Journal of Finsler Geometry and its Applications
Vol. 1, No. 1 (2020), pp 73-83
DOI: 10.22098/jfga.2020.1013

# The class of Matsumoto metrics with almost vanishing H-curvatures

M. Faghfouri [a*] and N. Jazer[b]

[a,b]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.

E-mail: faghfouri@tabrizu.ac.ir
E-mail: njazer@tabrizu.ac.ir

Abstract. In this paper, we are going to consider a class of $(\alpha, \beta)$-metrics which introduced by Matsumoto. We find a condition under which a Matsumoto metric of almost vanishing **H**-curvature reduces to a Berwald metric.

**Keywords:** Matsumoto metric, Almost vanishing **H**-curvature, Berwald curvature.

## 1. Introduction

The $(\alpha, \beta)$-metrics form an important class of Finsler metrics appearing iteratively in formulating Physics, Mechanics and Seismology, Biology, Ecology, Control Theory, etc, see for instance [4, 9, 10, 11, 13]. This class of metrics was first introduced by Matsumoto [5]. According to definition, an $(\alpha, \beta)$-metric is a Finsler metric of the form $F := \alpha\phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a $C^\infty$ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold $M$.

The Matsumoto metric is special and significant $(\alpha, \beta)$-metric which constitutes a majority of actual research, which is defined by

$$F = \frac{\alpha^2}{\alpha - \beta}. \tag{1.1}$$

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C30

The Matsumoto metric is an important metric in Finsler geometry, which is the Matsumoto's slope-of-a-mountain metric. In the Matsumoto metric, the 1-form $\beta = b_i y^i$ was original to be induced by earth gravity. Hence, we could regard $b_i$ as the infinitesimals. This metric was introduced by Matsumoto as a realization of Finsler's idea "a slope measure of a mountain with respect to a time measure" (see [6, 14]).

In [2], Bao-Robles-Shen proved that the time-optimal solutions of the well-known Zermelo navigation-moving that is the motion of a vehicle equipped with an engine with a fixed output power in presence of a wind current-are actually the geodesics of a Randers metric. One may find in [3], an analog work for pure pursuit navigation formed on Matsumoto metric. Notice that, Matsumoto gave an exact formulation of a Finsler surface to measure the time on the slope of a hill and introduced the Matsumoto metrics in [6].

In [1], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. Akbar-Zadeh proved that for a Finsler metric of scalar flag curvature, the flag curvature is a scalar function on the manifold if and only if $\mathbf{H} = \mathbf{0}$. One can remark that the mean Berwald curvature $\mathbf{E}$ obtained by taking a trace from the Berwald curvature $\mathbf{B}$. Then the quantity $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics, where $H_{ij} := E_{ij|m} y^m$. A Finsler metric $F$ is called of almost vanishing $\mathbf{H}$-curvature if

$$\mathbf{H} = \frac{n+1}{2} F^{-1} \theta \, \mathbf{h}, \tag{1.2}$$

where $\mathbf{h}_y(u,v) = \mathbf{g}_y(u,v) - F^{-2}(y) \mathbf{g}_y(y,u) \mathbf{g}_y(y,v)$ is the angular metric and $\theta = \theta_i(x) y^i$ is a 1-form on a manifold $M$.

**Theorem 1.** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha,\beta)$-metric on an $n$-dimensional manifold $M$ defined by (1.1). Suppose that $\beta$ satisfies*

$$r_{ij} = c(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0, \tag{1.3}$$

*where $c = c(x)$ is a scalar function on $M$. Suppose that $F$ has almost vanishing $\mathbf{H}$-curvature. Then $F$ reduces to a Berwald metric.*

## 2. Preliminary

Given a Finsler manifold $(M, F)$, then a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on $TM_0$ satisfying $G^i(x,\lambda y)=\lambda^2 G^i(x,y)$, $\lambda>0$, and given by

$$G^i=\frac{1}{4}g^{il}\Big[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k-\frac{\partial F^2}{\partial x^l}\Big].$$

$\mathbf{G}$ is called the associated spray to $(M,F)$. The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic in $M$. $F$ is called a Berwald metric if $G^i$ are quadratic in $y\in T_xM$ for any $x\in M$ [12, 13, 15].

For $y\in T_xM_0$, define $\mathbf{B}_y:T_xM\otimes T_xM\otimes T_xM\to T_xM$ by $\mathbf{B}_y(u,v,w):=B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where $B^i{}_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}$. $\mathbf{B}$ is called Berwald curvature. Then $F$ is called a Berwald metric if $\mathbf{B}=0$.

Define the mean of Berwald curvature by $\mathbf{E}_y:T_xM\otimes T_xM\to\mathbb{R}$, where

$$\mathbf{E}_y(u,v):=\frac{1}{2}\sum_{i=1}^n g^{ij}(y)g_y\Big(\mathbf{B}_y(u,v,e_i),e_j\Big). \tag{2.1}$$

The family $\mathbf{E}=\{\mathbf{E}_y\}_{y\in TM\setminus\{0\}}$ is called the *mean Berwald curvature* or *E-curvature*. In local coordinates, $\mathbf{E}_y(u,v):=E_{ij}(y)u^iv^j$, where

$$E_{ij}:=\frac{1}{2}B^m{}_{mij}.$$

By definition, $\mathbf{E}_y(u,v)$ is symmetric in $u$ and $v$ and we have $\mathbf{E}_y(y,v)=0$. $\mathbf{E}$ is called the mean Berwald curvature. $F$ is called a weakly Berwald metric if $\mathbf{E}=\mathbf{0}$.

The quantity $\mathbf{H}_y=H_{ij}dx^i\otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics. More precisely, $H_{ij}:=E_{ij|m}y^m$. In local coordinates,

$$H_{ij}=\frac{1}{2}\Big[y^m\frac{\partial^4G^k}{\partial y^i\partial y^j\partial y^k\partial x^m}-2G^m\frac{\partial^4G^k}{\partial y^i\partial y^j\partial y^k\partial y^m}-\frac{\partial G^m}{\partial y^i}\frac{\partial^3G^k}{\partial y^j\partial y^k\partial y^m}-\frac{\partial G^m}{\partial y^j}\frac{\partial^4G^k}{\partial y^i\partial y^k\partial y^m}\Big]. \tag{2.2}$$

A Finsler metric $F$ is called of almost vanishing H-curvature if $\mathbf{H}=(n+1)/(2F)\theta\mathbf{h}$, where $\theta:=\theta_i(x)y^i$ is a 1-form on $M$ and $\mathbf{h}=h_{ij}dx^i\otimes dx^j$ is the angular metric.

An $(\alpha,\beta)$-metric is a Finsler metric of the form $F:=\alpha\phi(\frac{\beta}{\alpha})$, where $\phi=\phi(s)$ is a $C^\infty$ on $(-b_0,b_0)$ with certain regularity, $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on $M$ (see [8]). For an $(\alpha,\beta)$-metric $F:=\alpha\phi(s)$, $s=\beta/\alpha$, let us define $b_{i|j}$ by $b_{i|j}\theta^j:=db_i-b_j\theta_i^j$, where $\theta^i:=dx^i$ and $\theta_i^j:=\Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of $\alpha$. Let

$$r_{ij}:=\frac{1}{2}\Big[b_{i|j}+b_{j|i}\Big],s_{ij}:=\frac{1}{2}\Big[b_{i|j}-b_{j|i}\Big].$$

Thus $\beta$ is a closed form if $s_{ij}=0$ and is a Killing form if $r_{ij}=0$. Also, $\beta$ reduces to a parallel 1-form if $s_{ij}=r_{ij}=0$. Let us define

$$r_j:=b^ir_{ij},s_j:=b^is_{ij},r_{i0}:=r_{ij}y^j,s_{i0}:=s_{ij}y^j,r_0:=r_jy^j,s_0:=s_jy^j.$$

To satisfy that $F$ is positive and strongly convex on $TM_0$, it is known that if and only if

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (B - s^2)\phi''(s) > 0, \quad |s|^2 \leq B < b_0,$$

where $B := b^2 = \|\beta\|^2_\alpha$.

Let $G^i = G^i(x, y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x, y)$ denote the coefficients of $F$ and $\alpha$ respectively in the same coordinate system. Then, we have

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \Big(r_{00} - 2Q\alpha s_0\Big)\Big(\Theta\frac{y^i}{\alpha} + \Psi b^i\Big), \tag{2.3}$$

where

$$\begin{aligned}
Q :=& \frac{\phi'}{\phi - s\phi'},\\
\Delta :=& 1 + sQ + (b^2 - s^2)Q',\\
\Theta :=& \frac{Q - sQ'}{2\Delta},\\
\Psi :=& \frac{Q'}{2\Delta}.
\end{aligned}$$

If $\beta$ is a parallel 1-form, namely $s_{ij} = r_{ij} = 0$, then (2.3) reduces to $G^i = G^i_\alpha$. In this case, $F$ reduces to a Berwald metric.

In this section, we are going to find the formula of **H**-curvature of $(\alpha, \beta)$-metrics. First, we remark the formula of **E**-curvature of $(\alpha, \beta)$-metrics.

**Lemma 2** ([8], Proposition 3.1.)**.** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric. Put $\Omega := \Phi/(2\Delta^2)$, where

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''.$$

Then the **E**-curvature of $F$ is given by the following

$$\begin{aligned}
E_{ij} =& C_1 b_i b_j + C_2(b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5(r_{i0} b_j + r_{j0} b_i) + C_6(r_{i0} y_j + r_{j0} y_i)\\
&+ C_7 r_{ij} + C_8(s_i b_j + s_j b_i) + C_9(s_i y_j + s_j y_i) + C_{10}(r_i b_j + r_j b_i) + C_{11}(r_i y_j + r_j y_i),
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
C_1 :=& \frac{1}{2\alpha^3\Delta^2}\Big\{\Phi\alpha Q'' s_0 + 2\alpha\Delta^2\Psi'' r_0 - \Delta^2\Omega'' r_0 + 2\Delta^2\alpha\Omega'' Q s_0 + 4\Delta^2\alpha\Omega' Q' s_0 + 2\alpha\Delta^2\Psi'' s_0\Big\},\\
C_2 :=& \frac{-1}{2\alpha^4\Delta^2}\Big\{2\alpha\Delta^2\Psi'' s_0 - 2\Omega'\Delta^2 r_0 + 2\Omega'\Delta^2\alpha Q s_0 - \Delta^2\Omega'' s r_0 + 2\Delta^2\alpha\Omega'' s Q s_0\\
&+ 4\Delta^2\alpha\Omega' Q' s_0 s + 2\alpha\Delta^2\Psi' r_0 + 2\alpha\Delta^2\Psi'' s r_0 + 2\alpha\Delta^2\Psi'' s s_0 + \Phi\alpha Q' s_0 + \Phi\alpha Q'' s_0 s\Big\},
\end{aligned}$$

$$
\begin{aligned}
C_3 :=&\frac{1}{4\alpha^5\Delta^2}\Big\{4\Delta^2 s^2\Omega''\alpha Q s_0 - 2\Delta^2 s^2\Omega'' r_0 + 12\alpha\Delta^2\Psi' s r_0 + 12\alpha\Delta^2\Psi' s s_0 + 4\alpha\Delta^2\Psi'' s^2 r_0\\
&+ 4\alpha\Delta^2\Psi'' s^2 s_0 + 8\Delta^2 s^2\Omega'\alpha Q' s_0 + 2\Phi\alpha Q'' s_0 s^2 - 10\Omega'\Delta^2 s r_0 + 12\Omega'\Delta^2 s\alpha Q s_0\\
&\quad + 6\Phi\alpha Q' s_0 s - 3\Phi r_0\Big\},\\
C_4 :=& -\frac{1}{4\alpha^3\Delta^2}\Big\{4\alpha\Delta^2\Psi' s s_0 - \Phi r_0 - 2\Omega'\Delta^2 s r_0 + 4\Omega'\Delta^2 s\alpha Q s_0 + 4\alpha\Delta^2\Psi' s r_0 + 2\Phi\alpha Q' s_0 s\Big\},\\
C_5 :=& -\frac{\Omega'}{\alpha^2},\\
C_6 :=&\frac{2\Delta^2 s\Omega' + \Phi}{2\alpha^3\Delta^2},\\
C_7 :=& -\frac{\Phi}{2\alpha\Delta^2},\\
C_8 :=&\frac{1}{2\alpha\Delta^2}\Big\{2\Omega'\Delta^2 Q + 2\Delta^2\Psi' + \Phi Q'\Big\},\\
C_9 :=& -\frac{s}{\alpha}C_8,\\
C_{10} :=&\frac{\Psi'}{\alpha},\\
C_{11} :=& -\frac{s}{\alpha}C_{10}.
\end{aligned}
$$

By Lemma 2, we get the following.

**Proposition 3.** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha,\beta)$-metric on a manifold $M$. Then, the* **H***-curvature of $F$ is given by the following*

$$
\begin{aligned}
H_{ij} =&C_1[(r_{i0}+s_{i0})b_j + (r_{j0}+s_{j0})b_i] + C_2 b_i b_j + C_3[(r_{i0}+s_{i0})y_j + (r_{j0}+s_{j0})y_i]\\
&+ C_4(b_i y_j + b_j y_i) + C_5 y_i y_j + C_6 a_{ij} + C_7[r_{i0|0}b_j + r_{j0|0}b_i + r_{i0}(r_{j0}+s_{j0}) + r_{j0}(r_{i0}+s_{i0})]\\
&+ C_8(r_{i0}b_j + r_{j0}b_i) + C_9[r_{i0|0}y_j + r_{j0|0}y_i] + C_{10}(r_{i0}y_j + r_{j0}y_i) + C_{11}r_{ij|0} + C_{12}r_{ij}\\
&+ C_{13}[s_{i|0}b_j + s_{j|0}b_i + s_i(r_{j0}+s_{j0}) + s_j(r_{i0}+s_{i0})] + C_{14}(s_i b_j + s_j b_i)\\
&+ C_{15}[s_{i|0}y_j + s_{j|0}y_i] + C_{16}(s_i y_j + s_j y_i) + C_{17}[(r_{i|0}b_j + r_{j|0}b_i)\\
&+ r_i(r_{j0}+s_{j0}) + r_j(r_{i0}+s_{i0})] + C_{18}(r_i b_j + r_j b_i)\\
&+ C_{19}(r_{i|0}y_j + r_{j|0}y_i) + C_{20}(r_i y_j + r_j y_i),
\end{aligned}
\tag{2.5}
$$

*where*

$$
\begin{aligned}
C_1 :=&\frac{1}{2\alpha^3\Delta^2}\Big[\Phi\alpha Q_{ss}s_0 + \Delta^2(2\alpha\Psi_{ss}r_0 - \Omega_{ss}r_0 + 2\alpha\Omega_{ss}Q s_0 + 4\alpha\Omega_s Q_s s_0 + 2\alpha\Psi_{ss}s_0)\Big],\\
C_2 :=&\frac{-\Delta_{|0}}{\alpha^3\Delta^3}\Big[\Phi\alpha Q_{ss}s_0 + \Delta^2(2\alpha\Psi_{ss}r_0 - \Omega_{ss}r_0 + 2\alpha\Omega_{ss}Q s_0 + 4\alpha\Omega_s Q_s s_0 + 2\alpha\Psi_{ss}s_0)\Big]\\
&+ \frac{1}{2\alpha^3\Delta^2}\Big[\alpha(\Phi_{|0}Q_{ss}s_0 + \Phi Q_{ss|0}s_0 + \Phi Q_{ss}s_{0|0}) + 2\alpha\Delta\Big(2\Delta_{|0}\Psi_{ss}r_0 + \Delta\Psi_{ss|0}r_0 + \Delta\Psi_{ss}r_{0|0}\Big)\\
&- \Delta(2\Delta_{|0}\Omega_{ss}r_0 - \Delta\Omega_{ss|0}r_0 - \Delta\Omega_{ss}r_{0|0}) + 2\alpha\Delta\Big(2\Delta_{|0}\Omega_{ss}Q + \Delta\Omega_{ss|0}Q + \Delta\Omega_{ss}Q_{|0}\Big)s_0
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\alpha\Delta^2\Omega_{ss}Qs_{0|0} + 4\alpha\Big(2\Delta\Delta_{|0}\Omega_s Q_s s_0 \quad + \Delta^2\Omega_{s|0}Q_s s_0 + \Delta^2\Omega_s Q_{s|0}s_0 + \Delta^2\Omega_s Q_s s_{0|0}\Big)\\
&+ 2\alpha\Big(2\Delta\Delta_{|0}\Psi_{ss}s_0 + \Delta^2\Psi_{ss|0}s_0 + \Delta^2\Psi_{ss}s_{0|0}\Big)\Big],\\
C_3 :=& \frac{-1}{2\alpha^4\Delta^2}\Big[2\alpha\Delta^2\Psi_{ss}s_0 - 2\Omega_s\Delta^2 r_0 + 2\Omega_s\Delta^2\alpha Q s_0 - \Delta^2\Omega_{ss}sr_0 + 2\Delta^2\alpha\Omega_{ss}sQs_0 + 4\Delta^2\alpha\Omega_s Q_s s_0 s\\
&+ 2\alpha\Delta^2\Psi_s r_0 + 2\alpha\Delta^2\Psi_{ss}sr_0 + 2\alpha\Delta^2\Psi_{ss}ss_0 + \Phi\alpha Q_s s_0 + \Phi\alpha Q_{ss}s_0 s\Big],\\
C_4 :=& \frac{-1}{2\alpha^4\Delta^2}\Big[2\alpha\Big(2\Delta\Delta_{|0}\Psi_{ss}s_0 + \Delta^2\Psi_{ss|0}s_0 + \Delta^2\Psi_{ss}s_{0|0}\Big) - 2\Omega_{s|0}\Delta^2 r_0 - 4\Omega_s\Delta\Delta_{|0}r_0 - 2\Omega_s\Delta^2 r_{0|0}\\
&+ 2\alpha\Big(\Omega_{s|0}\Delta^2 Qs_0 + 2\Omega_s\Delta\Delta_{|0}Qs_0 + \Omega_s\Delta^2 Q_{|0}s_0 + \Omega_s\Delta^2 Qs_{0|0}\Big) - 2\Delta\Delta_{|0}\Omega_{ss}sr_0 - \Delta^2\Omega_{ss|0}sr_0\\
&- \Delta^2\Omega_{ss}\frac{r_{00}}{\alpha}r_0 - \Delta^2\Omega_{ss}sr_{0|0} + 2\alpha\Big(2\Delta\Delta_{|0}\Omega_{ss}sQs_0 + \Delta^2\Omega_{ss|0}sQs_0 + \Delta^2\Omega_{ss}sQ_{|0}s_0 + \Delta^2\Omega_{ss}sQs_{0|0}\Big)\\
&+ 4\alpha\Big(2\Delta\Delta_{|0}\Omega_s Q_s s_0 s + \Delta^2\Omega_{s|0}Q_s s_0 s + \Delta^2\Omega_s Q_{s|0}s_0 s + \Delta^2\Omega_s Q_s s_{0|0}s\Big) + 4\Delta^2\Omega_s Q_s s_0 r_{00}\\
&+ 2\Delta^2\Omega_{ss}Qr_{00}s_0 + 2\alpha\Big(2\Delta\Delta_{|0}\Psi_s r_0 + \Delta^2\Psi_{s|0}r_0 + \Delta^2\Psi_s r_{0|0} + 2\Delta\Delta_{|0}\Psi_{ss}sr_0 + \Delta^2\Psi_{ss|0}sr_0\\
&+ \Delta^2\Psi_{ss|0}sr_{0|0} + 2\Delta\Delta_{|0}\Psi_{ss}ss_0 + \Delta^2\Psi_{ss|0}ss_0 + \Delta^2\Psi_{ss}ss_{0|0}\Big) + 2\Delta^2\Psi_{ss}r_{00}s_0 + 2\Delta^2\Psi_{ss}r_{00}r_0\\
&+ \alpha\Big(\Phi_{|0}Q_s s_0 + \Phi(Q_{s|0}s_0 + \Phi Q_s s_{0|0} + \Phi_{|0}Q_{ss})s_0 s + \Phi Q_{ss|0}s_0 s + \Phi Q_{ss}s_{0|0}s\Big) + \Phi Q_{ss}s_0 r_{00}\Big],\\
&+ \frac{\Delta_{|0}}{\alpha^4\Delta^3}\Big[2\alpha\Delta^2\Psi_{ss}s_0 + 2\Omega_s\Delta^2\alpha Qs_0 + 2\Delta^2\alpha\Omega_{ss}sQs_0 + 4\Delta^2\alpha\Omega_s Q_s ss_0 + 2\alpha\Delta^2\Psi_{ss}ss_0\\
&+ \Phi\alpha Q_s s_0 + \Phi\alpha Q_{ss}ss_0 + 2\alpha\Delta^2\Psi_s r_0 + 2\alpha\Delta^2\Psi_{ss}sr_0 - \Delta^2\Omega_{ss}sr_0 - 2\Omega_s\Delta^2 r_0\Big],\\
C_5 :=& \frac{1}{4\alpha^5\Delta^2}\Big[4\alpha\Big(2\Delta\Delta_{|0}s^2\Omega_{ss}Qs_0 + 2\Delta^2 s\frac{r_{00}}{\alpha}\Omega_{ss}Qs_0 + \Delta^2 s^2\Omega_{ss|0}Qs_0 + \Delta^2 s^2\Omega_{ss}Q_{|0}s_0 + \Delta^2 s^2\Omega_{ss}Qs_{0|0}\Big)\\
&- 2\Big(2\Delta\Delta_{|0}s^2\Omega_{ss}r_0 + 2\Delta^2 s\frac{r_{00}}{\alpha}\Omega_{ss}r_0 + \Delta^2 s^2\Omega_{ss|0}r_0 + \Delta^2 s^2\Omega_{ss}r_{0|0}\Big) + 12\alpha\Big(2\Delta\Delta_{|0}\Psi_s sr_0\\
&+ \Delta^2\Psi_{s|0}sr_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}r_0 + \Delta^2\Psi_s sr_{0|0} + 2\Delta\Delta_{|0}\Psi_s ss_0 + \Delta^2\Psi_{s|0}ss_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}s_0 + \Delta^2\Psi_s ss_{0|0}\Big)\\
&+ 4\alpha\Big(2\Delta\Delta_{|0}\Psi_{ss}s^2 r_0 + \Delta^2\Psi_{ss|0}s^2 r_0 + 2\Delta^2\Psi_{ss}s\frac{r_{00}}{\alpha}r_0 + \Delta^2\Psi_{ss}s^2 r_{0|0} + 2\Delta\Delta_{|0}\Psi_{ss}s^2 s_0 + \Delta^2\Psi_{ss|0}s^2 s_0\\
&+ 2\Delta^2\Psi_{ss}s\frac{r_{00}}{\alpha}s_0 + \Delta^2\Psi_{ss}s^2 s_{0|0}\Big) + 2\alpha s^2\Big(8\Delta\Delta_{|0}\Omega_s Q_s s_0 + 4\Delta^2\Omega_{s|0}Q_s s_0 + 4\Delta^2\Omega_s Q_{s|0}s_0\\
&+ 4\Delta^2\Omega_s Q_s s_{0|0} + \Phi_{|0}Q_{ss}s_0 + \Phi Q_{ss|0}s_0 + \Phi Q_{ss}s_{0|0}\Big) + 4ss_0\Big(4\Delta^2\Omega_s Q_s + \Phi Q_{ss}\Big)r_{00}\\
&- 10\Big(\Omega_{s|0}\Delta^2 sr_0 + 2\Omega_s\Delta\Delta_{|0}sr_0 + \Omega_s\Delta^2\frac{r_{00}}{\alpha}r_0 + \Omega_s\Delta^2 sr_{0|0}\Big) + 6\alpha(2\Omega_{s|0}\Delta^2 sQs_0 + 4\Omega_s\Delta\Delta_{|0}sQs_0\\
&+ 2\Omega_s\Delta^2\frac{r_{00}}{\alpha}Qs_0 + 2\Omega_s\Delta^2 sQ_{|0}s_0 + 2\Omega_s\Delta^2 sQs_{0|0} + \Phi_{|0}Q_s s_0 s + \Phi Q_{s|0}s_0 s + \Phi Q_s s_{0|0}s\Big)\\
&- 3\Phi_{|0}r_0 - 3\Phi r_{0|0} + 6\Phi Q_s s_0 r_{00}\Big] - \frac{\Delta_{|0}}{2\alpha^5\Delta^3}\Big[4\Delta^2 s^2\Omega_{ss}\alpha Qs_0 - 2\Delta^2 s^2\Omega_{ss}r_0 + 12\alpha\Delta^2\Psi_s sr_0\\
&+ 12\alpha\Delta^2\Psi_s ss_0 + 4\alpha\Delta^2\Psi_{ss}s^2 r_0 + 4\alpha\Delta^2\Psi_{ss}s^2 s_0 + 8\Delta^2 s^2\Omega_s\alpha Q_s s_0 + 2\Phi\alpha Q_{ss}s_0 s^2 - 10\Omega_s\Delta^2 sr_0\\
&+ 12\Omega_s\Delta^2 s\alpha Qs_0 + 6\Phi\alpha Q_s s_0 s - 3\Phi r_0\Big],\\
C_6 :=& \frac{-1}{4\alpha^3\Delta^2}\Big[4\alpha\Big(2\Delta\Delta_{|0}\Psi_s ss_0 + \Delta^2\Psi_{s|0}ss_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}s_0 + \Delta^2\Psi_s ss_{0|0}\Big) - \Phi_{|0}r_0 - \Phi r_{0|0}\\
&- 2\Big(\Omega_{s|0}\Delta^2 sr_0 + 2\Omega_s\Delta\Delta_{|0}sr_0 + \Omega_s\Delta^2\frac{r_{00}}{\alpha}r_0 + \Omega_s\Delta^2 sr_{0|0}\Big) + \alpha\Big[4\Big(\Omega_{s|0}\Delta^2 sQs_0 + \Omega_s\Delta^2\frac{r_{00}}{\alpha}Qs_0\\
&+ \Omega_s\Delta^2 sQ_{|0}s_0 + \Omega_s\Delta^2 sQs_{0|0} + 2\Omega_s\Delta\Delta_{|0}sQs_0 + 2\Delta\Delta_{|0}\Psi_s sr_0 + \Delta^2\Psi_{s|0}sr_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}r_0\\
&+ \Delta^2\Psi' sr_{0|0}\Big) + 2\Phi_{|0}Q_s s_0 s + 2\Phi Q_{s|0}s_0 s + 2\Phi Q_s s_{0|0}s + 2\Phi Q_s s_0 r_{00}\Big] + \frac{\Delta_{|0}}{4\alpha^3\Delta^3}\Big[4\alpha\Delta^2\Psi_s ss_0\\
&- \Phi r_0 - 2\Omega_s\Delta^2 sr_0 + 4\Omega_s\Delta^2 s\alpha Qs_0 + 4\alpha\Delta^2\Psi_s sr_0 + 2\Phi\alpha Q_s s_0 s\Big],\\
C_7 :=& -\frac{1}{\alpha^2}\Omega_s,\\
C_8 :=& -\frac{1}{\alpha^2}\Omega_{s|0},\\
C_9 :=& \frac{1}{2\alpha^3\Delta^2}(2\Delta^2 s\Omega_s + \Phi),
\end{aligned}
$$

$$
\begin{aligned}
C_{10} :=& \frac{4\Delta\Delta_{|0} s\Omega_s + 2\Delta^2 s\Omega_{s|0} + \Phi_{|0}}{2\alpha^3\Delta^2} + \frac{r_{00}\Omega_s}{\alpha^4} - \frac{(2\Delta^2 s\Omega_s + \Phi)\Delta_{|0}}{\alpha^3\Delta^3},\\
C_{11} :=& \frac{-\Phi}{2\alpha\Delta^2},\\
C_{12} :=& \frac{-\Phi_{|0}}{2\alpha\Delta^2} + \frac{\Phi\Delta_{|0}}{\alpha\Delta^3},\\
C_{13} :=& \frac{1}{2\alpha\Delta^2}\Big[2\Omega_s\Delta^2 Q + 2\Delta^2\Psi_s + \Phi Q_s\Big],\\
C_{14} :=& \frac{1}{2\alpha\Delta^2}\Big[2\Omega_{s|0}\Delta^2 Q + 4\Omega_s\Delta\Delta_{|0} Q + 2\Omega_s\Delta^2 Q_{|0} + 4\Delta\Delta_{|0}\Psi_s + 2\Delta^2\Psi_{s|0} + \Phi_{|0}Q_s + \Phi Q_{s|0}\Big],\\
&- \frac{\Delta_{|0}}{\alpha\Delta^3}\Big[2\Omega_s\Delta^2 Q + 2\Delta^2\Psi_s + \Phi Q_s\Big],\\
C_{15} :=& -\frac{s}{\alpha}C_{13}, \quad C_{16} := -\frac{s}{\alpha}C_{14} - \frac{r_{00}}{\alpha^2}C_{13}, \quad C_{17} := \frac{1}{\alpha}\Psi_s,\\
C_{18} :=& \frac{1}{\alpha}\Psi_{s|0}, \quad C_{19} := -\frac{s}{\alpha}C_{17}, \quad C_{20} := -\frac{r_{00}}{\alpha^2}C_{17} - \frac{s}{\alpha}C_{18},
\end{aligned}
$$

*and*

$$
\begin{aligned}
\Delta_{|0} :=& Q_{|0}s + \frac{r_{00}}{\alpha}Q + 2\Big[r_0 + s_0 - \frac{sr_{00}}{\alpha}\Big]Q_s + (b^2 - s^2)Q_{s|0}\\
\Phi_{|0} :=& -\Big[n\Delta_{|0} + \frac{r_{00}}{\alpha}Q + sQ_{|0}\Big](Q - sQ_s) - (Q_{|0} - \frac{r_{00}}{\alpha}Q_s - sQ_{s|0})(n\Delta + 1 + sQ)\\
&- 2\Big[r_0 + s_0 - \frac{sr_{00}}{\alpha}\Big](1 + sQ)Q_{ss} - (b^2 - s^2)\Big[\frac{r_{00}}{\alpha}QQ_{ss} + sQ_{|0}Q_{ss} + (1 + sQ)Q_{ss|0}\Big].
\end{aligned}
$$

## 3. Proof of the Theorem 1

For the Finsler metric defined by (1.1), we get

$$
Q := \frac{1}{-1 + 2s}, \qquad \Psi = \frac{1}{1 - 3s + 2B}. \tag{3.1}
$$

Using the Maple program, we can obtain the quantity **H** for the Finsler metric defined by (1.1). By using Proposition 3, putting $s_{ij} = 0$ and $r_{ij} = c(b^2 a_{ij} - b_i b_j)$ in (2.5) and decomposing of the rational (Rat) and irrational (Irrat) parts, we have the following

$$
H_{jk} = (Rat)_{jk} + \alpha(Irrat)_{jk}, \tag{3.2}
$$

where $Rat$ and $Irrat$ are listed in Appendix 4. It is remarkable that, if $\mathbf{H} = (n+1)/(2F)\theta\mathbf{h}$ then by equating the parts of rational and irrational parts in two sides of equation, we get two simple equations. By the obtained equations, one can get the desired result.

Now, suppose that $F$ has almost vanishing **H**-curvature on a $n$-dimensional manifold $M$. Then

$$
H_{ij} = \frac{n+1}{2}F^{-1}\theta h_{ij}, \tag{3.3}
$$

where $\theta := \theta_i(x)y^i$ is a 1-form on $M$. Substituting (3.2) in (3.3), we get

$$
(n+1)\theta h_{jk} = 2F\big[(Rat)_{jk} + \alpha(Irrat)_{jk}\big]. \tag{3.4}
$$

Multiplying (3.4) with $b^ib^j$ and using Maple program, we get the following

$$
\begin{aligned}
&\Pi\Big[\Big(-32B^3c^2b^4-32nB^3c^2b^4+52B^4c^2-20nB^4c^2-2B^3c^2-2nB^3c^2\Big)\alpha^{12}+B\Big(9B\theta\\
&+96nB^2c^2b^4\beta-188B^3c^2\beta+4nB^2c^2n\beta+100B^3c^2n\beta-2Bc^2n\beta+44B^3n\theta-32B^2c^2\beta\\
&+30B^2n\theta-2Bc^2\beta+44B^3\theta+96B^2c^2b^4\beta+9B\theta+\theta+n\theta+30B^2\theta+24B^4\theta+24B^4n\theta\Big)\alpha^{11}\\
&+\Big(-10Bn\beta\theta-148B^4c^2n\beta^2+284B^4c^2\beta^2-69B^2\beta\theta-96B^3c^2b^4n\beta^2+64B^2c^2b^4n\beta^2\\
&+64B^2c^2b^4\beta^2+96B^3c^2\beta^2-96B^3c^2b^4\beta^2-69B^2n\beta\theta+96B^3c^2n\beta^2+30B^2c^2\beta^2-156B^3n\beta\theta\\
&-116B^4n\beta\theta-156B^3\beta\theta-116B^4\beta\theta+30B^2c^2n\beta^2-10B\beta\theta\Big)\alpha^{10}+\Big(123B^2\beta^2\theta-192B^2c^2b^4\beta^3\\
&-n\beta^2\theta+2Bc^2n\beta^3-76B^2c^2\beta^3+2Bc^2\beta^3-476B^3c^2n\beta^3-116B^3c^2\beta^3-48B^4\beta^2\theta+32B^3c^2b^4n\beta^3\\
&+32B^3c^2b^4\beta^3-148B^2c^2n\beta^3-192B^2c^2b^4n\beta^3-48B^4n\beta^2\theta+44B^4c^2n\beta^3+123B^2n\beta^2\theta+24Bn\beta^2\theta\\
&+118B^3\beta^2\theta-244B^4c^2\beta^3+118B^3n\beta^2\theta+24B\beta^2\theta-\beta^2\theta\Big)\alpha^9+\Big(810B^3c^2n\beta^4+90B^3c^2\beta^4\\
&+192B^2c^2b^4n\beta^4+64B^2c^2\beta^4+192B^2c^2b^4\beta^4+280B^2c^2n\beta^4-28Bc^2n\beta^4-32Bc^2b^4n\beta^4\\
&+128B^4c^2\beta^4+256B^3\beta^3\theta+66Bn\beta^3\theta+177B^2\beta^3\theta-28Bc^2\beta^4+14n\beta^3\theta+56B^4c^2n\beta^4\\
&+256B^3n\beta^3\theta+177B^2n\beta^3\theta+66B\beta^3\theta-32Bc^2b^4\beta^4+14\beta^3\theta\Big)\alpha^8+\Big(+96Bc^2b^4\beta^5-12B^3c^2\beta^5\\
&-88B^3n\beta^4\theta+108Bc^2\beta^5-32B^4c^2\beta^5-64B^2c^2b^4\beta^5+66B^2c^2\beta^5-88B^3\beta^4\theta+96Bc^2b^4n\beta^5\\
&-64B^2c^2b^4n\beta^5-552B^3c^2n\beta^5-636B^2n\beta^4\theta-435B\beta^4\theta-78B^2c^2n\beta^5-636B^2\beta^4\theta-83\beta^4\theta\\
&-32B^4c^2n\beta^5-83n\beta^4\theta+144Bc^2n\beta^5-435Bn\beta^4\theta\Big)\alpha^7+\Big(88B^3c^2n\beta^6-230B^2c^2\beta^6-212Bc^2\beta^6\\
&-356Bc^2n\beta^6+160B^3n\beta^5\theta+948B\beta^5\theta+160B^3\beta^5\theta-96Bc^2b^4\beta^6-56B^3c^2\beta^6-374B^2c^2n\beta^6\\
&+948Bn\beta^5\theta+636B^2\beta^5\theta-96Bc^2b^4n\beta^6+281\beta^5\theta+636B^2n\beta^5\theta+281n\beta^5\theta\Big)\alpha^6+\Big(220B^2c^2\beta^7\\
&+238\beta^7c^2B+454n\beta^7c^2B+32B^3c^2n\beta^7-120B^3\beta^6\theta+32B^3c^2\beta^7-120B^3n\beta^6\theta+436B^2c^2n\beta^7\\
&-627\beta^6\theta-1404B\beta^6\theta+32Bc^2b^4\beta^7-900B^2\beta^6\theta+32Bc^2b^4n\beta^7-900B^2n\beta^6\theta-627n\beta^6\theta\\
&-1404Bn\beta^6\theta\Big)\alpha^5+\Big(-72B^2c^2\beta^8+48B^3\beta^7\theta-144Bc^2\beta^8-144B^2c^2n\beta^8+1656Bn\beta^7\theta\\
&+48B^3n\beta^7\theta-288Bc^2n\beta^8+612B^2n\beta^7\theta+612B^2\beta^7\theta+1656B\beta^7\theta+978n\beta^7\theta+978\beta^7\theta\Big)\alpha^4\\
&+\Big(72Bc^2n\beta-1032Bn\theta-1032B\theta-8B^3n\theta-1000\theta+36Bc^2\beta-8B^3\theta\\
&-1000n\theta-228B^2n\theta-228B^2\theta\Big)\beta^8\alpha^3+\Big(576n+360B+360Bn+36B^2n\\
&+36B^2+576\Big)\beta^9\theta\alpha^2-\Big(189n+189+54(n+1)B\Big)\beta^{10}\theta\alpha+27(n+1)\beta^{11}\theta\Big]=0, \qquad (3.5)
\end{aligned}
$$

where

$$\Pi := -\frac{1}{2(\alpha-\beta)^3\big[(1+2B)\alpha-3\beta\big]^3\alpha^6}.$$

By (3.5), it follows that $27\beta^{11}\theta(n+1)$ should be a multiple of $\alpha^2$ which is impossible. So $\theta=0$. By putting $\theta=0$ in (3.5), the coefficient of $\alpha^3$ reduces to $(2n+1)Bc^2\beta^9=0$. Thus $c=0$ and $\beta$ is parallel with respect to $\alpha$. Therefore, $F$ reduces to a Berwald metric. $\square$

## 4. Appendix: Coefficients in (3.2)

$$\begin{aligned}
(Rat)_{ij} :=& A\Big(96nB^2c^2b^4b_iy_j + 96nB^2c^2b^4b_jy_i - 6nBc^2b_ib_j\beta - 192Bc^2b^4b_ib_j\beta + 176nB^3c^2b_ib_j\beta \\
&+ 36nB^2c^2b_ib_j\beta - 48B^4c^2a_{ij}\beta - 8nB^4c_{|0}a_{ij} + 96B^3c^2a_{ij}\beta - 12nB^3c_{10}a_{ij} + 32B^3c_{|0}b_ib_j \\
&+ 6B^2c^2a_{ij}\beta - 6B^2c_{|0}a_{ij}n + 36B^2c_{|0}b_ib_j - c_{|0}Ba_{ij}n + 12c_{|0}Bb_ib_j + c_{10}b_ib_jn + 18B^2y_jc^2b_i \\
&+ 18B^2y_ic^2b_j - 216B^3y_jc^2b_i - 216B^3y_ic^2b_j - 2c^2b_i\beta b_j - 96Bc^2b^4b_ib_jn\beta + c_{|0}b_ib_j \\
&- 36B^3c_{|0}a_{ij} - c_{|0}Ba_{ij} - 12B^2c_{|0}a_{ij} - 32B^4c_{|0}a_{ij} - 78Bc^2b_ib_j\beta - 2c^2b_ib_jn\beta + 6nB^2y_jc^2b_i \\
&+ 144B^2c^2b^4b_jy_i - 96B^4c^2a_{ij}n\beta - 60B^3c^2a_{ij}n\beta - 16B^3c^2b_ib_j\beta + 144B^2c^2b^4b_iy_j \\
&+ 6nB^2y_ic^2b_j + 60nB^3y_jc^2b_i + 60nB^3y_ic^2b_j + 6nc_{|0}Bb_ib_j + 8B^3c_{|0}b_ib_jn - 6B^2c^2a_{ij}n\beta \\
&+ 72B^2c^2b_ib_j\beta + 12B^2c_{|0}b_ib_jn\Big)\alpha^6 + \Big( - 56B^3c_{|0}b_iny_j\beta - 56B^3c_{|0}b_jny_i\beta + 240nB^2c_{|0}b_ib_j\beta^2 \\
&- 96c^2b^4b_ib_jn\beta^3 - 288B^2c^2b^4y_iy_j\beta + 176B^4c^2ny_iy_j\beta - 12n\beta b_jc_{|0}By_i + 12\beta^2nb_jc^2By_i \\
&- 12n\beta b_ic_{10}By_j + 12\beta^2nb_ic^2By_j - 12B^2c^2n\beta y_iy_j - 72n\beta B^2b_jc_{|0}y_i + 318\beta^2nBb_jc_{|0}b_i \\
&- 696\beta^3nBb_jc^2b_i + 24\beta^2nB^2b_jc^2y_i - 72n\beta B^2b_ic_{|0}y_j + 24\beta^2nB^2b_ic^2y_j - 24B^3c^2n\beta y_iy_j \\
&- 32\beta^2nB^3y_jc^2b_i - 32\beta^2nB^3y_ic^2b_j - 2ny_j\beta c^2By_i + 48\beta^2b^4y_jc^2b_iB + 48\beta^2b^4y_ic^2b_jB \\
&- 672B^2c^2b_ib_jn\beta^3 + 96c_{|0}b^4b_ib_jn\beta^2 - 48c_{|0}Bb^4b_iy_j\beta - 48c_{|0}Bb^4b_jy_i\beta + 2y_jc_{|0}By_i \\
&- y_jc_{|0}\beta b_i + 2y_jc^2b_i\beta^2 - y_ic_{|0}\beta b_j + 2y_ic^2b_j\beta^2 + 48B^3c^2a_{ij}\beta^3 - 96B^2c^2a_{ij}\beta^3 + 64B^4c_{|0}y_iy_j \\
&- 120B^3c_{|0}a_{ij}\beta^2 - 6Bc^2a_{ij}\beta^3 - 228B^2c_{|0}a_{ij}\beta^2 + 24B^2y_jc_{|0}y_i + 72B^3y_jc_{|0}y_i - 57\beta^2c_{|0}Ba_{ij} \\
&+ 45b_jc_{|0}\beta^2b_i - 36c^2b_ib_j\beta^3 - 192B^2c^2b^4ny_iy_j\beta - 48c_{|0}Bb^4b_iny_j\beta - 48c_{|0}Bb^4b_jny_i\beta \\
&+ 48\beta^2nb^4y_jc^2b_iB + 48\beta^2nb^4y_ic^2b_jB + 96c_{|0}b^4b_ib_j\beta^2 + 16B^4c_{|0}ny_iy_j - 144B^3c_{|0}a_{ij}n\beta^2 \\
&+ 6Bc^2a_{ij}n\beta^3 - 216Bc^2b_ib_j\beta^3 - 32B^3c_{|0}b_iy_j\beta - 32B^3c_{|0}b_jy_i\beta - 198B^2c_{|0}a_{ij}n\beta^2 \\
&+ 120B^2c_{|0}b_ib_j\beta^2 + 96B^3c^2a_{ij}n\beta^3 + 16\beta^2B^3y_jc^2b_i + 16\beta^2B^3y_ic^2b_j + 2ny_ic^2b_j\beta^2 \\
&+ 60B^2y_jc_{|0}\beta b_i + 72B^2y_jc^2b_i\beta^2 + 168B^3y_j\beta c^2y_i + 60B^2y_ic_{|0}\beta b_j + 72B^2y_ic^2b_j\beta^2 \\
&- 2y_j\beta c^2By_i + 75\beta^2nb_jc_{|0}b_i - 156\beta^3nb_jc^2b_i - 6\beta b_jc_{|0}By_i + 24b_j\beta^2c^2By_i - 6\beta b_ic_{|0}By_j \\
&+ 24b_i\beta^2c^2By_j - 72B^2c^2\beta y_iy_j - 63\beta^2nc_{|0}Ba_{ij} + 12nB^2y_jc_{|0}y_i + 24nB^3y_jc_{|0}y_i + 2ny_jc_{|0}By_i \\
&- ny_jc_{|0}\beta b_i + 2ny_jc^2b_i\beta^2 - ny_ic_{|0}\beta b_j + 36c_{|0}Bb_ib_j\beta^2 + 32B^4c^2y_iy_j\beta + 60B^2c^2a_{ij}n\beta^3 \\
&- 42\beta^3y_jc^2By_i - 48\beta^3b^4y_jc_{|0}b_i - 48\beta^3b^4y_ic_{|0}b_j + 108n\beta^4b_jc_{|0}b_i - 216n\beta^5b_jc^2b_i \\
&- 132\beta^3b_jc_{|0}By_i + 288b_j\beta^4c^2By_i - 132\beta^3b_ic_{|0}By_j + 288b_i\beta^4c^2By_j - 312B^2c^2\beta^3y_iy_j \\
&- 108n\beta^4c_{|0}Ba_{ij} - 69\beta^3ny_jc_{|0}b_i + 144n\beta^4y_jc^2b_i - 69\beta^3ny_ic_{|0}b_j + 144n\beta^4y_ic^2b_j \\
&+ 264\beta^2B^2y_jc_{|0}y_i + 240\beta^2B^3y_jc_{|0}y_i)\alpha^2 + 108\beta^6y_jc^2b_i + 108\beta^4y_jc_{|0}By_i - 108\beta^5y_jc^2By_i \\
&- 54\beta^5y_jc_{|0}b_i + 216n\beta^6y_ic^2b_j - 108n\beta^5y_jc_{|0}b_i + 216n\beta^6y_jc^2b_i - 108n\beta^5y_ic_{|0}b_j \\
&- 216n\beta^5y_jc^2By_i + 108\beta^6y_ic^2b_j + 216n\beta^4y_jc_{|0}By_i - 54\beta^5y_ic_{|0}b_j\Big),
\end{aligned}$$

$$\begin{aligned}
(Irrat)_{ij} :=& A\Big[\Big(144\beta^4y_jc^2By_i + 144\beta^4By_ic_{|0}b_j - 144\beta^3y_jc_{|0}By_i - 288n\beta^5y_jc^2b_i + 72\beta^4y_jc_{|0}b_i \\
&+ 144\beta^4ny_jc_{|0}b_i - 288\beta^3B^2y_jc_{|0}y_i + 144\beta^4By_jc_{|0}b_i - 288b_j\beta^5c^2By_i - 288b_i\beta^5c^2By_j \\
&+ 288\beta^4B^2y_jc^2y_i - 288n\beta^5y_ic^2b_j + 144\beta^4ny_ic_{|0}b_j + 72\beta^4y_ic_{|0}b_j - 144\beta^5y_jc^2b_i \\
&- 144\beta^5y_ic^2b_j + 576\beta^4nB^2y_jc^2y_i - 576n\beta^5b_ic^2By_j + 288\beta^4nBy_ic_{|0}b_j - 576n\beta^5b_jc^2By_i
\end{aligned}$$

$$
\begin{aligned}
&- 288\beta^3 ny_j c_{|0} By_i + 288\beta^4 ny_j c^2 By_i + 288\beta^4 nBy_j c_{|0} b_i - 576\beta^3 nB^2 y_j c_{|0} y_i\Big)\alpha^7 \\
&+ \Big(72B^2c^2b_ib_j - 64B^2c^2b^4b_ib_j + 12B^3c^2a_{ij}n - 32B^4c^2b_ib_j + 32nB^5c^2a_{ij} + nB^2c^2a_{ij} \\
&- 3Bc^2b_ib_j + 32B^5c^2a_{ij} - 36B^4c^2a_{ij} - 24B^3c^2a_{ij} - 32B^4c^2b_ib_jn + B^2c^2a_{ij} + 36B^4c^2a_{ij}n \\
&- 32Bc^2b^4b_ib_j - 3nBc^2b_ib_j - 36nB^2c^2b_ib_j - 76B^3c^2b_ib_jn - 64B^2c^2b^4b_ib_jn + 140B^3c^2b_ib_j \\
&- 32Bc^2b^4b_ib_jn\Big)\alpha^6 + \Big(192Bc^2b^4b_ib_jn\beta^2 - 32B^2c^2b^4b_iny_j\beta - 32B^2c^2b^4b_jny_i\beta \\
&- 16\beta nb^4y_jc^2b_iB - 16\beta nb^4y_ic^2b_jB - 64Bc_{|0}b^4b_ib_jn\beta - 18\beta nB^2y_jc^2b_i - 18\beta nB^2y_ic^2b_j \\
&- 128\beta nB^3y_jc^2b_i - 128\beta nB^3y_ic^2b_j - 16\beta b^4y_jc^2b_iB - 16\beta b^4y_ic^2b_jB + 396B^2c^2b_ib_jn\beta^2 \\
&- 64Bc_{|0}b^4b_ib_j\beta + 16Bc_{|0}b^4b_iny_j + 16Bc_{|0}b^4b_jny_i - 32c_{|0}b^4b_ib_jn\beta - 32B^3c_{|0}b_ib_jn\beta \\
&+ 213Bc^2b_ib_jn\beta^2 - 124B^2c_{|0}b_ib_jn\beta - 84c_{|0}Bb_ib_jn\beta - 32B^2c^2b^4b_iy_j\beta - 32B^2c^2b^4b_jy_i\beta \\
&- Bc^2a_{ij}\beta^2 + b_jc_{|0}By_i - 15b_jc_{|0}\beta b_i + 33b_jc^2b_i\beta^2 + b_ic_{|0}By_j - 2B^2c^2y_iy_j + 13c_{|0}Ba_{ij}\beta \\
&- 32B^4c^2a_{ij}\beta^2 + 72B^4c^2y_iy_j + 36B^3c^2a_{ij}\beta^2 - 64B^5c^2y_iy_j + 32B^4c_{|0}a_{ij}\beta + 48B^3c^2y_iy_j \\
&- 3B^2c^2a_{ij}\beta^2 + 156B^3c_{|0}a_{ij}\beta - 52B^3c_{|0}b_iy_j - 52B^3c_{|0}b_jy_i + 96B^2c_{|0}a_{ij}\beta - 24B^2c_{|0}b_iy_j \\
&- 24B^2c_{|0}b_jy_i + 192Bc^2b^4b_ib_j\beta^2 + 96c^2b^4b_ib_jn\beta^2 + 32B^4c^2b_iny_j\beta + 32B^4c^2b_jny_i\beta \\
&+ 96B^3c^2b_ib_jn\beta^2 + 32B^2c_{|0}b^4b_iny_j + 32B^2c_{|0}b^4b_jny_i - Bc^2a_{ij}n\beta^2 + 54B^2\beta y_jc^2b_i \\
&+ 54B^2\beta y_ic^2b_j + 16\beta B^3y_jc^2b_i + 16\beta B^3y_ic^2b_j + 13c_{|0}Ba_{ij}n\beta - 48c_{|0}Bb_ib_j\beta \\
&+ b_ic_{|0}Bny_j + b_jc_{|0}Bny_i - 15c_{|0}b_ib_jn\beta + 32B^4c_{|0}a_{ij}n\beta - 24B^3c^2ny_iy_j - 3B^2c^2a_{ij}n\beta^2 \\
&- 36B^2c^2b_ib_j\beta^2 + 16Bc_{|0}b^4b_iy_j + 16Bc_{|0}b^4b_jy_i - 32c_{|0}b^4b_ib_j\beta + 84B^3c_{|0}a_{ij}n\beta \\
&+ 20B^3c_{|0}b_iny_j + 20B^3c_{|0}b_jny_i - 2B^2c^2ny_iy_j - 32B^3c_{|0}b_ib_j\beta - 3Bc^2b_ib_j\beta^2 + 33b_jc^2b_in\beta^2 \\
&+ 60B^2c_{|0}a_{ij}n\beta - 52B^2c_{|0}b_ib_j\beta + 12B^2c_{|0}b_iny_j + 12B^2c_{|0}b_jny_i - 64B^5c^2ny_iy_j \\
&+ 96c^2b^4b_ib_j\beta^2 - 32B^4c^2a_{ij}n\beta^2 + 32B^4c^2b_iy_j\beta + 32B^4c^2b_jy_i\beta - 72B^4c^2ny_iy_j \\
&+ 36B^3c^2a_{ij}n\beta^2 + 96B^3c^2b_ib_j\beta^2 + 32B^2c_{|0}b^4b_iy_j + 32B^2c_{|0}b^4b_jy_i\Big)\alpha^4 + \Big(32\beta^2y_jc^2By_i \\
&- 120\beta^3B^2y_ic^2b_j - 28\beta y_jc_{|0}By_i + 16\beta^2b^4y_jc_{|0}b_i - 48\beta^3b^4y_jc^2b_i + 16\beta^2b^4y_ic_{|0}b_j \\
&- 153\beta^3nb_jc_{|0}b_i - 48\beta^3b^4y_ic^2b_j + 14n\beta^2y_jc_{|0}b_i - 30\beta^3ny_jc^2b_i + 14n\beta^2y_ic_{|0}b_j \\
&+ 297\beta^4nb_jc^2b_i + 45\beta^2b_jc_{|0}By_i - 96b_j\beta^3c^2By_i + 45\beta^2b_ic_{|0}By_j - 96b_i\beta^3c^2By_j \\
&- 6B^2c^2\beta^2y_iy_j - 30\beta^3ny_ic^2b_j - 144B^2\beta y_jc_{|0}y_i - 208\beta B^3y_jc_{|0}y_i + 104\beta^2B^2y_jc_{|0}b_i \\
&- 120\beta^3B^2y_jc^2b_i + 32\beta^2B^3y_jc^2y_i + 104\beta^2B^2y_ic_{|0}b_j + 135c_{|0}Ba_{ij}n\beta^3 - 144c_{|0}Bb_ib_j\beta^3 \\
&- 9Bc^2a_{ij}n\beta^4 + 128B^4c^2y_iy_j\beta^2 - 96B^3y_jc^2b_i\beta^3 - 96B^3y_ic^2b_j\beta^3 - 72B^2c^2a_{ij}n\beta^4 \\
&+ 288Bc^2b_ib_j\beta^4 - 64B^4c_{|0}y_iy_j\beta + 32B^3c_{|0}b_iy_j\beta^2 + 32B^3c_{|0}b_jy_i\beta^2 + 216B^2c_{|0}a_{ij}n\beta^3 \\
&+ 99c_{|0}Ba_{ij}\beta^3 + 27Bc^2a_{ij}\beta^4 + 144B^2c_{|0}a_{ij}\beta^3 + 14\beta^2y_jc_{|0}b_i - 30\beta^3y_jc^2b_i + 14\beta^2y_ic_{|0}b_j \\
&- 30\beta^3y_ic^2b_j - 45b_jc_{|0}\beta^3b_i + 117b_jc^2b_i\beta^4 + 128B^2c^2b^4ny_iy_j\beta^2 - 96Bc^2b^4b_iny_j\beta^3 \\
&- 96Bc^2b^4b_jny_i\beta^3 - 64B^2c_{|0}b^4ny_iy_j\beta + 32Bc_{|0}b^4b_iny_j\beta^2 + 32Bc_{|0}b^4b_jny_i\beta^2 \\
&+ 64n\beta^2b^4y_jc^2By_i - 32n\beta b^4y_jc_{|0}By_i - 96Bc^2b^4b_jy_i\beta^3 + 128B^4c^2ny_iy_j\beta^2 \\
&- 96nB^3y_jc^2b_i\beta^3 - 96nB^3y_ic^2b_j\beta^3 + 210B^2c^2n\beta^2y_iy_j - 144\beta nB^2y_jc_{|0}y_i \\
&- 208\beta nB^3y_jc_{|0}y_i + 16n\beta^2b^4y_jc_{|0}b_i - 48\beta^3nb^4y_jc^2b_i + 16n\beta^2b^4y_ic_{|0}b_j - 48\beta^3nb^4y_ic^2b_j \\
&- 32\beta b^4y_jc_{|0}By_i + 64\beta^2b^4y_jc^2By_i - 28\beta ny_jc_{|0}By_i + 32n\beta^2y_jc^2By_i - 64B^2c_{|0}b^4y_iy_j\beta
\end{aligned}
$$

$$+ 32B^3c_{|0}b_jny_i\beta^2 + 32Bc_{|0}b^4b_iy_j\beta^2 + 32Bc_{|0}b^4b_jy_i\beta^2 - 64B^4c_{|0}ny_iy_j\beta + 32B^3c_{|0}b_iny_j\beta^2$$
$$+ 128B^2c^2b^4y_iy_j\beta^2 - 96Bc^2b^4b_iy_j\beta^3 + 648\beta^4nBb_jc^2b_i - 264\beta^3nB^2b_jc^2y_i + 176n\beta^2B^2b_ic_{|0}y_j$$
$$- 264\beta^3nB^2b_ic^2y_j + 464B^3c^2n\beta^2y_iy_j + 81n\beta^2b_jc_{|0}By_i - 168\beta^3nb_jc^2By_i + 81n\beta^2b_ic_{|0}By_j$$
$$- 168\beta^3nb_ic^2By_j + 176n\beta^2B^2b_jc_{|0}y_i - 360\beta^3nBb_jc_{|0}b_i\Big)\alpha^2\Big],$$

where
$$A := -\frac{1}{2\alpha^3\big[(1+2B)\alpha - 3\beta\big]^4}.$$

## References


1. H. Akbar-Zadeh, *Sur les espaces de Finsler a courbures sectionnelles constantes*, Acad Roy Belg Bull Cl Sci. **80**(1988), 271-322.
2. D. Bao, C. Robles and Z. Shen, *Zermelo navigation on Riemannian manifolds*, J Diff Geom. **66**(2004), 377-435.
3. B. Bidabad and M. Rafie-Rad, *Pure pursuit navigation on Riemannian manifolds*, Nonlinear Analysis: RWA. **10**(2009), 1265-1269.
4. M.H. Emamian and A. Tayebi, *Generalized Douglas-Weyl Finsler spaces*, Iranian J Math Sci Inform. **10**(2015), 67-75.
5. M. Matsumoto, *The Berwald connection of Finsler with an $(\alpha,\beta)$-metric*, Tensor (N.S). **50**(1991), 18-21.
6. M. Matsumoto, *A slope of a hill is a Finsler surface with respect to a time measure*, J Math Kyoto Univ. **29**(1980), 17-25.
7. B. Najafi, Z. Shen and A. Tayebi, *Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties*, Geom Dedicata. **131**(2008), 87-97.
8. B. Najafi and A. Tayebi, *A class of isotropic mean Berwald metrics*, Acta Math Acad Paedagogicae Nyiregyhaziensis. **32**(2016), 113-123.
9. H. Shimada and V. S. Sabau, *An introduction to Matsumoto metric*, Nonlinear Analysis: RWA. **63**(2005), 165-168.
10. A. Tayebi and H. Sadeghi, *Generalized P-reducible $(\alpha,\beta)$-metrics with vanishing S-curvature*, Ann Polon Math. **114**(2015), 67-79.
11. A. Tayebi and H. Sadeghi, *On generalized Douglas-Weyl $(\alpha,\beta)$-metrics*, Acta Math Sinica English Series. **31**(2015), 1611-1620.
12. A. Tayebi and M. Shahbazi Nia, *A new class of projectively at Finsler metrics with constantag curvature $K = 1$*, Differ. Geom Appl. **41**(2015), 123-133.
13. A. Tayebi and T. Tabatabeifar, *Dougals-Randers manifolds with vanishing stretch tensor*, Publ Math Debrecen. **86**(2015), 423-432.
14. A. Tayebi and T. Tabatabeifar, *Matsumoto metric of reversible curvatures*, Acta Math Acad Paedagogicae Nyiregyhaziensis. **32**(2016), 165-200.
15. A. Tayebi and T. Tabatabeifar, *Unicorn metrics with almost vanishing $H$- and $\Xi$-curvatures*, Turkish J Math. **41**(2017), 998-1008.


Received: 19.01.2020
Accepted: 23.05.2020