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S-Curvature of left invariant Randers metrics on some simple Lie groups

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Abstract. In this paper we study the Riemannian geometry of simple Lie groups $SO(3,\mathbb{R})$, $SL(2,\mathbb{R})$ and $SO(1,3)$, equipped with a left invariant Riemannian metric. We consider left invariant Randers metrics induced by these left invariant Riemannian metrics. Then, in each case, we obtain the S-curvature and show that although these Randers metrics are not of Berwald or Douglas type but in the case of $SO(3,\mathbb{R})$ it is of almost isotropic S-curvature. Finally, we give the S-curvature of left invariant Randers metrics on four-dimensional Einstein Lie groups.

Keywords: S-curvature, isotropic S-curvature, Lie groups, Randers space.

1. Introduction

Among Finsler metrics, (α,β)-metrics are very interesting, since they are computable and have many applications. An (α,β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For example the Randers metric, which is a special type of (α,β)-metrics, has been introduced because of its applications in physics (see [2]). An important special class of Finsler metrics is the family of left invariant Finsler metrics on Lie groups. On such spaces,

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the algebraic structures of the space help the study of geometric concepts, such as connection and curvature.

In the recent years studying invariant Finsler metrics on Lie groups and homogeneous spaces has a very fast development (for more details see [6]). A way for constructing left invariant Finsler metrics on a certain Lie group G is the use of a left invariant Riemannian metric and a left invariant vector field. In this way we can define a left invariant (α,β)-metric on G.

In this paper, we are going to consider the Riemannian geometry of simple Lie groups $SO(3,\mathbb{R})$, $SL(2,\mathbb{R})$ and $SO(1,3)$, equipped with a left invariant Riemannian metric. Thus we study left invariant Randers metrics induced by these left invariant Riemannian metrics. In each case, we get the S-curvature and show that although these Randers metrics are not of Berwald or Douglas type but in the case of $SO(3,\mathbb{R})$ it is of almost isotropic S-curvature. Then, we give the S-curvature formula of left invariant Randers metrics on 4-dimensional Einstein Lie groups.

2. Preliminaries

Here we give some preliminaries of Finsler geometry which we will need during the paper.

A Finsler manifold (see [3]) is a pair (M,F) consisting of a smooth manifold M and a function $F : TM \longrightarrow [0,\infty)$ such that

i) F is a differentiable function on $TM\backslash\{0\}$,

ii) $F(x,\lambda y)=\lambda F(x,y)$, for any $x\in M, y\in T_xM$ and $\lambda>0$,

iii) The matrix

$$(g_{ij})=\left(\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j}\right),$$

which is named the hessian matrix, is positive definite for all $(x,y)\in TM\backslash\{0\}$.

A special family of Finsler metrics is the family of (α,β)-metrics. Let a be a Riemannian metric and β be a 1-form on a manifold M. An (α,β)-metric on the manifold M is a Finsler metric of the form $F=\alpha\phi(\frac{\beta}{\alpha})$, where $\alpha(x,y)=\sqrt{\mathsf{a}(y,y)}$ and $\phi:(-b_0,b_0)\longrightarrow\mathbb{R}^+$ is a C^∞ function. F is a Finsler metric if

$$\phi(s)-s\phi^{'}(s)+(b^2-s^2)\phi^{''}(s)>0, \qquad |s|\leq b<b_0, \tag{2.1}$$

and $\|\beta\|_\alpha<b_0$ (see [4] and [7]).

An important type of (α,β)-metrics appears when $\phi(s)=1+s$. In this case F is called a Randers metric, then we have $F=\alpha+\beta$. It can be shown that a Randers metric is a Finsler metric if and only if $\|\beta\|_\alpha<1$. Another important (α,β)-metric happens when $\phi(s)=1/s$. In this case F is of the form $F=\alpha^2/\beta$.

Let G be a Lie group and F be a Finsler metric on G. The Finsler metric F is called a left invariant Finsler metric on G, if for any $x \in G$ and $y \in T_xG$ we have

$$F(x,y) = F(e, dl_{\,x^{-1}}y) \ \text{ for every } x \in G, y \in T_xG, \tag{2.2}$$

where e is the unit element of G and l_x denotes the left translation.

In the case of left invariant (α, β)-metrics, it is very useful to rewrite the 1-form β as follows

$$\mathsf{a}\big(y, X(x)\big) = \beta(x,y), \tag{2.3}$$

for a left invariant vector field X on G such that

$$\|X\|_\alpha = \mathsf{a}(X,X) < b_0.$$

For more details, see [5].

So easily we can see, for a left invariant Riemannian metric a and a left invariant vector field X on a Lie group G with $\mathsf{a}(X,X) < 1$, the Randers metric

$$F(x,y) = \sqrt{\mathsf{a}\,(y,y)} + \mathsf{a}\,(X\,(x)\,,y)\,, \tag{2.4}$$

is a left invariant Finsler metric on G.

3. The Riemannian geometry of Simple Lie groups $SO(3,\mathbb{R})$, $SL(2,\mathbb{R})$ and $SO(1,3)$

In this section, we first introduce the Lie brackets given in [8]. Then, we calculate the sectional curvature of the left invariant Riemannian metrics corresponded to the inner products on the Lie algebras, such that the basis $\{E_1, \cdots, E_n\}$ is orthonormal.

If $\{E_1, E_2, E_3\}$ is an orthonormal base for the Lie algebras $so(3,\mathbb{R})$ and $sl(2,\mathbb{R})$, and if $\{E_1, \cdots, E_6\}$ is an orthonormal base for the Lie algebra $so(1,3)$, then the Lie bracket of each case is as follows:

In case so(3,$\mathbb{R}$):

$$[E_1, E_2] = E_3,$$
$$[E_1, E_3] = -E_2,$$
$$[E_2, E_3] = E_1.$$

In case sl(2,$\mathbb{R}$):

$$[E_1, E_2] = 2E_1,$$
$$[E_1, E_3] = -E_2,$$
$$[E_2, E_3] = 2E_3.$$

In case so(1,3):

$$\begin{aligned}
[E_2, E_3] &= [E_6, E_5] = E_1, \\
[E_3, E_1] &= [E_4, E_6] = E_2, \\
[E_1, E_2] &= [E_5, E_4] = E_3, \\
[E_2, E_6] &= [E_5, E_3] = E_4, \\
[E_3, E_4] &= [E_6, E_1] = E_5, \\
[E_1, E_5] &= [E_4, E_2] = E_6.
\end{aligned}$$

Theorem 3.1. *Suppose that $\langle , \rangle$ denotes the inner product induced by the above left invariant Riemannian metric on the Lie algebras. If we put $R_{ijk} := R(E_i, E_j)E_k$ then in each case we have:*

(i) On three-dimensional Lie algebra $so(3, \mathbb{R})$:

$$R_{122} = R_{133} = \frac{E_1}{4}, \qquad R_{211} = R_{233} = \frac{E_2}{4}, \qquad R_{311} = R_{322} = \frac{E_3}{4}.$$

(ii) On three-dimensional Lie algebra $sl(2, \mathbb{R})$:

$$\begin{aligned}
&\frac{4}{13}R_{133} = E_1, \qquad \frac{4}{13}R_{311} = E_3, \\
&R_{122} = \frac{-15}{4}E_1 - 2E_3, \qquad R_{232} = 2E_1 + \frac{15}{4}E_3, \\
&\frac{1}{2}R_{123} = \frac{1}{2}R_{321} = \frac{4}{15}R_{121} = \frac{4}{15}R_{323} = E_2.
\end{aligned}$$

(iii) On six-dimensional Lie algebra $so(1, 3)$:

$$\begin{aligned}
E_1 &= 4R_{122} = 4R_{133} = 4R_{155} = 4R_{166} = 4R_{245} \\
&= 4R_{346} = 2R_{524} = 2R_{634} = \frac{4}{3}R_{542} = \frac{4}{3}R_{643} \\
E_2 &= 4R_{211} = 4R_{154} = 4R_{233} = 4R_{244} = 4R_{266} \\
&= 4R_{356} = 2R_{415} = 2R_{635} = \frac{4}{3}R_{451} = \frac{4}{3}R_{653} \\
E_3 &= 4R_{311} = 4R_{164} = 4R_{322} = 4R_{265} = 4R_{344} \\
&= 4R_{355} = 2R_{416} = 2R_{526} = \frac{4}{3}R_{461} = \frac{4}{3}R_{562} \\
E_4 &= 4R_{512} = 4R_{613} = 4R_{422} = 4R_{433} = 2R_{251} \\
&= 2R_{361} = \frac{4}{7}R_{545} = \frac{4}{7}R_{646} = \frac{4}{3}R_{215} = \frac{4}{3}R_{316}
\end{aligned}$$

$$\begin{aligned}
E_5 &= 4R_{511} = 4R_{421} = 4R_{623} = 4R_{533} = 2R_{142}\\
&= 2R_{362} = \frac{4}{7}R_{454} = \frac{4}{7}R_{656} = \frac{4}{3}R_{124} = \frac{4}{3}R_{326}\\
E_6 &= 4R_{611} = 4R_{622} = 4R_{431} = 4R_{532} = 2R_{143}\\
&= 2R_{253} = \frac{4}{7}R_{464} = \frac{4}{7}R_{565} = \frac{4}{3}R_{134} = \frac{4}{3}R_{235}
\end{aligned}$$

Proof. The Riemannian metric $\langle , \rangle$ is left-invariant, so for the Levi-Civita connection we have,

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \langle \nabla_{E_i} E_j, E_k \rangle E_k,$$

and

$$\Big\langle \nabla_{E_i} E_j, E_k \Big\rangle = \frac{1}{2}\Bigg\{ -\Big\langle E_i, [E_j, E_k] \Big\rangle + \Big\langle E_j, [E_k, E_i] \Big\rangle + \Big\langle E_k, [E_i, E_j] \Big\rangle \Bigg\}.$$

We obtain the Levi-Civita connections of each case as follows:

(a) For $so(3, \mathbb{R})$:

$$\begin{aligned}
\nabla_{E_1} E_2 = -\nabla_{E_2} E_1 &= \frac{E_3}{2},\\
-\nabla_{E_1} E_3 = \nabla_{E_3} E_1 &= \frac{E_2}{2}\\
\nabla_{E_2} E_3 = -\nabla_{E_3} E_2 &= \frac{E_1}{2}.
\end{aligned}$$

(b) For $sl(2, \mathbb{R})$:

$$\begin{aligned}
&-2\nabla_{E_2} E_3 = E_1,\\
&2\nabla_{E_1} E_2 = 4E_1 + E_3,\\
&-2\nabla_{E_3} E_2 = 4E_3 + E_1,\\
&2\nabla_{E_2} E_1 = E_3,\\
&-\frac{1}{2}\nabla_{E_1} E_1 = \frac{1}{2}\nabla_{E_3} E_3 = -2\nabla_{E_1} E_3 = 2\nabla_{E_3} E_1 = E_2.
\end{aligned}$$

(c) For $so(1, 3)$:

$$\begin{aligned}
&\nabla_{E_2} E_3 = -\nabla_{E_3} E_2 = -\nabla_{E_5} E_6 = \nabla_{E_6} E_5 = \frac{E_1}{2},\\
&\nabla_{E_3} E_1 = -\nabla_{E_1} E_3 = -\nabla_{E_6} E_4 = \nabla_{E_4} E_6 = \frac{E_2}{2},\\
&\nabla_{E_1} E_2 = -\nabla_{E_2} E_1 = -\nabla_{E_4} E_5 = \nabla_{E_5} E_4 = \frac{E_3}{2},\\
&\frac{\nabla_{E_2} E_6}{3} = \frac{-\nabla_{E_3} E_5}{3} = -\nabla_{E_5} E_3 = \nabla_{E_6} E_2 = \frac{E_4}{2},
\end{aligned}$$

$$\frac{\nabla_{E_3}E_4}{3} = \frac{-\nabla_{E_1}E_6}{3} = -\nabla_{E_6}E_1 = \nabla_{E_4}E_3 = \frac{E_5}{2},$$
$$\frac{\nabla_{E_1}E_5}{3} = \frac{-\nabla_{E_2}E_4}{3} = -\nabla_{E_4}E_2 = \nabla_{E_5}E_1 = \frac{E_6}{2}.$$

Let us consider the following formula

$$R_{ijk} := R(E_i, E_j)E_k = \nabla_{E_i}\nabla_{E_j}E_k - \nabla_{E_j}\nabla_{E_i}E_k - \nabla_{[E_i,E_j]}E_k.$$

This completes the proof. □

We mention that in the case of $so(3,\mathbb{R})$ we have considered the bi-invariant metric so we have:

$$\nabla_{E_i}E_j = \frac{1}{2}[E_i, E_j],$$
$$R(E_i, E_j)E_k = -\frac{1}{4}\Big[[E_i, E_j], E_k\Big], \qquad \forall E_i, E_j, E_k \in \mathfrak{g}$$

Then we have the following.

Theorem 3.2. *Suppose that $(G, \langle,\rangle)$ denotes the Riemannian Lie group corresponded to the Lie algebras $so(3,\mathbb{R})$, $sl(2,\mathbb{R})$ and $so(1,3)$, where $\langle,\rangle$ is the above left-invariant Riemannian metric. If $\{U, V\}$ is a two-dimensional orthonormal subspace of the tangent space at the unit element of G, where $U = \sum_{i=1}^{n} a_iE_i$ and $V = \sum_{i=1}^{n} b_iE_i$, then the sectional curvature $K(U, V)$ in each case is as follows:*

For $so(3,\mathbb{R})$ we have

$$K(U, V) = \frac{1}{4}$$

For $sl(2,\mathbb{R})$:

$$\begin{aligned} K(U, V) = &-\frac{15}{4}(a_1b_2 - a_2b_1)^2 + \frac{13}{4}(a_1b_3 - a_3b_1)^2 - \frac{15}{4}(a_2b_3 - a_3b_2)^2 \\ &+ 4\Big(a_1a_2b_2b_3 - a_1a_3b_2^2 - a_2^2b_1b_3 + a_2a_3b_1b_2\Big) \end{aligned}$$

For $so(1,3)$ we get

$$\begin{aligned} K(U, V) = &\frac{1}{4}\Big\{(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_5 - a_5b_1)^2 + (a_1b_6 - a_6b_1)^2 \\ &+ (a_2b_3 - a_3b_2)^2 + (a_2b_4 - a_4b_2)^2 + (a_2b_6 - a_6b_2)^2 + (a_3b_4 - a_4b_3)^2 \\ &+ (a_3b_5 - a_5b_3)^2\Big\} - \frac{7}{4}\Big\{(a_4b_5 - a_5b_4)^2 + (a_4b_6 - a_6b_4)^2 + (a_5b_6 - a_6b_5)^2\Big\} \\ &+ \frac{1}{2}\Big\{ - a_4\Big(4a_1b_2b_5 + 4a_1b_3b_6 - 5a_2b_1b_5 - 5a_3b_1b_6 + a_5b_1b_2 + a_6b_1b_3\Big) \end{aligned}$$

$$- a_5\Big(- 5b_2(a_1b_4 + a_3b_6) + 4a_2(b_1b_4 + b_3b_6) + a_6b_2b_3\Big) - a_1a_2b_4b_5$$
$$- a_1a_3b_4b_6 + 5a_1a_6b_3b_4 - a_2a_3b_5b_6 + 5a_2a_6b_3b_5 - 4a_3a_6b_1b_4 - 4a_3a_6b_2b_5\Bigg\}.$$

Proof. According to Theorem 3.1 and the curvature formula, one can get the proof. □

4. S-curvature of Randers metrics of Simple Lie groups $SO(3,\mathbb{R})$, $SL(2,\mathbb{R})$ and $SO(1,3)$

In this section, we consider the Randers metrics defined by the left invariant Riemannian metrics of the previous section and left invariant vector fields. Note that because these Lie algebras are simple, these Randers metrics are neither of Berwald type nor of Douglas type. To obtain the desired result, first we recall the following theorem from [6].

Theorem 4.1. ([6]) *Let G be an n-dimensional connected Lie group with Lie algebra $\mathfrak{g}$. Let $\langle,\rangle$ be an inner product on $\mathfrak{g}$ and $u \in \mathfrak{g}$ with $\langle u,u\rangle < 1$. Then the left-invariant Randers metric F on G defined by $\langle,\rangle$ and u has S-curvature*

$$\mathbf{S}\,(e,y) = \frac{n+1}{2}\left\{\frac{\langle [u,y], \langle y,u\rangle\, u - y\rangle}{F(y)} - \langle [u,y],u\rangle\right\}. \tag{4.1}$$

Here F has almost isotropic S-curvature if and only if F has vanishing S-curvature, if and only if the linear endomorphism $ad(u)$ of $\mathfrak{g}$ is skew-symmetric with respect to the inner product $\langle,\rangle$.

By using [6], we are going to prove the following.

Theorem 4.2. *For the simple Lie algebras $so(3,\mathbb{R})$, $sl(2,\mathbb{R})$ and $so(1,3)$, for $y := \sum_{i=1}^{n} y_iE_i$, $u := \sum_{i=1}^{n} u_iE_i$ with $\langle u,u\rangle < 1$, the S-curvature of the induced left invariant Randers metric in each case is as follows:*

On three-dimensional Lie algebra $so(3,\mathbb{R})$:

$$\mathbf{S}(e,y) = 0.$$

On three-dimensional Lie algebra $sl(2,\mathbb{R})$:

$$\mathbf{S}(e,y) = \frac{2}{\alpha + u_1y_1 + u_2y_2 + u_3y_3}\Bigg\{y_2\Big(- 2u_1^2\alpha + u_1(y_3 - 2y_1) + 2u_3(u_3\alpha + y_3)$$
$$- u_3y_1\Big) + u_2\Big(y_1(2u_1 - u_3)\alpha + y_3\big((u_1 - 2u_3)\alpha - 2y_3\big) + 2y_1^2\Big)\Bigg\},$$

where $\alpha = \sqrt{y_1^2 + y_2^2 + y_3^2}$.

On six-dimensional Lie algebra $so(1,3)$*:*

$$\begin{aligned}\mathbf{S}(e,y) =7\Bigg\{ & \frac{1}{\alpha+\beta}\Big[\beta\Big(y_4(u_3u_5 - u_2u_6) + y_5(u_1u_6 - u_3u_4) + y_6(u_2u_4 - u_1u_5)\Big) \\ & - u_4y_2y_6 + u_4y_3y_5 + u_5y_1y_6 - u_5y_3y_4 - u_6y_1y_5 + u_6y_2y_4\Big] \\ & + u_1u_5y_6 - u_1u_6y_5 - u_2u_4y_6 + u_2u_6y_4 + u_3u_4y_5 - u_3u_5y_4\Bigg\}\end{aligned}$$

where

$$\alpha = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2}, \quad \textit{and} \quad \beta = u_1y_1 + \cdots + u_6y_6.$$

Proof. For $F = \alpha + \beta = \sqrt{\langle y, y\rangle} + \langle y, u\rangle$, according to the theorem 4.1 we have the following:

(i) For $so(3,\mathbb{R})$ we have

$$\begin{aligned}&\big\langle [u,y], u\big\rangle = 0,\\ &[u,y] = (u_2y_3 - u_3y_2)E_1 + (u_3y_1 - u_1y_3)E_2 + (u_1y_2 - u_2y_1)E_3,\end{aligned}$$

and

$$\begin{aligned}&\langle y,u\rangle\, u - y = ((u_1^2-1)y_1 + u_1(u_2y_2 + u_3y_3))E_1 + (u_1u_2y_1 + (u_2^2-1)y_2\\ &+ u_2u_3y_3)E_2 + (u_1u_3y_1 + u_2u_3y_2 + (u_3^2-1)y_3)E_3.\end{aligned}$$

So we get that

$$\Big\langle [u,y],\ \langle y,u\rangle\, u - y\Big\rangle = 0.$$

Therefore, if we put the above relations in (4.1), then the verdict for $n = 3$ is proved.

(ii) For $sl(2,\mathbb{R})$, we get

$$\begin{aligned}[u,y] &= (2u_1y_2 - 2u_2y_1)E_1 + (u_3y_1 - u_1y_3)E_2 + (2u_2y_3 - 2u_3y_2)E_3,\\ \langle [u,y], u\rangle &= 2u_1^2y_2 - u_1u_2\Big(2y_1 + y_3) + u_3(u_2(y_1 + 2y_3) - 2u_3y_2\Big),\end{aligned}$$

and

$$\begin{aligned}\langle y,u\rangle\, u - y = &\Big((u_1^2-1)y_1 + u1(u_2y_2 + u_3y_3)\Big)E_1 + \Big(u_1u_2y_1 + (u_2^2-1)y_2\\ &+ u_2u_3y_3)E_2 + (u_1u_3y_1 + u_2u_3y_2 + (u_3^2-1)y_3\Big)E_3.\end{aligned}$$

So, we get

$$\Big\langle [u,y],\ \langle y,u\rangle u-y\Big\rangle = 2u_1^3y_1y_2 - y_2\Big(2u_1y_1+u_3y_1-u_1y_3-2u_3y_3\Big)$$
$$+2u_2(y_1^2-y_3^2)+u_3(u_2y_2+u_3y_3)\Big(u_2(y_1+2y_3)-2u_3y_2\Big)+u_1^2\Big(2u_3y_2y_3$$
$$-u_2(2y_1^2-2y_2^2+y_1y_3)\Big)-u_1\Big(2u_3^2y_1y_2+u_2^2y_2(2y_1+y_3)+u_2u_3(-y_1^2+y_3^2)\Big).$$

Therefore, if we put the above relations in (4.1), then the verdict for $n=3$ is proved.

(iii) For $so(1,3)$, we obtain

$$\begin{aligned}[u,y] &= (u_2y_3-u_3y_2+u_6y_5-u_5y_6)E_1+(u_3y_1-u_1y_3-u_6y_4+u_4y_6)E_2\\ &\quad+(u_1y_2-u_2y_1+u_5y_4-u_4y_5)E_3+(u_5y_3-u_6y_2-u_3y_5+u_2y_6)E_4\\ &\quad+(u_6y_1-u_4y_3+u_3y_4-u_1y_6)E_5+(u_4y_2-u_5y_1-u_2y_4+u_1y_5)E_6,\end{aligned}$$
$$\langle [u,y],u\rangle = 2(u_3u_5y_4-u_2u_6y_4-u_3u_4y_5+u_1u_6y_5+u_2u_4y_6-u_1u_5y_6),$$

and

$$\begin{aligned}\langle y,u\rangle\, u-y = &\Big((u_1^2-1)y_1+u_1(u_2y_2+u_3y_3+u_4y_4+u_5y_5+u_6y_6)\Big)E_1\\ &+\Big(u_1u_2y_1+(u_2^2-1)y_2+u_2(u_3y_3+u_4y_4+u_5y_5+u_6y_6)\Big)E_2\\ &+(u_1u_3y_1+u_2u_3y_2-y_3+u_3^2y_3+u_3u_4y_4+u_3u_5y_5+u_3u_6y_6)E_3\\ &+(u_1u_4y_1+u_2u_4y_2+u_3u_4y_3-y_4+u_4^2y_4+u_4u_5y_5+u_4u_6y)E_4\\ &+(u_1u_5y_1+u_2u_5y_2+u_3u_5y_3+u_4u_5y_4-y_5+u_5^2y_5+u_5u_6y_6)E_5\\ &+(u_1u_6y_1+u_2u_6y_2+u_3u_6y_3+u_4u_6y_4+u_5u_6y_5-y_6+u_6^2y_6)E_6.\end{aligned}$$

So we get

$$\begin{aligned}\Big\langle [u,y],\langle y,u\rangle u-y\Big\rangle = 2\Big\{&u_6y_2y_4-u_5y_3y_4-u_6y_1y_5+u_4y_3y_5+u_5y_1y_6-u_4y_2y_6\\ &+(u_3u_5y_4-u_2u_6y_4-u_3u_4y_5+u_1u_6y_5+u_2u_4y_6-u_1u_5y_6)\\ &\times(u_1y_1+u_2y_2+u_3y_3+u_4y_4+u_5y_5+u_6y_6)\Big\}.\end{aligned}$$

Now the substitution of the above relations in (4.1) completes the proof. □

5. S-curvature of left invariant Randers metrics on four-dimensional Einstein Lie groups

In reference [1], it is shown that a four-dimensional Einstein Lie group admits a left invariant Randers metric of Berwald or Douglas type if and only if for the Lie brackets and the left invariant vector field $u := \sum\limits_{i=1}^{4} u_iE_i$ we have

Case 1: Berwaldian, $u = u_1 E_1 + u_2 E_2$

$$[E_1, E_3] = E_4,$$
$$[E_1, E_4] = -E_3.$$

Case 2: Non-Berwaldian metric of Douglas type, $u = u_1 E_1$

$$\begin{aligned}
&[E_1, E_2] = E_2 - tE_3, \\
&[E_1, E_3] = tE_2 + E_3, \\
&[E_1, E_4] = 2E_4, \\
&[E_2, E_3] = 2E_4, \qquad 0 \le t < \infty.
\end{aligned}$$

Case 3: Non-Berwaldian metric of Douglas type, $u = u_1 E_1$

$$\begin{aligned}
&[E_1, E_2] = E_2, \\
&[E_1, E_3] = E_3 - tE_4, \\
&[E_1, E_4] = tE_3 + E_4, \qquad 0 \le t < \infty.
\end{aligned}$$

Case 4: Non-Berwaldian metric of Douglas type, $u = u_1 E_1 + u_2 E_2$

$$[E_1, E_3] = E_3,$$
$$[E_2, E_4] = E_4.$$

Now, for $y := \sum_{i=1}^{4} y_i E_i$, we calculate the S-curvature of each case according to the formula (4.1):

Case 1:

$$\mathbf{S}(e, y) = 0,$$

Case 2:

$$\mathbf{S}(e, y) = \frac{-5}{2} \; \frac{u_1 \left(y_2^2 + y_3^2 + 2y_4^2\right)}{\sqrt{\sum_{i=1}^{4} {y_i}^2} + u_1 y_1},$$

Case 3:

$$\mathbf{S}(e, y) = \frac{-5}{2} \; \frac{u_1 \left(y_2^2 + y_3^2 + y_4^2\right)}{\sqrt{\sum_{i=1}^{4} {y_i}^2} + u_1 y_1},$$

Case 4:

$$\mathbf{S}(e, y) = \frac{-5}{2} \; \frac{\left(u_1 y_3^2 + u_2 y_4^2\right)}{\sqrt{\sum_{i=1}^{4} {y_i}^2} + u_1 y_1 + u_2 y_2}.$$

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