\tau = \tau(x, y)$ is a non-Riemannian quantity that is determined by the Busemann-Hausdorff volume form. The vertical and horizontal derivations of distortion τ on each tangent space give rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^s} dx^s$ and S-curvature $\mathbf{S} = \tau_{|t} y^t$. The horizontal derivative of \mathbf{I} along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|s} y^s$. Finsler metrics with $\mathbf{J} = 0$ are called weakly Landsberg metrics. The mean Landsberg curvature \mathbf{J}_y is the rate of change of \mathbf{I}_y along geodesics for any $y \in T_e M_0$. It has been shown that on a weakly Landsberg manifold, the volume function $V = Vol(x)$ is a constant [3]. The constancy of the volume function is required to establish a Gauss-Bonnet theorem for Finsler manifolds [2]. In [7], Shen showed that if $\mathbf{J} = 0$, then all the slit tangent spaces $T_e M_0$ are minimal in TM_0 . Some rigidity problems also lead to weakly Landsberg manifolds. For example, for a closed Finsler manifold with non-positive flag curvature, if the S-curvature is a constant, then it is weakly Landsbergian [8]. We remark that, S-curvature is constructed by Shen for the given comparison theorems on Finsler manifolds. Apparently, the S-curvature and mean Landsberg curvature deserve further investigation.

There is a relation between an m -th root metric and an (α, β) -metric. In [4], Matsumoto-Numata studied the class of cubic (α, β) -metrics and found a complete form of these Finsler metrics on a manifold of dimension $n \geq 3$. Inspired by their results, we characterize 5-th root (α, β) -metrics and investigate the explicit form of these metrics (Lemma 3.1). Then, we show that every weakly Landsberg 5-th root (α, β) -metric has vanishing S-curvature (Theorem 3.3). Using it, we prove that weakly Landsberg 5-th root (α, β) -metrics are Berwaldian.

Theorem 1.1. *Let $F = \sqrt[5]{c_1 \alpha^4 \beta + c_2 \alpha^2 \beta^3 + c_3 \beta^5}$ be a 5-th root (α, β) -metric on a manifold M . Then, F is a weakly Landsberg metric if and only if it is a Berwald metric.*

A Finsler metric F on a manifold M is called relatively isotropic mean Landsberg metric if $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M . From Theorem 1.1, we obtain the following.

Corollary 1.2. *Every 5-th root (α, β) -metric has relatively isotropic mean Landsberg curvature if and only if it is a Berwald metric.*

The Ξ -curvature $\Xi = \Xi_j dx^j$ on the tangent bundle TM is defined by $\Xi_j := \mathbf{S}_{\cdot j|m} y^m - \mathbf{S}_{|j}$, where “ \cdot ” and “ $|$ ” denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of F , respectively [9]. It is obvious that $\mathbf{S} = 0$ implies $\Xi = 0$. We show that for quintic (α, β) -metrics, the converse is true.

Theorem 1.3. *Let $F = \sqrt[5]{c_1 \alpha^4 \beta + c_2 \alpha^2 \beta^3 + c_3 \beta^5}$ be a 5-th root (α, β) -metric on a manifold M . Then $\Xi = 0$ if and only if $\mathbf{S} = 0$.*

2. Preliminaries

Let M be a n -dimensional C^∞ manifold and $TM = \bigcup_{e \in M} T_e M$ be the tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form \mathbf{g}_y on $T_e M$ is called the fundamental tensor

$$\mathbf{g}_y(v, u) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} \left[F^2(y + tv + su) \right] \Big|_{t=s=0}, \quad u, v \in T_e M.$$

Let $e \in M$ and $F := F|_{T_e M}$. To measure the non-Euclidean feature of F_e , one can define $\mathbf{C}_y : T_e M \times T_e M \times T_e M \rightarrow \mathbb{R}$ by

$$\mathbf{C}_y(w, v, u) := \frac{1}{2} \frac{d}{ds} [\mathbf{g}_{y+su}(w, v)]_{s=0} = \frac{1}{4} \frac{\partial^3}{\partial r \partial t \partial s} [F^2(y + rw + tv + su)]_{r=t=s=0},$$

where $w, v, u \in T_e M$. By definition, \mathbf{C}_y is a symmetric trilinear form on $T_e M$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

For $y \in TM_0$, define $\mathbf{I}_y : T_e M \rightarrow \mathbb{R}$ by

$$\mathbf{I}_y(w) = \sum_{i=1}^n g^{mt}(y) \mathbf{C}_y(w, \partial_m, \partial_t),$$

where $g^{mt} = (g_{mt})^{-1}$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion.

For a Finsler manifold (M, F) of dimension n , F induced spray \mathbf{G} on $TM_0 := TM - \{0\}$, in local coordinates in TM_0 , it is given by

$$\mathbf{G} = y^t \frac{\partial}{\partial x^t} - 2G^t \frac{\partial}{\partial y^t},$$

where $G^i = G^i(x, y)$ are local functions on TM_0 expressed by

$$G^i := \frac{1}{4} g^{is} \left\{ \frac{\partial^2 [F^2]}{\partial x^t \partial y^s} y^t - \frac{\partial [F^2]}{\partial x^s} \right\}, \quad y \in T_e M.$$

\mathbf{G} is called the associated spray to (M, F) .

For a Finsler manifold (M, F) , the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \dots dx^n$ is defined as follows:

$$\sigma_F(x) := \frac{\text{Vol}(B^n(1))}{\text{Vol}\{(y^t) \in \mathbb{R}^n | F(y^t \frac{\partial}{\partial x^t}|_x) < 1\}}.$$

Then, for $y = y^m \partial / \partial x^m|_e \in T_e M$, the S -curvature is defined by

$$\mathbf{S}(y) := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} [\ln \sigma_F(x)]. \quad (2.1)$$

The S -curvature has been introduced by Shen for the formulation of a comparison theorem on Finsler manifolds .

Let (M, F) be an n -dimensional Finsler manifold. The non-Riemannian quantity Ξ -curvature $\Xi = \Xi_j dx^j$ on the tangent bundle TM is defined by

$$\Xi_j := \mathbf{S}_{\cdot j|m} y^m - \mathbf{S}_{|j},$$

where “ \cdot ” and “ $|$ ” denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of F , respectively. F is said to be of almost vanishing Ξ -curvature if

$$\Xi_j = -(n+1)F^2 \left(\frac{\theta}{F} \right)_{y^j},$$

where $\theta := t_s(x)y^s$ is a 1-form on M

For a non-zero vector $y \in T_e M$, define $\mathbf{B}_y : T_e M \times T_e M \times T_e M \rightarrow T_e M$ by $\mathbf{B}_y(v, u, w) = B_{ijl}^m v^i u^j w^l \frac{\partial}{\partial x^m} |l$, where

$$B_{ijl}^m := \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^l}.$$

\mathbf{B} is called the Berwald curvature, and F represents a Berwald metric if $\mathbf{B} = 0$.

The mean of Berwald curvature is defined by $\mathbf{E}_y : T_e M \times T_e M \rightarrow \mathbb{R}$, were

$$\mathbf{E}_y(v, w) = \sum_{i=1}^n g^{ij}(y) g_y \mathbf{B}_y(v, w, e_i, e_j).$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in T_e M_0}$ is called the mean Berwald curvature or E-curvature. In local coordinates, $\mathbf{E}_y(u, v) := E_{sl}(x, y)v^s u^l$, were

$$E_{sl} = \frac{1}{2} \mathbf{S}_{y^s y^l}(x, y) = \frac{1}{2} B_{ijm}^m,$$

If $\mathbf{E} = 0$, then F is a weakly Berwald metric. By (??), one can get the following equation

$$\mathbf{S}_{y^s y^l} = [G^m]_{y^s y^l y^m} = E_{sl}.$$

Thus $\mathbf{S} = 0$ implies that $\mathbf{E} = 0$.

To measure the changes of the Cartan torsion \mathbf{C} along geodesics, we define $\mathbf{L}_y : T_e M \otimes T_e M \otimes T_e M \rightarrow \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := \frac{d}{ds} \left[\mathbf{C}_{\dot{c}(s)}(U(s), V(s), W(s)) \right] \Big|_{s=0},$$

where $c(s)$ is a geodesic and $U(s), V(s), W(s)$ are parallel vector fields along $c(s)$ with $\dot{c}(0) = y, U(0) = u, V(0) = v, W(0) = w$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM \setminus \{0\}}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

For $y \in T_e M$ define $\mathbf{J}_y : T_e M \rightarrow \mathbb{R}$ by $\mathbf{J}_y(u) := J_t(y)u^t$, where $J_t := I_{t|s} y^s$. \mathbf{J} is called the mean Landsberg curvature or \mathbf{J} -curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$.

3. Proof of the Theorem 1.1

In this section, we are going to prove the Theorem 1.1. In order to prove it, we need to note some necessary facts. In [5], Matsumoto-Numata studied the class of cubic metrics and found the explicit form of a cubic (α, β) -metric. Here, we prove the following results.

Lemma 3.1. *Let $F = \sqrt[5]{A}$ be a 5-th Finsler metric on a manifold M . Then, we have:*

- (1): *Let $\dim(M) = 2$. In this case, by choosing a suitable quadratic form $\alpha = \sqrt{a_{jt}(x)y^j y^t}$ and one form $\beta = b_j(x)y^j$, F is always written in the form*

$$F = \sqrt[5]{c_1 \alpha^4 \beta + c_2 \alpha^2 \beta^3},$$

where c_1 and c_2 are real constants and α^2 may be degenerate.

- (2): *If $\dim(M) \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{a_{jt}(x)y^j y^t}$ and a 1-form $\beta = \beta_j(x)y^j$, then it is written in the following form*

$$F = \sqrt[5]{c_1 \alpha^4 \beta + c_2 \alpha^2 \beta^3 + c_3 \beta^5},$$

where c_1, c_2 and c_3 are real constants.

Proof. By the same argument used by Matsumoto-Numata to obtain the explicit form of a cubic (α, β) -metric in [5], we get the proof. \square

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{jt}(x)y^j y^t}$ is a Riemannian metric and $\beta = b_j(x)y^j$ is a 1-form over the manifold M . For an (α, β) -metric, let us define $b_{j;k}$ by $b_{j;k}\theta^k := db_j - b_k\theta_j^k$, where $\theta^j := dx^j$ and $\theta_j^k := \Gamma_{js}^k dx^s$ denote the Levi-Civita connection form of α . Let

$$\begin{aligned} r_{it} &:= \frac{1}{2}(b_{i;t} + b_{t;i}), & s_{it} &:= \frac{1}{2}(b_{i;t} - b_{t;i}), & r_{i0} &:= r_{it}y^t, \\ r_{00} &:= r_{it}y^i y^t, & r_t &:= b^i r_{it}, & s_{i0} &:= s_{it}y^t, & s_t &:= b^i s_{it}, \\ s^i_t &= a^{is} s_{st}, & s^i_0 &= s^i_t y^t, & r_0 &:= r_t y^t, & s_0 &:= s_t y^t. \end{aligned}$$

where $a^{it} = (a_{it})^{-1}$ and $b^i := a^{it}b_t$. Put

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &:= \frac{\phi\phi' - s(\phi'\phi' + \phi\phi'')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ \Psi &:= \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \end{aligned} \tag{3.1}$$

where $B := \|\beta\|_\alpha^2$. Let $G^t = G^t(x, y)$ and $G_\alpha^t = G_\alpha^t(x, y)$ denote the coefficients of F and α , respectively, in the same coordinate system. By definition, we have

$$G^t = G_\alpha^t + \alpha Q s^t_0 + (r_{00} - 2Q\alpha s_0)(\alpha^{-1}\Theta y^t + \Psi b^t). \quad (3.2)$$

where

$$P := \left[-2Q\alpha s_0 + r_{00} \right] \Theta \alpha^{-1}, \quad Q^t := \Psi \left[r_{00} - 2\alpha Q s_0 \right] b^t + \alpha Q s^t_0.$$

Clearly, if β is parallel with respect to α , that is $r_{ij} = 0$ and $s_{ij} = 0$, then $P = 0$ and $Q^i = 0$. In this case, $G^i = G_\alpha^i$ are quadratic in y . In this case, F is a Berwald metric. Put

$$\Phi := (sQ' - Q)\{n\Delta + sQ + 1\} - (B - s^2)(sQ + 1)Q''.$$

By direct computation, we can obtain a formula for the mean Cartan torsion of (α, β) -metrics as follows

$$I_j = -\frac{(\phi - s\phi')\Phi}{2\Delta\phi\alpha^2}(\alpha b_j - sy_j). \quad (3.3)$$

Thus $\mathbf{I} = 0$ if and only if $\Phi = 0$.

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on an n -dimensional manifold M . Then the S -curvature of F is given by

$$\mathbf{S} = \left[2\Psi - \frac{f'(b)}{bf(b)} \right] (s_0 + r_0) - \frac{\Phi}{2\Delta^2\alpha} (r_{00} - 2Q\alpha s_0),$$

where

$$f(b) := \frac{\int_0^\pi \sin^{n-2} t T(b \cos t) dt}{\int_0^\pi \sin^{n-2} t dt},$$

$$T(s) := \phi(\phi - s\phi')^{n-2} [(\phi - s\phi') + (b^2 - s^2)\phi''].$$

Here, we calculate the S -curvature of 5-th root (α, β) -metric and obtain the following.

Lemma 3.2. *The S-curvature of 5-th root (α, β) -metric is given by*

$$\begin{aligned} \mathbf{S} = & \frac{1}{2s^2\varphi\mu} \left\{ 3c_2^2s^2 + 13c_2c_3s^4 + 10c_3^2s^6 + 3c_1c_2 + 10c_1c_3s^2 - \frac{f'(b)}{bf(b)} \right\} (s_0 \\ & + r_0) - \frac{1}{4\alpha\varphi^2\mu^2s^3} \left\{ 8c_2c_3^3s^{12} - 3c_2^4b^2s^4 - 20c_3^4b^2s^{12} - 60nc_1c_2^2c_3s^6 \right. \\ & - 112nc_2c_1c_3^2s^8 + 36nc_1c_2^4b^2s^6 + 120nc_1c_2c_3^3b^2s^{12} + 640nc_1c_3^4b^2s^{14} \\ & + 348nc_2^4c_3b^2s^{10} + 120nc_2^3c_3^2b^2s^{12} + 2456nb^2c_2^2c_3^3s^{14} + 2240nb^2c_2c_3^4s^{16} \\ & - 56nc_1^2c_2^2c_3s^8 - 128nc_1^2c_2c_3^2s^{10} - 304nc_1c_2^3c_3s^{10} - 94nc_1c_2^2c_3^2s^{12} \\ & - 120nc_1c_2c_3^3s^{14} - 6c_2^4s^6 - 20nc_1^2c_2c_3s^4 - 58c_1c_3b^2c_2^2s^4 - 136c_1c_3^2c_2s^6 \\ & - 42nc_2^3c_3s^8 - 104nc_2^2c_3^2s^{10} - 108nc_2c_3^3s^{12} - 64nc_3^3c_1s^{10} - 34b^2c_1^2c_2c_3s^2 \\ & + 24c_1^2c_2c_3s^4 + 800nc_3^5b^2s^{18} - 24nc_1^2c_3^2s^6 + 36nb^2c_2^5s^8 + 4c_1c_2^3s^4 \\ & - 800nc_3^5s^{20} + 4c_1^2c_2^2s^2 - 8nc_1^2c_2^3s^6 - 10nc_2^3c_1s^4 - 2456nc_2^2c_3^3s^{16} \\ & - 36nc_1c_2^4s^8 - 96nc_1^2c_3^3s^{12} - 640nc_1c_3^4s^{16} - 348nc_2^4c_3s^{12} - 62c_2c_3^3b^2s^{10} \\ & - 2240nc_2c_3^4s^{18} + 28c_1c_2^2c_3s^6 + 80c_1c_2c_3^2s^8 + 8c_2^2c_3^2s^{10} - 9c_1c_2^3b^2s^2 \\ & - 21c_2^3c_3b^2s^6 - 60b^2c_2^2c_3^2s^8 - 80c_1c_3^3b^2s^8 - 60c_1^2c_3^2b^2s^4 - 40c_3^4s^{14} \\ & - 130nc_2^3c_3^2s^{14} - 36c_2^5s^{10} + 48c_1c_3^3s^{10} + 48c_1^2c_3^2s^6 + 56b^2c_1^2c_2^2c_3s^6 \\ & + 128nc_1^2c_2c_3^2b^2s^8 - 4nc_1^2c_2^2s^2 + 8nb^2c_1^2c_2^3s^4 + 944nb^2c_1c_2^2c_3^2s^{10} \\ & \left. + 304nb^2c_1c_2^3c_3s^8 + 96nc_1^2c_3^3b^2s^{10} \right\} \left(r_{00} + (c_1 + 3c_2s^2 + 5c_3s^4)s_0 \right), \end{aligned}$$

where

$$\begin{aligned} \varphi : & = -c_2s^2 - c_3s^4 + 2c_1c_2b^2s^2 + 4c_1c_3b^2s^4 + 6c_2^2b^2s^4 + 22c_2c_3b^2s^6 + 20c_3^2b^2s^8 \\ & - 2c_1c_2s^4 - 4c_1c_3s^6 - 6c_2^2s^6 - 22c_2c_3s^8 - 20c_3^2s^{10}, \\ \mu : & = c_2 + 2c_3s^2, \\ T : & = (c_1s + c_2s^3 + c_3s^5)(c_1s + c_2s^3 + c_3s^5 - s(c_1 + 3c_2s^2 + 5c_3s^4))^{n-2} \left\{ c_1s \right. \\ & \left. + c_2s^3 + c_3s^5 - s(c_1 + 3c_2s^2 + 5c_3s^4) + (b^2 - s^2)(6c_2s + 20c_3s^3) \right\}. \end{aligned}$$

Now, we study weakly Landsberg 5-th root (α, β) -metrics and prove the following.

Theorem 3.3. *Every weakly Landsberg 5-th root (α, β) -metric has vanishing S-curvature.*

Proof. For an (α, β) -metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$\begin{aligned} J_t = & -\frac{1}{2\Delta\alpha^4} \left[\frac{2\alpha^2}{b^2 - s^2} \left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (r_0 + s_0) h_t \right. \\ & + \frac{\alpha}{b^2 - s^2} (\Psi_1 + s \frac{\Phi}{\Delta}) (r_{00} - 2\alpha Q s_0) h_t + \alpha \left[-\alpha Q' s_0 h_t + \alpha Q (\alpha^2 s_t - y_t s_0) \right. \\ & \left. \left. + \alpha^2 \Delta s_{i0} + \alpha^2 (r_{i0} - 2\alpha Q s_t) - (r_{00} - 2\alpha Q s_0) y_t \right] \frac{\Phi}{\Delta} \right], \end{aligned} \quad (3.4)$$

where

$$\Psi_1 := \sqrt{b^2 - s^2} \left[\frac{\sqrt{b^2 - s^2} \Phi}{\Delta^{\frac{3}{2}}} \right]' \Delta^{\frac{1}{2}}, \quad h_t := b_t - \alpha^{-1} s y_t, \quad B = b^2.$$

contracting (3.4) with b^t and simplifying it, we have $\mathbf{J} = b^t J_t = 0$. It is equal to following

$$d_7 \alpha^7 + d_5 \alpha^5 + d_4 \alpha^4 + d_3 \alpha^3 + d_2 \alpha^2 + d_1 \alpha + d_0 = 0, \quad (3.5)$$

where

$$\begin{aligned} d_0 := & \left(-7\beta^6 b^2 c_1^3 c_2^4 + 33\beta^6 b^4 c_1^2 c_2^5 + 4\beta^6 n c_1^4 c_2^3 + 52\beta^6 b^4 c_1^3 c_2^3 c_3 + 150\beta^6 b^6 c_1^3 c_2^5 \right. \\ & - 128\beta^6 b^4 c_1^4 c_2 c_3^2 + 78\beta^6 c_1^4 c_2^2 c_3 b^2 - 48\beta^6 b^8 c_1^4 c_2^5 - 2\beta^6 b^2 n c_1^3 c_2^4 \\ & - 96\beta^6 b^4 c_1^4 c_2^4 + 32\beta^6 b^4 n c_1^4 c_2^4 + 348\beta^6 b^6 c_1^4 c_2^3 c_3 - 12b^4 \beta^5 c_1^3 c_2^4 - 4\beta^6 c_1^4 c_2^3 \\ & \left. - 28\beta^6 b^2 n c_1^4 c_2^2 c_3 \right) \left((2\beta^5 c_3 + \beta^3 c_2 \alpha^2) r_{00} - (\alpha^6 c_1 + 3\alpha^4 c_2 \beta^2 + 5\alpha^2 c_3 \beta^4) s_0 \right), \\ d_1 := & 8b^4 c_1^2 \beta^5 \left(-15b^2 c_2^2 - 56b^2 c_1 c_3 + 14c_1 c_2 + 21b^4 c_1 c_2^2 - 2n c_1 c_2 \right) \left((2\beta^5 c_3 \right. \\ & \left. + \beta^3 c_2 \alpha^2) r_{00} - (\alpha^6 c_1 + 3\alpha^4 c_2 \beta^2 + 5\alpha^2 c_3 \beta^4) s_0 \right), \\ d_2 := & \left(1620\beta^6 b^4 c_1^3 c_2^3 c_3 + 1424\beta^6 b^4 c_1^4 c_2 c_3^2 - 386\beta^6 c_1^4 c_2^2 c_3 b^2 + 108\beta^6 b^2 n c_1^3 c_2^4 \right. \\ & - 140c_1^4 c_2^3 c_3 \beta^6 + 112b^4 \beta^5 c_1^3 c_2^4 - 135\beta^6 b^2 c_1^3 c_2^4 + 198\beta^6 b^4 c_1^2 c_2^5 - 12\beta^6 n c_1^4 c_2^3 \\ & + 180c_1^4 c_2^4 \beta^6 + 200\beta^6 b^2 n c_1^4 c_2^2 c_3 + 12\beta^6 c_1^4 c_2^3 - 45\beta^6 b^6 c_1 c_2^6 + 342\beta^6 b^8 c_1^2 c_2^6 \\ & - 120\beta^5 b^6 c_2^5 c_1^2 + 168\beta^5 b^8 c_1^3 c_2^5 + 496\beta^6 b^6 n c_1^4 c_2^3 c_3 - 272\beta^6 b^4 n c_1^4 c_2 c_3^2 \\ & - 48\beta^6 b^8 n c_1^4 c_2^5 - 2352\beta^6 b^6 c_1^3 c_2^2 c_3^2 - 90\beta^6 b^4 n c_1^2 c_2^5 + 1688\beta^6 b^8 c_1^4 c_2^2 c_3^2 \\ & - 144\beta^6 b^4 n c_1^4 c_2^4 - 714\beta^6 b^6 c_1^3 c_2^5 - 600\beta^6 b^6 c_1^4 c_3^3 - 504\beta^6 b^4 n c_1^3 c_2^3 c_3 \\ & + 2436\beta^6 b^8 c_1^3 c_2^4 c_3 + 312n c_1^3 c_2^5 \beta^6 - 873\beta^6 b^6 c_1^2 c_2^4 c_3 - 448\beta^5 b^6 c_1^3 c_2^3 c_3 \\ & \left. - 16\beta^5 n c_1^3 c_2^4 b^4 \right) \left(b^2 s_0 c_1^4 c_2^3 + 3b^4 r_0 c_1^3 c_2^4 - 4n s_0 c_1^4 c_2^3 b^2 - 46b^4 s_0 c_1^4 c_2^3 c_3 \right. \\ & \left. + 27b^6 s_0 c_1^4 c_2^4 - 9b^4 s_0 c_1^3 c_2^4 \right) \beta^4, \end{aligned}$$

$$\begin{aligned}
d_3 := & 396b^8\beta^4s_0c_1^4c_2^3c_3 - 132b^6\beta^4s_0c_1^4c_2^4 + 186b^4\beta^4s_0c_1^3c_2^4 - 624\beta^4s_0c_1^3c_2^3c_3 \\
& - 24b^4r_0\beta^4c_1^3c_2^4 - 42b^2\beta^4s_0c_1^4c_2^3 - 100b^4n\beta^4s_0c_1^4c_2^2c_3 + 48b^6n\beta^4s_0c_1^4c_2^4 \\
& - 520c_1^4c_2c_3^2\beta^4s_0 + 198c_1^5c_2^3\beta^4s_0 + 436b^4\beta^4s_0c_1^4c_2^2c_3 + 24n\beta^4s_0c_1^4c_2^3b^2 \\
& - 48\beta^6b^6c_1^3c_2^5r_0\alpha^2 - 48\beta^3b^6c_1^4c_2^4s_0\alpha^6 - 144\beta^5b^6c_1^3c_2^5s_0\alpha^4 \\
& - 240\alpha^2\beta^7b^6c_1^3c_2^4c_3s_0 - 54b^4n\beta^4s_0c_1^3c_2^4 - 81b^6\beta^4c_1^2c_2^5s_0 - 96\beta^8b^6c_1^3c_2^4r_0c_3,
\end{aligned}$$

$$d_4 := -6b^4c_1^4c_2^3\beta^2s_0,$$

$$d_5 := 3b^4c_1^3c_2^2\beta^2(12b^4c_1c_2^2 - 4nc_1c_2 - 21b^2c_2^2 - 54b^2c_1c_3 + 16c_1c_2)s_0,$$

$$d_7 := -18b^6c_1^4c_2^3s_0. \quad (3.6)$$

(3.5) implies that

$$d_7\alpha^6 + d_5\alpha^4 + d_3\alpha^2 + d_1 = 0, \quad (3.7)$$

$$d_4\alpha^4 + d_2\alpha^2 + d_0 = 0. \quad (3.8)$$

By (3.7), we find that there exists a non-zero function $\gamma = \gamma(x, y)$ such that

$$r_{00} = \frac{c_1\alpha^6 + 3c_2\alpha^4\beta^2 + 5c_3\alpha^2\beta^4}{2c_3\beta^5 + c_2\beta^3\alpha^2}s_0 + \gamma\alpha^2. \quad (3.9)$$

Similarly, (3.8) implies that there exists a non-zero function $\delta = \delta(x, y)$ such that

$$r_{00} = \frac{c_1\alpha^6 + 3c_2\alpha^4\beta^2 + 5c_3\alpha^2\beta^4}{2c_3\beta^5 + c_2\beta^3\alpha^2}s_0 + \delta\alpha^2. \quad (3.10)$$

Since $\gamma \neq \delta$ and also γ is not a multiple of δ , then by (3.9) and (3.10) we get

$$r_{00} = \left(\frac{c_1\alpha^6 + 3\alpha^4c_2\beta^2 + 5\alpha^2c_3\beta^4}{2\beta^5c_3 + \beta^3c_2\alpha^2} \right) s_0. \quad (3.11)$$

Taking a vertical derivation of (3.11) give us the following

$$\begin{aligned}
r_{i0} = & \left\{ \frac{6c_1\alpha^4y_i + 12c_2\alpha^2y_i\beta^2 + 6c_2\alpha^4\beta b_i + 10c_3\beta^4y_i + 20c_3\alpha^2\beta^3b_i}{2\beta^5c_3 + \beta^3c_2\alpha^2} \right. \\
& \left. - \frac{(\alpha^6c_1 + 3\alpha^4c_2\beta^2 + 5\alpha^2c_3\beta^4)(10c_3\beta^4b_i + 3c_2\beta^2\alpha^2b_i + 2c_2\beta^3y_i)}{(2\beta^5c_3 + \beta^3c_2\alpha^2)^2} \right\} s_0 \\
& + \left(\frac{\alpha^6c_1 + 3\alpha^4c_2\beta^2 + 5\alpha^2c_3\beta^4}{2\beta^5c_3 + \beta^3c_2\alpha^2} \right) s_i. \quad (3.12)
\end{aligned}$$

Contracting (3.12) with b^i yields

$$\begin{aligned}
r_0 = & \left\{ \frac{6c_1\alpha^4\beta + 12c_2\alpha^2\beta^3 + 6c_2\alpha^4\beta b^2 + 10c_3\beta^5 + 20c_3\alpha^2\beta^3b^2}{2\beta^5c_3 + \beta^3c_2\alpha^2} \right. \\
& \left. - \frac{(\alpha^6c_1 + 3\alpha^4c_2\beta^2 + 5\alpha^2c_3\beta^4)(10c_3\beta^4b^2 + 3c_2\beta^2\alpha^2b^2 + 2c_2\beta^4)}{(2\beta^5c_3 + \beta^3c_2\alpha^2)^2} \right\} s_0. \quad (3.13)
\end{aligned}$$

By putting (3.11) and (3.13) into (3.6) and simplifying the result, we have

$$\eta(x, y)s_0 = 0. \quad (3.14)$$

where

$$\eta(x, y) := (15c_3^2b^4 + 5c_3c_1 - 66c_3c_2b^2 + 42c_2^2)\alpha^4 + (-15c_3^2\beta^2b^2 + 22c_3\beta^2c_2)\alpha^2 + 5c_3^2\beta^4.$$

By (3.14), it is obvious that $\eta = 0$ or $s_i = 0$. Let $\eta(x, y) = 0$. One can rewrite $\eta = 0$ as follows

$$\theta\alpha^4 + \gamma\alpha^2\beta^2 + \varepsilon\beta^4 = 0, \quad (3.15)$$

where $\theta = \theta(x, y)$, $\gamma = \gamma(x, y)$ and $\varepsilon = \varepsilon(x, y)$ are functions on TM . (3.15) implies that

$$\alpha^2 = \left(\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\varepsilon}}{2\theta} \right) \beta^2. \quad (3.16)$$

This contradicts with the positive-definiteness of α . Thus $\eta \neq 0$ and $s_i = 0$. Putting it into (3.11) gives $r_{ij} = 0$. By putting these relations in (3.4), we obtain $\mathbf{S} = 0$. \square

Proof of Theorem 1.1: In [6] Najafi-Tayebi showed that every weakly Landsberg (α, β) -metric with vanishing S-curvature on a manifold M of dimension $n \geq 3$ is a Berwald metric. By Theorem 1.1, every weakly Landsberg 5-th root metric on M of dimension $n \geq 3$ is a Berwald metric. We consider the class 5-th (α, β) -metrics of dimension $n = 2$. We know that Every 2-dimensional Finsler manifold is C -reducible

$$C_{ijt} = \frac{1}{3} \left\{ h_{ij}I_t + h_{jt}I_i + h_{ti}I_j \right\}. \quad (3.17)$$

Taking a horizontal derivation of (3.17) along Finslerian geodesic yields

$$L_{ijt} = \frac{1}{3} \left\{ h_{ij}J_t + h_{jt}J_i + h_{ti}J_j \right\}. \quad (3.18)$$

By putting $\mathbf{J} = 0$ in (3.18) implies that $\mathbf{L} = 0$. On the other hand, the Berwald curvature Finsler manifold of dimensional $n = 2$ can be written as follows

$$B^i_{jkt} = -\frac{2}{F}L_{jkt}l^i + \frac{2}{3} \left\{ E_{jk}h_t^i + E_{kt}h_j^i + E_{tj}h_k^i \right\}. \quad (3.19)$$

By Putting $\mathbf{L} = 0$ and $\mathbf{E} = 0$ in (3.19), we conclude that F is a Berwald metric. The proof is complete. \square

Proof of Corollary 1.2: Let $F = \sqrt[5]{A}$ 5-th root metric on manifold M , where $A := a_{ijklm}(x)y^i y^j y^k y^l y^m$, with a_{ijklm} symmetric in all its indices. Put

$$A_j = \frac{\partial A}{\partial y^j}, \quad A_{jt} = \frac{\partial^2 A}{\partial y^j \partial y^t}, \quad A_{xt} = \frac{\partial A}{\partial x^t}, \quad A_0 = A_{x^t} y^t, \quad A_{0j} = A_{x^t y^j} y^t.$$

Assuming that (A^{jt}) is the inverse of the definite positive tensor (A_{jt}) . In this case we have

$$g_{jt} = \frac{1}{25}A^{-\frac{8}{5}}\mathbb{A}, \quad g^{jt} = A^{-\frac{2}{5}}\mathbb{A}^{jt}, \quad y_i = \frac{1}{5}A^{-\frac{3}{5}}A_i,$$

where

$$\mathbb{A}_{jt} := 5AA_{jt} - 3A_jA_t$$

and

$$\mathbb{A}^{jt} := 5AA^{jt} + \frac{3}{4}y^jy^t.$$

The Cartan tensor of F is given by

$$C_{ijs} = \frac{1}{5}A^{-\frac{12}{5}}\mathbb{C}_{ijs}, \quad (3.20)$$

where

$$\mathbb{C}_{ijs} := A^2A_{ijs} + \frac{24}{25}A_iA_jA_s - \frac{3}{5}A\{A_iA_{js} + A_jA_{si} + A_sA_{ij}\}.$$

Thus the mean Cartan torsion is as follows

$$I_s = g^{ij}C_{ijs} = \frac{1}{5}A^{-3}\mathbb{A}^{ij}\mathbb{C}_{ijs}. \quad (3.21)$$

In [14], Yu and You found that the spray coefficients of F are given by

$$G^i = \frac{1}{2}(A_{0s} - A_{xs})A^{is}. \quad (3.22)$$

It is easy to see that G^i are rational functions in y . Since

$$L_{ijs} = \frac{1}{2}y_t G^t_{y^i y^j y^s},$$

then we have

$$L_{ijs} = -\frac{1}{10}A^{-\frac{3}{5}}A_t G^t_{y^i y^j y^s}.$$

Therefore, we have

$$J_s = g^{ij}L_{ijs} = -\frac{1}{10}A^{-1}\mathbb{A}^{ij}A_t G^t_{y^i y^j y^s}. \quad (3.23)$$

Since F has relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$, then by (3.23), (3.21) and $F = \sqrt[5]{A}$, we have

$$A^2A_t G^t_{y^i y^j y^s} = -2c\sqrt[5]{A}\mathbb{C}_{ijs}, \quad (3.24)$$

The left hand side of (3.24) is a rational function in y , while the other side is an irrational function in y . So $c = 0$ and F reduces to a weakly Landsberg metric. By Theorem 1.1, we get the proof. \square

4. Proof of Theorem 1.3

In these section, we will prove a generalized version of Theorem 1.3. Indeed we study 5-th root (α, β) -metrics with almost vanishing Ξ -curvature. More precisely, we prove the following.

Theorem 4.1. *Let $F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3 + c_3\beta^5}$ be a 5-th root (α, β) -metric on a manifold M . Then F has almost vanishing Ξ -curvature if and only if $\mathbf{S} = 0$.*

for proving Theorem 4.1, we calculate the Ξ -curvature of 5-th root (α, β) -metrics. For any (α, β) -metric, the Ξ -curvature is given by

$$\Xi_j := H_{j;t}y^t - H_{;j} - 2H_{.j.t}H^t, \quad (4.1)$$

where “;” denotes the horizontal covariant derivative with respect to α . By calculating the right side of the (4.1) and gaining the following

$$H_{;j} := c_1 \frac{r_{00;j}}{\alpha} + c_2 \frac{r_{j0} - s_{j0}}{\alpha} + c_3 s_{0;j} + 2c_4(r_j + s_j) + 2\Psi r_{0;j},$$

where

$$\begin{aligned} A &:= r_{00} - 2\alpha Q s_0, \\ c_1 &:= (n+1)(\Psi' + \Theta), \\ c_2 &:= \left\{ (n+1)\Theta' + (B-s^2)\Psi'' - 2s\Psi' \right\} \frac{A}{\alpha} + 2\Psi' r_0 \left\{ 2(sQ' - Q)\Psi \right. \\ &\quad \left. + -Q'' - sQ - 2(n+1)Q'\Theta - (Q' + 2Q'\Psi' + 2(B-s^2)\Psi Q'') \right\} s_0, \\ c_3 &:= Q' - 2s\Psi Q - 2(n+1)Q\Theta - 2(B-s^2)(Q'\Psi + \Psi'Q), \\ c_4 &:= \Psi' \frac{A}{\alpha} - 2Q'\Psi s_0. \end{aligned}$$

Also, we have

$$\begin{aligned} H_{j;t}y^t &:= p_{5j} \frac{r_{00}}{\alpha} + p_6 s_{j;0} + 2p_{7j}(r_0 + s_0) + 2\Psi r_{j;0} \\ &\quad + \Lambda \left((r_{j0} + s_{j0}) \frac{1}{\alpha} - \frac{r_{00}y_j}{\alpha^3} \right) + \Lambda_{;t}y^t(\alpha b_j - s y_j) \frac{1}{\alpha^2}, \end{aligned}$$

where

$$\begin{aligned} p_{5j} &:= 2\Psi' r_j + \left\{ Q'' - \left((B-s^2)Q'' + Q - sQ' \right) \right\} s_j \\ &\quad + \left\{ (n+1)\Theta' - 2\Psi' s + \Psi''(B-s^2) \right\} \left(\frac{2r_{j0}}{\alpha} - r_{00} \frac{y_j}{\alpha^3} - 2Qs_j \right), \\ p_6 &:= Q' - (B-s^2)Q' - sQ, \\ p_{7j} &:= \left(\frac{2r_{j0}}{\alpha} - r_{00} \frac{y_j}{\alpha^3} - 2Qs_j \right) \Psi' + Q' s_j, \end{aligned}$$

$$\Lambda := \frac{A}{\alpha} \left\{ 2\Psi' r_0 + (n+1)\Theta' + \Psi''(B-s^2) - 2\Psi' s \right\} + \left\{ Q'' - 2Q\Psi \right. \\ \left. + 2(Q'\Psi - Q\Psi')s - 2(n+1)Q'\Theta - 2(2\Psi'Q' - \Psi Q'')(B-s^2) \right\} s_0,$$

$$\Lambda_{,t} y^t := p_{11} r_{00;0} + \left(\frac{1}{\alpha} p_{12} - 2p_{11} Q' s_0 \right) r_{00} + (Q'' - 2(n+1)Q'\Theta - 2\alpha Q p_{11}) s_{0;0} \\ + 2\Psi'' \frac{A}{\alpha} (r_0 + s_0) + 2\Psi' r_{0;0} - 2s \frac{\Psi'}{\alpha} (r_{00;0} - 2Q' s_0 r_{00} - 2\alpha Q s_{0;0}) \\ + p_{21} \frac{r_{00}}{\alpha} + p_{22} s_{0;0} + 2p_{23} (r_0 + s_0) + p_{31} \frac{r_{00}}{\alpha} + p_{32} s_{0;0} - 4Q'' \Psi (r_0 + s_0),$$

$$p_{11} := \frac{1}{\alpha} \left\{ (n+1)\Theta' + (B-s^2)\Psi'' \right\},$$

$$p_{12} := \frac{A}{\alpha} \left\{ (n+1)\Theta'' - 2s\Psi'' + (B-s^2)\Psi''' \right\} - \left\{ 2(n+1)(\Theta'Q' + Q''\Theta) - Q''' \right\} s_0,$$

$$p_{21} := 2\Psi'' r_0 - 2 \left\{ 3\Psi' Q' s + (\Psi' + \Psi'' s) Q \right\} s_0 - 2 \frac{A}{\alpha} (\Psi' + \Psi'' s),$$

$$p_{22} := -2\Psi' Q s - 4(B-s^2)\Psi' Q',$$

$$p_{23} := -4Q' \Psi' s_0,$$

$$p_{31} := \left\{ 2(Q'\Psi' - Q\Psi' + 3Q''\Psi) s - 2(B-s^2)(Q''\Psi' + Q'''\Psi) \right\} s_0,$$

$$p_{32} := 2 \left\{ sQ' - Q - (B-s^2)Q'' \right\} \Psi,$$

instead of $H_{,j,t} H^t$ in (4.1) is given by

$$H_{,j,t} H^t = Q \left\{ c_{1j} s_0 + c_{2j} \alpha + c_{3j} \alpha^2 + \Lambda \left(-\frac{s_0}{\alpha^2} y_j - \frac{ss_{j0}}{\alpha} \right) + \Lambda_{,m} s^m \left(b_j - \frac{sy_j}{\alpha} \right) \right\} \\ + A \Psi \left\{ \frac{c_{1j}(B-s^2)}{\alpha} + \Lambda \left[\left(3\frac{s^2}{\alpha^3} - \frac{B}{\alpha^3} \right) y_j - 2\frac{s}{\alpha^2} b_j \right] + \Lambda_{,m} b^m \left(\frac{b_j}{\alpha} - \frac{sy_j}{\alpha^2} \right) \right\},$$

where

$$\Lambda_{,t} s^t_0 := 2(\alpha p_{11} - 2\Psi' s) \left(\frac{q_{00}}{\alpha} - Q t_0 - \frac{Q'}{\alpha} s_0^2 \right) \\ + \frac{1}{\alpha} (p_{12} + p_{21} + p_{31}) s_0 + (p_{41} + p_{22} + p_{32}) t_0 + 2\Psi' q_0,$$

$$\Lambda_{,t} b^t := (\alpha p_{11} - 2\Psi' s) \left(2\frac{r_0}{\alpha} - \frac{sr_{00}}{\alpha^2} - 2(B-s^2)Q' \frac{s_0}{\alpha} \right) \\ + \frac{(B-s^2)}{\alpha} (p_{12} + p_{21} + p_{31}) + 2\Psi' r,$$

$$p_{41} := Q'' - 2(n+1)Q'\Theta,$$

$$c_{1j} := 2\Psi' r_j + \left[(n+1)\Theta' + (B-s^2)\Psi'' - 2s\Psi' \right] \left(2\frac{r_{j0}}{\alpha} - \frac{r_{00}y_j}{\alpha^3} - 2Qs_j \right) \\ - \left\{ 2Q' \left((n+1)\Theta + (B-s^2)\Psi' \right) - Q'' + \left(Q + sQ' + Q''(B-s^2) - 2Q's \right) \right\} s_j,$$

$$c_{2j} := \left[(B-s^2)\Psi' + (n+1)\Theta \right] \left(2\frac{q_{j0}}{\alpha} - 2\frac{q_{00}y_j}{\alpha^3} - \frac{r_{00}y_j}{\alpha^3} s_{j0} \right),$$

$$c_{3j} := \left[(B-s^2)\Psi' + (n+1)\Theta \right] \left(2\frac{r_j}{\alpha} - 2\frac{r_{j0}}{\alpha^2} s + 3\frac{r_{00}sy_j}{\alpha^4} - 2\frac{r_0y_j}{\alpha^3} - \frac{r_{00}b_j}{\alpha^3} \right).$$

Lemma 4.2. *Let $F = F(x, y)$ be a 5-th root (α, β) -metric on a manifold M of dimension $n \geq 3$. Suppose that F is of almost vanishing Ξ -curvature. Then F has vanishing Ξ -curvature.*

Proof. Let $F = \sqrt[5]{a_{ijklm}y^i y^j y^k y^l y^m}$ be a 5-th root metric with almost vanishing Ξ -curvature on an n -dimensional manifold M . Then its Ξ -curvature can be expressed as

$$\Xi_j = -(n+1)F^2 \left(\frac{\theta}{F} \right)_{y^j}. \quad (4.2)$$

where $\theta = t_j(x)y^j$ is a 1-form on M . By Lemma 2.1 in [14], the spray coefficients of an m -th root metric is rational function in y . By definition, the S-curvature and then the Ξ -curvature of F are rational functions in y . It follows that the left side of (4.2) is a rational function in y . while the right side of (4.2) is an irrational function in y . Thus we get $\Xi = 0$. \square

Proof of Theorem 4.1: Let $F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3 + c_3\beta^5}$ be a 5-th root (α, β) -metric on a manifold M . Suppose that F has almost vanishing Ξ -curvature. Then, by Lemma 4.2, we have $\Xi_j = 0$. Let us define $\Xi := \Xi_j b^j$. So multiplying (4.1) with b^j yields

$$f_7\alpha^7 + f_6\alpha^6 + f_5\alpha^5 + f_4\alpha^4 + f_3\alpha^3 + f_2\alpha^2 + f_1\alpha + f_0 = 0. \quad (4.3)$$

where

$$\begin{aligned} f_0 &:= -12960c_3^3\beta^6r_{00}^2, \\ f_1 &:= -5400c_3^3\beta^6r_{00}^2, \\ f_2 &:= 1290c_3^3\beta^5r_{00,0} - 8568c_2c_3^2\beta^4r_{00}^2 + 1260c_3^3\beta^5r_{00,0} + 5610c_3^3\beta^4r_{00}^2 \\ &\quad - 1620\beta^5c_3^3r_{00}s_0 + 6370c_3^3\beta^4r_{00}^2 + 100(n+1)c_3^3\beta^4r_{00}^2, \\ f_3 &:= 38880c_3^3\beta^5s_0^2 - 100c_3^3\beta^4r_{00}^2 - 270c_3^3\beta^5r_{00}s_0 + 120c_3^3\beta^5r_{00} + 4200c_3^3\beta^4r_{00}^2 \\ &\quad + 200c_3^3B\beta^4r_{00}^2 + 38880c_3^3\beta^5r_0s_0 + 15552c_3^3\beta^5r_{00} - 35640c_2c_3^2\beta^4r_{00}^2 \\ f_4 &:= -9020\beta^4c_3^3s_{0,0} + 40500\beta^4c_3^3s_0^2 + 1080\beta^3c_3^3r_{00}^2 + 31104c_3^3\beta^5s_0 + 31104\beta^5c_3^3r_0 \\ &\quad + 64800(n+1)\beta^4c_3^3s_{0,0} + 12960\beta^4c_3^3r_{00} + 2052\beta^3c_3^3r_{00}s_0 \\ &\quad + 1420c_3^3\beta^3r_0r_{00} + 4100\beta^3c_3^3Br_{00,0} - 1080(n+1)c_3^3\beta^3r_{00}^2 - 808c_2^2\beta^3r_{00,0}c_2 \\ &\quad - 970\beta^2c_3^3B^2r_{00}^2 - 380\beta^2c_3^2r_{00}^2c_1 + 4184\beta^2c_3^2r_{00}^2c_2 - 254\beta^2c_3r_{00}^2c_2^2 \\ &\quad + 5400(n+1)c_3^3\beta^3r_{00}s_0 - 34560(n+1)c_3^3B\beta^3r_{00,0} - 47520c_3^3\beta^3r_0r_{00} + \\ &\quad - 1080c_2^2\beta^3r_{00}s_0c_2 - 55980c_3^3B\beta^2r_{00}^2 + 97452(n+1)c_3^2\beta^2r_{00}^2c_2 \\ &\quad + 371088c_2^2B\beta^2r_{00}^2c_2 - 130c_3^3\beta^2Br_{00}^2 + 700\beta^3c_3^3Br_{00}s_0 + 81216\beta^3c_3^2r_{00,0}c_2, \\ f_5 &:= 3600\beta^2c_3^3Br_{00}^2 - 266976\beta^3c_3^2s_0^2c_2 + 858\beta^3c_3^2r_{00}c_2 + 120\beta^3c_3^3Br_{00}s_0 \\ &\quad - 560c_3^3\beta^3Br_{00} + 324\beta^3c_3^3r_0r_{00} + 266976\beta^3c_3^2r_0s_0c_2 + 150\beta^2c_3^2Br_{00}^2c_2 \end{aligned}$$

$$\begin{aligned}
& -430c_3^3\beta^2B^2r_{00}^2 + 1030\beta^3c_3^3r_{00}c_2 - 330\beta^4c_3^3s_0^2 - 640\beta^3c_3^3Br_{00} \\
& + 240c_3^3\beta^2r_{00}^2c_2 - 18\beta^3c_3^3Br_{00}s_0 - 180900\beta^3c_3^3r_{00}s_0c_2 + 181440\beta^3c_3^3Bs_0^2 \\
& + 23400c_3^3\beta^3r_{00}s_0 - 9200\beta^2c_3^3Br_{00}^2 - 162\beta^2c_3^3r_{00}^2c_1 - 105426\beta^2c_3r_{00}^2c_2^2 \\
& - 63\beta^2c_3^2r_{00}^2c_2 - 9450\beta^3c_3^3r_{00}s_0, \\
f_6 = & 4516r_{00}^2c_2^3 + 840c_3^3B^3r_{00}^2 + 640\beta^3c_3^3q_0 - 639720c_3^2B^2r_{00}^2c_2 + 120\beta^2c_3^3r_{00} \\
& + 20970c_3^3B^2r_{00}^2 + 191160c_3^2r_{00}^2c_1 + 1288c_3r_{00}^2c_2^2 - 45360\beta^3c_3^3s_0 - 220\beta^3c_3^3r_0 \\
& + 113400\beta^3c_3^3t_0 - 92610\beta^2c_3^3s_0^2 + 51840\beta^2c_3^3r_0 + 106350(n+1)c_3^3B^2r_{00}^2 \\
& + 168480c_3^2Br_{00}^2c_1 - 879912c_3^2Br_{00}^2c_2 + 11720(n+1)c_3^2r_{00}^2c_1 + 279c_3r_{00}^2c_2^2 \\
& - 219024c_3r_{00}^2c_1c_2 + 1099152c_3Br_{00}^2c_2^2 - 32400(n+1)\beta^3c_3^3q_0 - 81\beta^3c_3^3t_0 \\
& + 32400(n+1)\beta^3c_3^3s_0 + 420\beta^2c_3^3r_0s_0 - 175500\beta^2c_3^3Bs_0^2 + 100\beta^2c_3^3Bq_{00} \\
& + 420\beta^2c_3^3rr_{00} + 210\beta^2c_3^3r_0r_{00} + 480\beta^2c_3^3r_0s_0 + 339120\beta^2c_3^3Bs_{0;0} \\
& + 607392\beta^2c_3^2s_{0;0}c_2 + 69480c_2c_3^2\beta r_{00}^2 - 120\beta^3c_3^3Br_0 + 42768\beta^2c_3^2s_{0;0}c_2 \\
& - 4320\beta^2c_3^2q_{00}c_2 + 276750\beta^2c_3^2s_0^2c_2 - 14400\beta c_3^3Br_{00}^2 - 37080\beta c_3^3B^2r_{00;0} \\
& - 380\beta c_3^2r_{00;0}c_1 - 260\beta c_3r_{00;0}c_2^2 - 316(n+1)c_3^2Br_{00}^2c_2 - 129600\beta^3c_3^3Bs_0 \\
& + 207360\beta^3c_3^3s_0c_2 + 20760\beta^3c_3^3r_0c_2 - 226c_3^3\beta^2Bs_{0;0} - 81\beta^2c_3^3r_{00}s_0 \\
& - 1510(n+1)\beta^2c_3^3r_0s_0 - 210\beta^2c_3^3B(n+1)q_{00} - 430\beta^2c_3^3Br_{00} \\
& + 8640(n+1)\beta^2c_3^2q_{00}c_2 + 84672\beta^2c_3^2(n+1)r_{00}c_2 - 121500\beta c_3^3B^2r_{00}s_0 \\
& - 628920\beta c_3^3Br_{00}s_0 + 9000\beta c_3^3Br_{00}^2 - 340\beta c_3^3Br_0r_{00} - 51840\beta c_3^2r_{00}s_0c_1 \\
& + 1402848\beta c_3^2r_{00}s_0c_2 + 160\beta c_3^2r_{00;0}c_1 + 264\beta c_3^2Br_{00;0}c_2 - 620\beta c_3^2r_{00}^2c_2 \\
& - 3216\beta c_3r_{00}s_0c_2^2 + 103500\beta c_3^3Br_{00}s_0 + 820(n+1)\beta c_3^3Br_0 \\
& + 15840(n+1)c_3^2\beta r_{00}s_0c_2 - 297072(n+1)\beta c_3^2r_0r_{00}c_2 - 2108c_3^2\beta Br_{00;0}c_2 \\
& + 2160\beta c_3^3B^2r_{00;0} + 944\beta c_3^2r_{00}c_2 + 220\beta c_3r_{00;0}c_2^2 + 470\beta c_3^2Br_{00}s_0c_2, \\
f_7 := & 184916c_2^2r_{00}^2 + 39125c_3^3B^3r_{00}^2 + 25800c_3^3B^2r_{00}^2 + 12960c_1c_3^2r_{00}^2 \\
& + 317250\beta^2c_3^3s_0^2 + 477c_3^3B^2(n+1)r_{00}^2 - 2860c_3^2B^2r_{00}^2c_2 + 729c_3^2Br_{00}^2c_1 \\
& + 324(n+1)c_3^2r_{00}^2c_1 - 9208c_3(n+1)r_{00}^2c_2^2 - 907c_3r_{00}^2c_1c_2 + 47c_3Br_{00}^2c_2^2 \\
& - 160(n+1)\beta^2c_3^3s_0^2 + 155\beta^2c_3^3Bs_0^2 - 220\beta^2c_3^2s_0^2c_2 + 90\beta c_3^3B^2r_{00} \\
& + 380\beta c_3^2r_{00}c_1 - 168\beta c_3^2s_0^2c_1 + 2744\beta c_3r_{00}c_2^2 - 828\beta c_3s_0^2c_2^2 - 90c_3^2Br_{00}^2c_2 \\
& - 2180\beta c_3^3B^2r_{00}s_0 + 351\beta c_3^3B^2r_0s_0 - 490\beta c_3^3Br_{00}s_0 + 108\beta c_3^3B^2r_{00} \\
& + 12432\beta c_3^2Bs_0^2c_2 + 46656\beta c_3^2r_{00}c_1 + 1368\beta c_3^2r_0s_0c_1 - 3788\beta c_3^2Br_{00}c_2 \\
& + 220\beta c_3^2r_0r_{00}c_2 + 17120\beta c_3^2r_{00}s_0c_2 + 8248\beta c_3r_0s_0c_2^2 + 3176\beta c_3r_{00}c_2^2 \\
& + 24950(n+1)\beta^2c_3^3s_0^2 + 1750(n+1)\beta c_3^3Br_{00}s_0 - 1242432\beta c_3^2Br_{00}s_0c_2 \\
& + 811800\beta c_3^2Br_{00}s_0c_2 - 600300\beta c_3^2(n+1)r_{00}s_0c_2 - 4240\beta c_3^2Br_{00}c_2 \\
& + 807864c_2^2c_3r_{00}^2 - 6260c_3^2Br_{00}^2c_2 + 81\beta^2c_3^3r_0s_0 - 35\beta c_3^3B^2s_0^2 \\
& - 80\beta c_3^2r_{00}s_0c_1 - 5430\beta c_3r_{00}s_0c_2^2 - 820\beta c_3^3Br_0r_{00},
\end{aligned}$$

By (4.3), we get

$$f_7\alpha^6 + f_5\alpha^4 + f_3\alpha^2 + f_1 = 0, \quad (4.4)$$

$$f_6\alpha^6 + f_4\alpha^4 + f_2\alpha^2 + f_0 = 0. \quad (4.5)$$

(4.4) implies that there exists a non-zero function $\mu = \mu(x, y)$ such that

$$\beta^6 c_3^3 r_{00}^2 = \mu\alpha^2. \quad (4.6)$$

Similarly, (4.5) implies that there exists a non-zero function $\nu = \nu(x, y)$ such that the following holds

$$\beta^6 c_3^3 r_{00}^2 = \nu\alpha^2. \quad (4.7)$$

Since $\mu \neq \nu$ and also μ is not a multiplication of ν , then by (4.6) and (4.7) we get

$$\beta^6 c_3^3 r_{00}^2 = 0. \quad (4.8)$$

Since $\beta^6 c_3^3 \neq 0$, then $r_{ij} = 0$ which implies that $r_i = 0$. Putting these relations in (4.3) yields

$$g_4\alpha^4 + g_3\alpha^3 + g_2\alpha^2 + g_1\alpha + g_0 = 0. \quad (4.9)$$

where

$$g_0 := +1440c_3^2s_0^2,$$

$$g_1 := 150c_3^2s_0^2\beta^3 - 152c_3^2s_0\beta^4 - 240(n+1)c_3^2\beta^3s_{0;0} + 360c_3^2\beta^3s_{0;0},$$

$$g_2 := 120c_3^2s_0^2\beta^3 - 670c_3^2\beta^2s_0^2 + 988c_2c_3\beta^2s_0^2,$$

$$\begin{aligned} g_3 := & 300(n+1)c_3^2\beta^2t_0 - 1200(n+1)c_3^2\beta^2s_0 - 7680c_2c_3\beta^2s_0 + 2496c_2c_3\beta s_{0;0} \\ & - 9250(n+1)c_3^2\beta s_0^2 - 4200c_3^2\beta^2t_0 + 1680c_3^2\beta^2s_0 + 3430c_3^2\beta s_0^2 \\ & + 8400(n+1)c_3^2B\beta s_{0;0} - 1260c_3^2\beta s_{0;0} + 480c_3^2B\beta^2s_0 - 1050c_2c_3\beta s_0^2 \\ & + 650c_3^2B\beta s_0^2 - 140(n+1)c_3\beta s_{0;0}c_2, \end{aligned}$$

$$\begin{aligned} g_4 := & 6250(n+1)c_3^2\beta s_0^2 - 460c_2c_3B s_0^2 - 5625c_3^2B\beta s_0^2 + 8500c_2c_3\beta s_0^2 \\ & - 11750c_3^2\beta s_0^2 + 130c_3^2B^2s_0^2 + 514c_1c_3s_0^2 + 3024c_2^2s_0^2, \end{aligned}$$

By (4.9), we get

$$g_3\alpha^2 + g_1 = 0, \quad (4.10)$$

$$g_4\alpha^4 + g_2\alpha^2 + g_0 = 0. \quad (4.11)$$

(4.11) implies that

$$\eta(x, y)s_0^2 = 0, \quad (4.12)$$

where

$$\begin{aligned} \eta(x, y) = & \left\{ 620(n+1)c_3^2\beta - 4606c_2c_3B - 5625c_3^2B\beta + 800c_2c_3\beta - 1150c_3^2\beta \right. \\ & + 1300c_3^2B^2 + 5184c_1c_3 + 30424c_2^2 \left. \right\} \alpha^4 + \left\{ 1250c_3^2\beta^3 - 6720c_3^2B\beta^2 \right. \\ & \left. + 988c_2c_3\beta^2 \right\} \alpha^2 + 140c_3^2\beta^4. \end{aligned}$$

By (4.12), one can find that $\eta = 0$ or $s_i = 0$. Let $\eta(x, y) = 0$. We rewrite η as follows

$$\theta\alpha^4 + \gamma\alpha^2\beta^2 + \varepsilon\beta^4 = 0, \quad (4.13)$$

where $\theta = \theta(x, y)$, $\gamma = \gamma(x, y)$ and $\varepsilon = \varepsilon(x, y)$ are scalar functions on TM . (4.13) give us

$$\alpha^2 = \left(\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\varepsilon}}{2\theta} \right) \beta^2. \quad (4.14)$$

This contradicts with the positive-definiteness of α . Thus

$$\eta \neq 0.$$

Hence, we get

$$s_i = 0.$$

Putting $r_{ij} = 0$ and $s_i = 0$ in (3.4) imply that $\mathbf{S} = 0$. The converse is trivial. \square

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Received: 02.25.2024

Accepted: 03.17.2024