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Some Rigidity Results on Complete Finsler Manifolds

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Abstract. We provide an extension of Obata's theorem to Finsler geometry and establish some rigidity results based on a second-order differential equation. Mainly, we prove that every complete simply connected Finsler manifold of positive constant flag curvature is isometrically homeomorphic to a Euclidean sphere endowed with a certain Finsler metric and vice versa. Based on these results, we present a classification of Finsler manifolds that admit a transnormal function. Specifically, we show that if a complete Finsler manifold admits a transnormal function with exactly two critical points, then it is homeomorphic to a sphere.

Keywords: Finsler metric, Rigidity, Constant curvature, Second-order differential equation, Adapted coordinate.

1. Introduction

Rigidity describes quite different concepts in mathematics. Historically, one of the first rigidity theorems, proved by Cauchy in 1813, states that if the faces of a convex polyhedron were made of metal plates and the edges were replaced by hinges, the polyhedron would be rigid [10]. Although rigidity problems are of immense interest to engineers, the intensive mathematical study of these types of problems has occurred only in the late 20th century, see [21]. In geometry sometimes an object is considered rigid if it has flexibility but not elasticity. In other words, geometrical rigidity implies invariant under isometries. In

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Riemannian geometry, the sectional curvature is invariant under isometries. Hence, a space of positive constant curvature is transformed into the same space by each isometry. This fact is sometimes described as the "strong rigidity" of a space of constant curvature.

In Finsler geometry, the encountered rigidity results are rather slightly weaker and they usually talk about under which assumptions on the flag curvature -analogous to the sectional curvature in Riemannian geometry- the underlying Finsler structure is either Riemannian or locally Minkowskian. A famous treatise in this area is by Akbar-Zadeh [1] where he established the following rigidity theorem for compact manifolds: *Let* (M, g) *be a compact without boundary Finsler manifold of constant flag curvature* K. *If* $K < 0$, *then* (M, g) *is Riemannian. If* $K = 0$, *then* (M, g) *is locally Minkowskian.*

There are several papers in Finsler geometry with results similar to Akbar-Zadeh's rigidity theorem but by considering different assumptions. Foulon addressed the case of strictly negative flag curvature in Akbar-Zadeh's theorem. In [18] he imposed the additional hypothesis that the curvature is covariantly constant along a distinguished vector field on the homogeneous bundle of tangent half-lines to show that the Finsler structure is Riemannian. Also, he presented a strong rigidity theorem for symmetric compact Finsler manifolds with negative curvature and proved that such manifolds are isometric to locally symmetric negatively curved Riemannian spaces [19]. This extends Akbar-Zadeh's rigidity theorem to a so-called "strong rigidity" one. Shen [30] considered the case of negative but not necessarily constant flag curvature by adding the assumption that the S-curvature is constant and showed that the Akbar-Zadeh's rigidity theorem still holds.

Following several rigidity theorems in the two joint papers [24] and [25], Kim in [26] proved that: "Any compact locally symmetric Finsler manifold with positive constant flag curvature is Riemannian". Also, Bidabad [8] established some rigidity theorems as an application of connection theory in Finsler geometry. Another rigidity result is presented by Wu [34] who proved that any locally symmetric Finsler manifold with nonzero flag curvature must be Riemannian.

Finsler manifolds of positive flag curvature have been studied and classified by several researchers and several results have been generalized from Riemannian spaces of positive sectional curvature to Finsler manifolds of positive flag curvature, see for instance [6, 33, 36, 37]. Also, Bidabad in [7], using the same idea as in [4], provided a classification of simply connected compact Finsler manifolds. In 2018 Boonnam et. al. [9] proved that a complete *Berwald manifold* with nowhere vanishing flag curvature must be Riemannian. Also, several results and open problems on Finsler manifolds with positive curvatures are addressed in [16].

In this paper, we apply the *adapted coordinate system* introduced in [4] to study the strong rigidity of Finsler manifolds of positive constant flag curvature. Particularly, we show that: *A simply connected complete n-dimensional Finsler manifold is of positive constant flag curvature if and only if it is isometrically homeomorphic to an n-sphere equipped with a certain Finsler metric.* This result complements Akbar-Zadeh's rigidity theorem by considering the case of $K > 0$. Note that under a pinching condition on the flag curvature, there is a well known topological classification by Rademacher, see [28].

Also, we provide an extension of Obata's theorem to Finsler geometry. Obata's theorem in Riemannian geometry says (see [35] for more details): *Let (M, g) be a complete connected Riemannian manifold of dimension $n \geq 1$ which admits a non-constant smooth solution of Obatas equation $\nabla dw + wg = 0$. Then (M, g) is isometric to the n-dimensional round sphere S^n.*
Here, we show that,
Theorem 4.2: *Let (M, g) be a complete simply connected Finsler manifold of dimension $n \geq 2$. In order to have a non-trivial solution of $\nabla^H \nabla^H \rho + C^2 \rho g = 0$ on M, it is necessary and sufficient that (M, g) be isometric to an n-sphere of radius $1/C$.*

Further, we apply adapted coordinates to extend some results from the Riemannian transnormal functions to Finsler geometry. A Finsler transnormal function is a natural generalization of distance functions. More precisely, a smooth function $\rho : M \to \mathbb{R}$ on a Finsler manifold (M, g) is called a Finsler transnormal function if the Finsler norm of the gradient of ρ is constant along each level set of ρ.

In the Riemannian geometry, transnormal functions have been studied for many years and some interesting results have been established, see for instance [22, 32]. However, transnormal functions, from the Finsler geometry point of view, have received less attention. This is despite several interesting problems that can be tackled in this area and applications of Finsler transnormal functions in Physics, particularly in modeling the propagation of waves of wildfire and water, see [12, 13, 14, 15, 17]. To the best of our knowledge, the only works on Finsler transnormal functions are [3, 23]. In [3] a generalization of some results of [32] to the Finsler geometry is presented, and in [23] classification of isoparametric functions on Randers-Minkowski spaces is presented.

Here, we extend the results of [23] to provide a classification of Finsler manifolds based on the number of critical points of a transnormal function defined on them: *If the transnormal function has no critical points, one critical point, or two critical points, then, respectively, it is conformal to the direct product of an open interval of the real line and some complete manifold, the Euclidean space, or the sphere.* Moreover, in Theorem 5.5 we prove: *If the transnormal function on a compact Finsler manifold has exactly two critical points, then the space is homeomorphic to the sphere.*

The remainder of this paper is structured as follows. In Section 2, we recall some basic definitions in Finsler geometry, including adapted coordinates for Finsler manifolds satisfying Eq. (2.1), and, Finsler transnormal functions. In Section 3, we study a special case of Eq. (2.1) which is important for establishing the main results of this work. In section 4, we proceed with generalizing Obata's theorem, and in Section 5, we focus on Finsler transnormal functions and prove that any complete Finsler transnormal function with two critical points is homeomorphic to a sphere.

2. Preliminaries

In this section, we review some definitions of Finsler geometry that we refer to through this paper. More details can be found in [29].

2.1. **Finsler Manifolds.** Let M be a real n-dimensional manifold of class C^∞ and TM its tangent bundle, i.e. $TM = \bigcup_{x\in M}\{(x,y) : y \in T_pM\}$. A *Finsler structure* on M is a function $F : TM \to [0,\infty)$, with the following properties:

(i) F is smooth on the tangent bundle of non-zero vectors $TM_0 := TM\backslash\{0\}$;

(ii) F is positively homogeneous of degree one in y, i.e. $F(x,\lambda y) = \lambda F(x,y)$, $\forall\lambda > 0$, where (x,y) is an element of TM;

(iii) The Hessian matrix of F^2, $(g_{ij}) := \left(\frac{1}{2}\left[\frac{\partial^2}{\partial y^i\partial y^j}F^2\right]\right)$, is positive definite on TM_0.

A *Finsler manifold* is a pair consisting of a differentiable manifold M and a Finsler structure F on M. The tensor field g with the components g_{ij} is called the *Finsler metric tensor* and we denote a Finsler manifold by (M,g). We denote the natural projection on TM_0 by π and its differential by π_*, i.e. $\pi_* : TTM_0 \to TM$. The *vertical vector bundle* on M is defined as $VTM := \bigcup_{\nu\in TM_0} V_\nu TM$ where $V_\nu TM = \ker(\pi_*)_\nu$. The complementary decomposition HTM where $HTM \oplus VTM = TTM_0$ is called a *non-linear connection* on TM_0. The coefficients of the nonlinear connection are denoted by $G^i_j(x,y)$, where $G^i_j := \dfrac{\partial G^i}{\partial y^j}$ and $G^i := \dfrac{1}{4}g^{ik}(\dfrac{\partial^2F^2}{\partial y^k\partial x^j}y^j - \dfrac{\partial F^2}{\partial x^k})$. By using the local coordinates (x^i,y^i) on TM, called the *line elements*, we have the local field of frames $\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\}$ on TTM. Given a non-linear connection, we can choose a local field of frames $\{\frac{\delta}{\delta x^i}\frac{\partial}{\partial y^i}\}$ on TTM_0 where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G^j_i\frac{\partial}{\partial y^j}$ and $\frac{\partial}{\partial y^i}$ are the set of vector fields on HTM and VTM, respectively.

A 1-form of the *Cartan connection* is given by $w^i_j = \Gamma^i_{jk}dx^k + C^i_{jk}\delta y^k$, where

$$\Gamma^i_{jk} = \frac{1}{2}g^{ir}(\frac{\delta g_{rk}}{\delta x^j} + \frac{\delta g_{jr}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^r}), \quad C^i_{jk} = \frac{1}{2}g^{ir}\frac{\partial g_{jk}}{\partial y^r}.$$

The coefficients Γ^i_{jk} and C^i_{jk} are called coefficients of *horizontal and vertical covariant derivatives* of the Cartan connection, respectively. Given a tensor field T with the components $T^i_{jk}(x,y)$ on TM, the components of the Cartan horizontal covariant derivative of T, $\nabla^H T$, are given by

$$\nabla_r T^i_{jk} := \frac{\delta}{\delta x^r} T^i_{jk} - T^i_{sk}\Gamma^s_{jr} - T^i_{js}\Gamma^s_{kr} + T^s_{jk}\Gamma^i_{sr}\,.$$

Assume that $\gamma : I \to M$ defined by $t \to x^i(t)$ be a smooth curve on M and $\tilde{\gamma}(t) = \big(x^i(t), \dfrac{dx^i}{dt}\big)$ its natural lift on TM. We say that γ is a *geodesic* of the Finsler manifold (M,g) if $\nabla_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}} = 0$. Here,

$$\dot{\tilde{\gamma}}(t) = \frac{d}{dt}\frac{\delta y^i}{\delta x^i} + \frac{\delta y^i}{dt}\frac{\partial}{\partial y^i},$$

where

$$\frac{\delta y^i}{dt} := \frac{dy^i}{dt} + G^i_j\big(x(t), \frac{dx}{dt}\big)\frac{\partial x^j}{dt}.$$

2.2. **Finsler Manifolds with a Non-trivial Solution of $\nabla^H\nabla^H\rho = \phi g$.** Let $\rho : M \to I\!R$ be a scalar function on M that satisfies the following second order differential equation

$$\nabla^H\nabla^H\rho = \phi g, \tag{2.1}$$

where ∇^H is the Cartan horizontal covariant derivative and ϕ is a function of x alone. The connected component of a regular hypersurface defined by $\rho = constant$ is called a *level set of* ρ. We denote by $\mathtt{grad}\rho$ the gradient vector field of ρ which is locally written in the form $\mathtt{grad}\rho = \rho^i\frac{\partial}{\partial x^i}$, where $\rho^i := g^{ij}\rho_j$, $\rho_j := \frac{\partial\rho}{\partial x^j}$ for $i, j, \cdots \in \{1,\ldots,n\}$. Note that the partial derivatives ρ_j are defined on the manifold M while ρ^i, the components of $\mathtt{grad}\rho$, are defined on its slit tangent bundle TM_0. Hence, $\mathtt{grad}\rho$ can be considered as a section of $\pi^*TM \to TM_0$, the pulled-back tangent bundle over TM_0, and its trajectories lie on TM_0. For more details, see [4] and references therein. One can easily verify that the canonical projection of the trajectories of the vector field $\mathtt{grad}\rho$ are geodesic arcs on M [4]. Therefore, we can choose local coordinates $(u^1 = t, u^2, ..., u^n)$ on M such that t is the parameter of the geodesic containing the projection of a trajectory of the vector field $\mathtt{grad}\rho$ and the level sets of ρ are given by t =constant. These geodesics are called *t-geodesics*. Since in this local coordinate system, the level sets of ρ are given by t =constant, ρ may be considered as a function of t only. In the sequel, we will refer to these level sets and these local coordinates as *t-levels* and *adapted coordinates*, respectively. Also, note that along any t-geodesic, Eq. (2.1) reduces to the second order differential equation

$$\frac{d^2\rho}{dt^2} = \phi(\rho), \tag{2.2}$$

where ϕ is a function of ρ which is differentiable at non-critical points.

Let (M, g) be a Finsler manifold and ρ is a non-trivial solution of Eq. (2.1) on M. Then, using the adapted coordinates, components of the Finsler metric tensor g are given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & g_{22} & \dots & g_{2n} \\ \vdots & & \dots & \\ 0 & g_{n2} & \dots & g_{nn} \end{pmatrix}, \tag{2.3}$$

and t may be regarded as the arc-length parameter of t-geodesics. It can be easily verified that the Finsler metric form of M is given by [4]

$$ds^2 = (dt)^2 + \rho'^2 f_{\gamma\beta} du^\gamma du^\beta, \tag{2.4}$$

where $f_{\gamma\beta}$ are components of a Finsler metric tensor on a t-level of ρ and $\rho'^2 f_{\gamma\beta}$ is the induced metric tensor of this t-level. Here, prime denotes the ordinary differentiation with respect to t. In this paper, the Greek indices $\alpha, \beta, \gamma, \dots$ run over the range $2, 3, \dots, n$.

A point o of (M, g) is called a *critical point* of ρ if the vector field $\mathtt{grad}\rho$ vanishes at o, or equivalently if $\rho'(o) = 0$, see [4]. If a non-trivial solution of Eq. (2.1) has some critical points, then M possess some interesting properties. For instance:

Lemma 2.1. [4] *Let (M, g) be an n-dimensional Finsler manifold which admits a non-trivial solution ρ of Eq.* (2.1) *with one critical point. Then any t-level set of ρ with Finsler metric form $\overline{ds}^2 = f_{\gamma\beta} du^\gamma du^\beta$, where $f_{\gamma\beta}$ is given by Eq. (2.4), has the positive constant flag curvature $\rho''^2(0)$.*

Proposition 2.2. [4] *Let (M, g) be a connected complete Finsler manifold of dimension $n \geq 2$. If M admits a non-trivial solution of Eq. (2.1), then depending on the number of critical points of ρ, i.e. zero, one or two respectively, it is conformal to*

(a) *A direct product $J \times M$ of an open interval J of the real line and an* $(n-1)$*-dimensional complete Finsler manifold M.*
(b) *An n-dimensional Euclidean space.*
(c) *An n-dimensional unit sphere in a Euclidean space.*

Lemma 2.3. [4] *Let (M, g) be a simply connected and compact Finsler manifold of dimension $n > 2$ which admits a solution of Eq. (2.1) with two critical points, then M is homeomorphic to an n-sphere.*

2.3. **Transnormal Functions.** Given a Finsler manifold (M, g) and a smooth function $\rho : M \longrightarrow \mathbb{R}$, if there exists a continuous function $\mathfrak{b} : \rho(M) \longrightarrow \mathbb{R}$ such that

$$g(\mathtt{grad}\rho, \mathtt{grad}\rho) = \mathfrak{b} \circ \rho, \tag{2.5}$$

then ρ is called a Finsler transnormal function.

It is not difficult to show that for any tangent vector v of M (see [29] for details),

$$g(v,\ \mathtt{grad}\rho) = d\rho\, v. \tag{2.6}$$

Recall that a critical point is a point o such that $\rho'(o) = 0$, we define a regular point as a point of M which is not critical. The regular and critical values are images of regular and critical points, respectively, under ρ. The connected component of the pre-image of a regular value, $\rho^{-1}(t)$, is called a *regular level set of* ρ and the connected component of the pre-image of a critical value is called a *singular level set of* ρ. From Eq. (2.5), one deduces that the function $\mathfrak{b}$ is smooth on $\rho(M^0)$, where M^0 is the subset of M containing the regular points [11].

Given a Finsler manifold (M, g) and any two points p, $q \in M$, the *Finsler distance* from p to q is defined as

$$d(p, q) := \inf_{\gamma} \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt, \tag{2.7}$$

where the infimum is taken over all piece-wise smooth curves $\gamma : [a, b] \longrightarrow M$ joining p to q. One special example of Finsler transnormal functions is the Finsler distance function: Given a compact subset $A \subset M$, the Finsler distance function from A to p is given by $\rho : M \longrightarrow \mathbb{R}$ where $\rho(p) = d(A, p)$. One can prove that ρ is locally Lipschitz continuous [29] and therefore it is differentiable almost everywhere. Also, it is not difficult to show that the Finsler distance function ρ satisfies $g(\mathtt{grad}\rho, \mathtt{grad}\rho) = 1$ (see Lemma 3.2.3 of [29]). So, the Finsler distance function associated to a given Finsler manifold is a transnormal function with $\mathfrak{b} = 1$ in Eq. (2.5).

Some interesting properties of Finsler transnormal functions are presented in [3]. For instance,

Proposition 2.4. [3] *Let (M, g) be a Finsler manifold and $\rho : M \to I\!R$ be a transnormal function. Then,*

(a) *Integral curves of the vector field* $\mathtt{grad}\rho$*, parameterized by arc-length, are geodesics orthogonal to regular leaves.*

(b) *If (M, g) is a complete Finsler manifold such that $[a, b] \subset \rho(M)$ does not have critical values, then, for every $p \in \rho^{-1}(b)$,*

$$d(\rho^{-1}(a), p) = l(\gamma) = d(\rho^{-1}(a), \rho^{-1}(b)) = r = \int_a^b \frac{ds}{\sqrt{\mathfrak{b}(s)}},$$

where $\gamma : [0, r] \to M$ is the integral curve of $\mathtt{grad}\rho$ *parameterized by arc-length joining $\rho^{-1}(a)$ to $p \in \rho^{-1}(b)$, and $l(\gamma)$ is the length of γ.*

Note that for a transnormal function $\rho : M \to [a, b]$ on the complete Finsler manifold M where a and b are the only critical values of ρ, one can extend the geodesic γ to $\rho^{-1}(a)$ and $\rho^{-1}(b)$ and so the results of Proposition 2.4 can be extended to the whole manifold M. That is, we have the following corollary.

Corollary 2.5. *If (M, g) is a complete Finsler manifold and $\rho : M \to [a, b]$ a transnormal function such that a and b are the only critical values of ρ, then, for every $c \in [a, b]$ and every $p \in \rho^{-1}(c)$,*

$$d\Big(\rho^{-1}(a), p\Big) = l(\gamma) = d\Big(\rho^{-1}(a), \rho^{-1}(c)\Big) = r = \int_a^c \frac{ds}{\sqrt{\mathfrak{b}(s)}},$$

where $\gamma : [0, r] \to M$ is the unit speed geodesic which joins $\rho^{-1}(a)$ to $p \in \rho^{-1}(c)$ and coincides with the reparametrization of integral curve of $\texttt{grad}\rho$ in $\rho^{-1}\{(a, b)\}$. Moreover, the geodesic γ is orthogonal to all the leaves $\rho^{-1}(c)$, $c \in [a, b]$.

We call the geodesic $\gamma : [0, r] \to M$, that is the unit speed geodesic whose trace coincides with the integral curve of $\texttt{grad}\rho$, a *horizontal geodesic.*

3. A Special Solution of $\nabla^H \nabla^H \rho = \phi g$

Let (M, g) be an n-dimensional Finsler manifold and $\rho : M \to I\!\!R$ a solution of Eq. (2.1). If ϕ is a linear function of ρ with constant coefficients, then we say that ρ is a *special solution* of Eq. (2.1). Hence, any special solution of Eq. (2.1) can be written in the form

$$\nabla^H \nabla^H \rho = (-K\rho + B)g, \tag{3.1}$$

where K and B are constants. The Eq. (3.1) along any geodesic with arc-length t reduces to the ordinary differential equation

$$\frac{d^2\rho}{dt^2} = -K\rho + B. \tag{3.2}$$

Now for the special case $K = C^2 > 0$ and $B = 0$, we have

$$\frac{d^2\rho}{dt^2} + C^2\rho = 0. \tag{3.3}$$

By a suitable choice of the arc-length t, a solution of Eq.(3.3) is given by

$$\rho(t) = A\cos(Ct), \tag{3.4}$$

and its first derivative is

$$\rho'(t) = -AC\sin(Ct). \tag{3.5}$$

So, we can see at a glance that Eq. (3.4) has two critical points corresponding to $t = 0$ and $t = \frac{\pi}{C}$ on M which are repeated periodically. Hence, if ρ is a non-trivial solution of Eq. (3.3), then it can be written in the following form

$$\rho(t) = \frac{-1}{C}\cos(Ct), \qquad (A = \frac{-1}{C}). \tag{3.6}$$

Taking Eq. (2.4) into account, the metric form of M becomes

$$ds^2 = dt^2 + \big(\sin(Ct)\big)^2 \overline{ds}^2, \tag{3.7}$$

where $\overline{ds}^2$ is the metric form of a t-level of ρ given by $\overline{ds}^2 = f_{\gamma\beta}du^\gamma du^\beta$. This is the polar form of a Finsler metric on a standard sphere of radius $\frac{1}{C}$, see [31].

4. **Finsler Manifolds of Positive Constant Flag Curvature**

Let (x, y) be the line element of TM and $P(y, X) \subset T_xM$ a 2-plane generated by the vectors y and X in T_xM. Then the *flag curvature* $K(x, y, X)$ with respect to the plane $P(y, X)$ at a point $x \in M$ is defined by

$$K(x, y, X) := \frac{g(R(X, y)\, y, X)}{g(X, X)g(y, y) - g(X, y)^2},$$

where $R(X, y)\, y$ is the h-curvature tensor of Cartan connection. If K is independent of X, then (M, g) is called *space of scalar curvature*. If K has no dependence on x or y, then the Finsler manifold is said to be of *constant (flag) curvature*, see for instance [2]. It can be easily verified that the components of the h-curvature tensor of Cartan connection in the adapted coordinate system are given by

$$\begin{aligned} R^{\alpha}{}_{1\gamma1} &= -R^{\alpha}{}_{\gamma11} = (\tfrac{\rho'''}{\rho'})\delta^{\alpha}_{\gamma}, \\ R^{1}{}_{1\gamma\beta} &= -R^{1}{}_{\gamma1\beta} = -\rho'\rho''' f_{\gamma\beta}, \qquad (4.1) \\ R^{\alpha}{}_{\delta\gamma\beta} &= \overline{R}^{\alpha}{}_{\delta\gamma\beta} - (\rho'')^2(f_{\gamma\beta}\delta^{\alpha}_{\delta} - f_{\delta\beta}\delta^{\alpha}_{\gamma}), \end{aligned}$$

where $\overline{R}^{\alpha}{}_{\delta\gamma\beta}$ are components of h-curvature tensor related to the metric form $\overline{ds}^2$ on a t-level of ρ, see [4] for more details.

Proposition 4.1. *The n-dimensional complete Finsler manifold (M, g) is of constant flag curvature $K = C^2 > 0$, if and only if, there is a non-trivial solution of $\nabla^H\nabla^H\rho = (-C^2\rho + B)g$ on M.*

Proof. A Finsler manifold (M, g) is of constant flag curvature K if and only if the components of the h-curvature tensor are given by the following, see [4] for more details.

$$R^{i}{}_{hjk} = K(\delta^i_h g_{jk} - \delta^i_j g_{hk}). \qquad (4.2)$$

Using Eq. (4.2), we can easily drive the differential equation

$$\ddot{A} + KAg = 0, \qquad (4.3)$$

where A is the Cartan torsion tensor,

$$\dot{A}_{ijk} := (\nabla^H_s A_{ijk})y^s, \quad \ddot{A}_{ijk} := (\nabla^H_s\nabla^H_t A_{ijk})y^s y^t.$$

See Section 1.4 of [5] for more details.

Assume that $X, Y, Z \in \pi^*TM$ are fixed at $v \in I_xM = \{w \in T_xM,\ g(w, w) = 1\}$. Let $c : I\!R \to M$ be the unit-speed geodesic on (M, g) with $\frac{dc}{dt}(0) = v$ and

$\hat{c} := \frac{dc}{dt}$ be the canonical lift of c to TM_0. Let $X(t)$, $Y(t)$ and $Z(t)$ denote the parallel sections along $\hat{c}$ with $X(0) = X$, $Y(0) = Y$ and $Z(0) = Z$. Put

$$A(t) = A(X(t), Y(t), Z(t)), \quad \dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$$

and

$$\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t)).$$

Indeed along geodesics, we have

$$\frac{dA}{dt} = \dot{A}, \qquad \frac{d\dot{A}}{dt} = \ddot{A}$$

Eq. (4.3) becomes

$$\frac{d^2A(t)}{dt^2} + KA(t) = 0. \tag{4.4}$$

The general solution of this differential equation is

$$A(t) = A_0 \cos\sqrt{K}t + B_0 \sin\sqrt{K}t.$$

Therefore, Eq. (3.3) which represents a special case of Eq. (3.1) along geodesics, has a non-trivial solution on M.

Conversely, let ρ given by Eq. (3.6) be a solution of Eq. (3.1) on M. Then, there is an adapted coordinate system on M for which the components of h-curvature are given by (4.1). Hence, first and second equations of (4.1) satisfy

$$R^i{}_{hjk} = \frac{-\rho'''}{\rho'}(\delta^i_h g_{jk} - \delta^i_j g_{hk}). \tag{4.5}$$

Differentiate (3.6) with respect to t and replace the first and third derivatives of ρ, we obtain $\frac{-\rho'''}{\rho'} = C^2$. Therefore, the first two equations of (4.1) satisfy Eq. (4.2).

For the third equation of (4.1), we recall that as we see in Section 3, ρ has critical points on M. Thus, from Lemma 2.1, the t-levels of ρ are spaces of positive constant curvature $\rho''^2(0) = C^2$. Therefore, the third equation of (4.1) becomes

$$R^\alpha{}_{\delta\gamma\beta} = (C^2 - \rho''^2)(f_{\gamma\beta}\delta^\alpha_\delta - f_{\delta\beta}\delta^\alpha_\gamma).$$

By substituting $g_{\alpha\beta} = \rho'^2 f_{\alpha\beta}$ and the first and second derivatives of ρ in the above equation, we obtain

$$R^\alpha{}_{\delta\gamma\beta} = C^2(g_{\gamma\beta}\delta^\alpha_\delta - g_{\delta\beta}\delta^\alpha_\gamma).$$

So, all three components of Cartan h-curvature tensor satisfy Eq. (4.2) and the Finsler manifold (M, g) is of constant flag curvature $K = C^2$. □

Now, we are in a position to prove an extension of *Obata's* theorem to Finsler manifolds.

Theorem 4.2. *Let (M, g) be a complete simply connected Finsler manifold of dimension $n \geq 2$. Then, (M, g) is isometric to an n-sphere of radius $\dfrac{1}{C}$ if and only if there is a non-trivial solution of the following equation on M:*

$$\nabla^H \nabla^H \rho + C^2 \rho g = 0. \tag{4.6}$$

Proof. Let (M, g) be a Finsler manifold which admits a non-trivial solution of Eq. (4.6). According to Proposition 4.1, (M, g) is of positive constant flag curvature C^2. So, as we see in Section 3, the metric form of (M, g) is given by (3.7) and so (M, g) is isometric to an n-sphere of radius $\dfrac{1}{C}$.

Conversely, if (M, g) is isometric to an n-sphere of radius $\frac{1}{C}$, then the metric form of M is given by $ds^2 = (dt)^2 + \sin^2(Ct)\overline{ds}^2$, where $\overline{ds}^2$ is the metric form of a hypersurface of M. This is the polar form of a Finsler metric on an n-sphere in $I\!R^{n+1}$ with the positive constant curvature C^2, see [31]. Now by substituting the derivative of $\rho(t) = -\dfrac{1}{C}\cos(Ct)$ in the metric form of M, we obtain $ds^2 = (dt)^2 + \rho'^2(t)\overline{ds}^2$. Hence, $\rho(t)$ is a non-trivial solution of the second order differential equation (3.3) or equivalently a non-trivial solution of Eq. (4.6) along geodesics. $\square$

Note that in Riemannian geometry, a unit sphere is characterized by the existence of a solution of the differential equation $\nabla\nabla f + fg = 0$, where f is a function on Riemannian manifold (M, g) and ∇ is the Levi-Civita connection associated to the Riemannian metric g [20]. In Finsler geometry, Cartan connection works similar to the Levi-Civita connection in Riemannian geometry. Therefore, for extending Obata's theorem to Finsler geometry, in Eq. (2.1) and Theorem 4.2, we applied the Cartan connection. Also, it is worth noting that since the horizontal part of the covariant derivative in the Cartan connection coincides with the covariant derivative in the Chern connection [5], the Chern connection can also be used in Eq. (2.1). However, the Chern connection will not necessarily provide the same results as presented here.

Now, by considering the number of critical points of ρ, we have the following result.

Corollary 4.3. *Let (M, g) be a complete simply connected Finsler manifold with dimension $n \geq 2$. Then, (M, g) is isometrically homeomorphic to an n-sphere if and only if $\nabla^H \nabla^H \rho + C^2 \rho g = 0$ has a non-trivial solution.*

Proof. Let (M, g) admit a non-trivial solution of $\nabla^H \nabla^H \rho + C^2 \rho g = 0$, then from Theorem 4.2 we know that it is isometric to an n-sphere of radius $\dfrac{1}{C}$. On the other hand, since M is complete, Proposition 4.1 results in (M, g) is of positive constant curvature. Therefore, by applying the extension of *Meyers's theorem* to Finsler manifolds, see [1], we can conclude that M is compact.

Thus, the function ρ admits its absolute maximum and minimum values on M. Consequently, ρ has two critical points on M and an extension of *Milnor theorem* to Finsler geometry, [27], implies that (M, g) is homeomorphic to an n-sphere.

Conversely, let (M, g) be isometrically homeomorphic to an n-sphere of radius $\frac{1}{C}$. Then, Theorem 4.2 implies that $\nabla^H\nabla^H\rho + C^2\rho g = 0$ has a non-trivial solution on M. $\square$

Following Obata's theorem in Riemannian geometry, a unit sphere is characterized by the existence of a solution of the differential equation $\nabla\nabla f + fg = 0$, where f is a certain function on Riemannian manifold (M, g) and ∇ is the Levi-Civita connection associated to the Riemannian metric g [20]. Similarly, Theorem 4.2 implies that in Finsler geometry a unit sphere can be characterized by existence of a solution of $\nabla^H\nabla^H\rho + \rho g = 0$, where ρ is a certain function on Finsler manifold (M, g) and ∇^H is the Cartan horizontal covariant derivative. In analogy with Riemannian geometry, this leads to a definition for an n-sphere in Finsler geometry as follows.

Definition 4.4. *A Finslerian n-sphere is a complete simply connected Finsler manifold which admits a non-trivial solution of Eq.* (4.6).

Equivalently, a *Finslerian n-sphere* is isometrically homeomorphic to an n-sphere endowed with a certain Finsler metric.

Theorem 4.5. *Let (M, g) be an n-dimensional complete simply connected Finsler manifold. Then, (M, g) has a positive constant flag curvature $K = C^2$, if and only if, (M, g) is isometrically homeomorphic to an n-sphere of radius $\frac{1}{C}$ endowed with the Finsler metric $ds^2 = (dt)^2 + \sin^2(Ct)\overline{ds}^2$. Here, $\overline{ds}^2$ is the metric form of a t-level set of ρ where ρ is a solution of the system of equation*

$$\frac{d^2\rho}{dt^2} + K\rho = 0. \tag{4.7}$$

.

Proof. Let (M, g) be of positive constant flag curvature C^2. As a consequence of Proposition 4.1, there is a non-trivial solution ρ of Eq. (3.3) on M. Thus, by means of Corollary 4.3 it is isometrically homeomorphic to an n-sphere of radius $\frac{1}{C}$ equipped with the Finsler metric form $ds^2 = (dt)^2 + \sin^2(Ct)\overline{ds}^2$, where $\overline{ds}^2$ is the metric form of a t-level set of ρ. See Section 3 for more details.

Conversely, let (M, g) be a Finsler manifold which is isometrically homeomorphic to an n-sphere of radius $\frac{1}{C} > 0$. Then, Corollary 4.3 implies that M admits a non-trivial solution of Eq. (3.3). So, from Proposition 4.1, (M, g) is of positive constant flag curvature C^2. $\square$

5. **Finsler Transnormal Functions**

We will assume throughout this section that $\rho : M \to I\!R$ is a non-null Finsler transnormal function (see Section 2.3) on the complete Finsler manifold (M, g). First, we show that there exists an adapted coordinate system on any Finsler manifold that admits a transnormal function.

Lemma 5.1. *Let $\rho : M \to [a, b]$ be a Finsler transnormal function on a complete Finsler manifold (M, g) with $g(\mathtt{grad}\rho, \mathtt{grad}\rho) = \mathfrak{b}\circ\rho$ and no critical values in (a, b). Then, there exists an adapted coordinate system $(u^1 = t, u^2, ..., u^n)$ on M, where t is parameter of the reparameterization of integral curve of $\mathtt{grad}\rho$.*

Proof. From Corollary 2.5, one deduces that the reparameterization of integral curve of $\mathtt{grad}\rho$ is a geodesic of (M, g) and it is orthogonal to every $\rho^{-1}(c)$, for $c \in [a, b]$. Furthermore, from the same corollary, all of these geodesics start from $\rho^{-1}(a)$ and meet $\rho^{-1}(c)$ at the same time $\int_a^c \frac{ds}{\sqrt{\mathfrak{b}(s)}}$. Therefore, inspired by Section 1 of [4], one can consider an adapted coordinate system on the set of regular points of M, $M^o := \rho^{-1}(a, b)$. In other words, there exists a local coordinate system $(u^1 = t, u^2, ..., u^n)$ on M^o such that t is the parameter of the unit speed geodesic whose trajectory coincides with the integral curve of $\mathtt{grad}\rho$. In this coordinate system all points belonging to each level set of ρ map into the same value. So, the value of ρ has no dependency on u^i, $i = 2, ..., n$, and just depends on t. Therefore, for every $p \in M^o$, $\rho(p) = \rho(\gamma(t_1))$, where γ is the reparameterization of integral curve of $\mathtt{grad}\rho$ and t_1 is the time when γ passes through p. Since we can extend each horizontal geodesic to the singular level sets $\rho^{-1}(a)$ and $\rho^{-1}(b)$, while it preserves its properties, we confirm the existence of the local coordinate system on the whole manifold M. □

Proposition 5.2. *Let $\rho : M \to [a, b]$ be a Finsler transnormal function on a complete Finsler manifold (M, g) with $g(\mathtt{grad}\rho, \mathtt{grad}\rho) = \mathfrak{b}\circ\rho$ and no critical values in (a, b). Then, in adapted coordinate system:*

(a) *Level sets of ρ are defined by $t = constant$, where $t \in [0, \int_a^b \frac{ds}{\sqrt{\mathfrak{b}(s)}}]$,*

(b) *In $\rho^{-1}\{(a, b)\}$, ρ satisfies the following equation*

$$\frac{d^2\rho}{dt^2} = \frac{1}{2}\mathfrak{b}'(\rho), \tag{5.1}$$

(c) *The Finsler metric form of M is given by $ds^2 = (dt)^2 + \rho'^2 f_{ij} du^i du^j$, $i, j = 2, 3, \ldots, n$, where f_{ij} is a Finsler metric tensor on a regular level set of ρ and $\rho'^2 f_{ij}$ is the induced metric tensor on this level set.*

Proof. In an adapted coordinate system on M, all points belonging to any level set $\rho^{-1}(c)$, for $c \in [a, b]$, only depend on t. Moreover, from Corollary 2.5, all horizontal geodesics from $\rho^{-1}(a)$ to $\rho^{-1}(c)$ reach to $\rho^{-1}(c)$ at the same time $t = \int_a^c \frac{ds}{\sqrt{\mathfrak{b}(s)}}$. So we have the proof of (a) at hand.

To prove (b), note that in an adapted coordinate system on M, γ is a unitary geodesic whose velocity vector coincides with the positive multiplication of $\mathtt{grad}\rho$, i.e. $\gamma' = \frac{\mathbf{grad}\rho}{\sqrt{(\mathbf{grad}\rho, \mathbf{grad}\rho)}}$. So,

$$\frac{d^2\rho(\gamma)}{dt^2} = \frac{d}{dt} d\rho\gamma' = \frac{d}{dt} d\rho \frac{\mathbf{grad}\rho}{\sqrt{g(\mathbf{grad}\rho, \mathbf{grad}\rho)}} = \frac{d}{dt}\sqrt{\mathfrak{b}(\rho)} = \frac{\mathfrak{b}'(\rho) d\rho\gamma'}{2\sqrt{\mathfrak{b}(\rho)}} = \frac{\mathfrak{b}'(\rho)}{2},$$

where the third equality comes from Eqs. (2.5) and (2.6).

To prove item (c), once we put the adapted coordinate system on M, the metric g can be written as Eq. (2.3) and the Finsler metric form is given by Eq. (2.4). Therefore, nothing else is left to be proved. $\square$

As a consequence of Lemma 5.1 and Proposition 5.2, one can say that given a complete Finsler manifold (M, g) and a transnormal function $\rho : M \to [a, b]$ on it, ρ is a solution of Eq. (2.2), at least in the regular part. Therefore, by using results in [3] and [4] one can establish several interesting results for Finsler transnormal functions.

Lemma 5.3. *Let (M, g) be a complete Finsler manifold and $\rho : M \to [a, b]$ a transnormal function on it. If ρ has only one critical point $o := \rho^{-1}(a)$, then each regular level set $\rho^{-1}(c)$ for $c \in (a, b)$ is a hypersphere of radius $r_c = \int_a^c \frac{ds}{\sqrt{\mathfrak{b}(s)}}$ with center o and constant sectional curvature $\frac{1}{2}\mathfrak{b}'(a)$. Also, the Finsler metric form of this level set is $\overline{ds}^2 = f_{ij} du^i du^j$, $i, j = 2, 3, \ldots, n$, where f_{ij} is given by Eq. (2.4).*

Proof. From Corollary 2.5, given any regular value $c \in (a, b)$, the horizontal geodesics (extensions of integral curves of $\mathtt{grad}\rho$) are the geodesics that minimize the distance from o to $\rho^{-1}(c)$. In fact, these geodesics start from o and reach orthogonally to $\rho^{-1}(c)$ at the same time. Therefore, all the points belonging to $\rho^{-1}(c)$ have the same distance $r_c = \int_a^c \frac{ds}{\sqrt{\mathfrak{b}(s)}}$ from o. That means $\rho^{-1}(c) = \{q \in M : \ d(o, q) = r_c\} = S_{r_c}(o)$ in which $S_{r_c}(o)$ is the Finsler sphere of radius r_c and center o. Also, according to Lemma 5.1, we can consider an adapted coordinate system on M. Hence, ρ satisfies Eq. (2.2) which is equivalent to Eq. (2.1) in an adapted coordinate system. Therefore, from Lemma 2.1, each $\rho^{-1}(c)$ with metric $\overline{ds^2} = f_{ij} du^i du^j$ is of positive constant sectional curvature $\rho''(0)$. Finally, from item (b) of Proposition 5.2, $\rho''(0) = \frac{1}{2}\mathfrak{b}'(a)$.

$\square$

Now, as special case of Proposition 2.2, we have the following classification result on the Finsler manifolds which admit a transnormal function.

Proposition 5.4. *Let (M,g) be a connected complete Finsler manifold of dimension $n \geq 2$ and $\rho : M \to I\!R$ a transnormal function on it. Then,*

(a) *If ρ has no critical points, M is conformal to a direct product $J \times M$ of an open interval J of the real line and an $(n-1)$-dimensional complete Finsler manifold $\overline{M}$.*
(b) *If ρ has one critical point, M is conformal to an n-dimensional Euclidean space.*
(c) *If ρ has two critical points, M is conformal to an n-dimensional unit sphere in an Euclidean space.*

For compact Finsler manifolds which admit a transnormal function we have the following theorem.

Theorem 5.5. *Let (M,g) be a simply connected and compact Finsler manifold of dimension $n > 2$ and $\rho : M \to [a,b]$ a transnormal function on it such that ρ has no critical values in (a,b). Then M is homeomorphic to an n-sphere.*

Proof. According to Lemma 5.1, there is an adapted coordinate system on M such that ρ satisfies Eq. (2.2) with $\phi = \frac{\mathfrak{b}'}{2}$. So, $\rho(t)$ has at most two critical points, see Section 3 of [4] for more details. Also, since M is compact, $\rho(t)$ takes its maximum and minimum on M. Consequently, ρ has exactly two critical points that might be repeated periodically. These critical points are corresponding to a and b. According to the fact that Eq. (2.1) is equivalent to Eq. (2.2) in the adapted coordinate system, the transnormal function ρ is a solution of Eq. (2.1) and the rest of proof is a direct result of Lemma 2.3. □

Proposition 5.6. *Let (M,g) be a complete simply connected Finsler manifold with dimension $n \geq 2$ and $\rho : M \to [a,b]$ a transnormal function with $g(\mathtt{grad}\rho, \mathtt{grad}\rho) = -C^2\rho^2 + d$, where C and d are constant positive numbers. Then, (M,g) is isometrically homeomorphic to an n-sphere of radius $\frac{1}{C}$.*

Proof. From Proposition 5.2, there exists an adapted coordinate system in which

$$\frac{d^2\rho}{dt^2} = \frac{1}{2}\mathfrak{b}'(\rho) = -C^2\rho. \tag{5.2}$$

So, the proof is a direct result of Corollary 4.3, where the equation $\nabla^H\nabla^H\rho + C^2\rho g = 0$ reduces to $\frac{d^2\rho}{dt^2} + C^2\rho = 0$ in the adapted coordinate system. □

5.1. **Example.** To see a simple example illustrating some results of Finsler transnormal functions, consider a calm pond of water where we throw some small piece of stone into it at some time slot. Assume a two-dimensional Euclidean coordinate system on the surface of the pond where the origin is the point where the stone entered the water. The only force perturbing the water surface is the wind $W(x,y) = \frac{1}{3}(y,-x)$ blowing across the pond. We want to find the equation of water waves at each time and also the path equation of water particles (molecules). First, we present the mathematical model of the problem. Assume the open disk $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < T\}$, where T is big enough such that D covers the pond. The associated metric to this problem is a special case of the Finsler metric which is called *Randers metric* and is given by

$$F(y) = \sqrt{\frac{h^2(y, W) + \lambda h(y, y)}{\lambda^2}} - \frac{h(y, W)}{\lambda},$$

where h is the canonical Euclidean metric and $\lambda = 1 - h(W, W)$, see [29]. Now we consider the function $\rho : D \longrightarrow \mathbb{R}$ defined by $\rho(x, y) = x^2 + y^2$. This is not difficult to show that $g(\mathtt{grad}\rho, \mathtt{grad}\rho) = 2\rho$, where g is the metric with components $(g_{ij}) = \left(\frac{1}{2}\left[\frac{\partial^2}{\partial y^i \partial y^j} F^2\right]\right)$. Hence ρ is a transnormal function. Also, it is easy to show that, for some $\tau < T$, $\rho^{-1}(\tau)$ coincides with the location of some water wave and therefore the locations of wave are given by pre-images of ρ, see Section 4.2 of [15] for the details. From Lemma 5.3, each regular level set of ρ, that is the location of the water wave at each time $t = x^2 + y^2$, is a circle of radius $r_t = \int_0^t \frac{ds}{\sqrt{2s}} = \sqrt{2t}$ with center 0.

Figure 1 illustrates the geodesic $\gamma(t) = \frac{\sqrt{2}t}{2}(\cos\frac{t}{3} - \sin\frac{t}{3}, \sin\frac{t}{3} + \cos\frac{t}{3})$ which is the track of a molecule of water from time 0 to time T; and also the path of an integral curve of $\mathtt{grad}\rho$. The figure also shows some t-levels of ρ, that is the location of waves at different time slots.

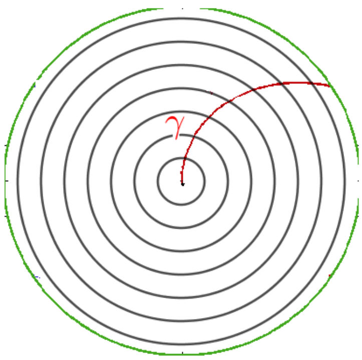

Figure 1. The waves of water and the path of a molecule of water.

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