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On Generalized 4-th Root Finsler Metrics

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Abstract. In this paper, we prove that every generalized cubic Finsler metric with vanishing Landsberg curvature is a Berwald metric. This yields an extension of Matsumoto theorem for the cubic metric. Then, we show that every generalized 4-th root Finsler metric with vanishing Landsberg curvature is a Berwald metric.

Keywords: 4-th root metrics, cubic metric, Landsberg metric, Berwald metric.

1. Introduction

Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle, and (x^i, y^i) the coordinates in a local chart on TM. Let F be the following function on M by

$$F = \sqrt[m]{A},$$

where $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ are non-singular and irreducible m-form with $a_{i_1 \dots i_m}$ symmetric in all its indices. Then (M, F) is called an m-th root Finsler manifold (see [12][15][16]). An m-th root metric is regarded as a direct generalization of the Riemannian metric in a sense, i.e., the second root metric is a Riemannian metric. The third and fourth root metrics are called the cubic metric $F = \sqrt[3]{A}$ and quartic metric $F = \sqrt[4]{A}$, respectively. The theory of m-th root metrics has been developed by Shimada [12] and applied to Biology as an ecological metric [1]. In four-dimension, the special fourth root metric in form $F = \sqrt[4]{y^1 y^2 y^3 y^4}$ is called the Berwald-Moór metric. This metric is singular in y and not positive definite. In the last two decades, physical studies due to

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Asanov, Pavlov and their co-workers emphasize the important role played by the Berwald-Moór metric in the theory of space-time structure and gravitation as well as in unified gauge field theories [2][9][10]. For quartic metrics, a study of the geodesics and of the related geometrical objects is made by Balan, Brinzei and Lebedev [3][4][7]. Also, Einstein equations for some relativistic models relying on such metrics are studied by Balan-Brinzei in two papers [5][6]. Recent studies have shown that physicists are interested in fourth root Finsler metrics because these metrics are related to the geometrization of space-time.

In [13], tensorial connections for such spaces have been studied by Tamassy. Li-Shen study locally projectively flat fourth root metrics under irreducibility condition [8]. Yu-You show that an m-th root Einstein Finsler metrics are Ricci-flat [18]. Tayebi-Najafi characterize locally dually flat and Antonelli m-th root Finsler metrics. They prove that every m-th root Finsler metric of isotropic mean Berwald curvature (resp, isotopic Lanbdsberg curvature) reduces to a weakly Berwald metric (resp, Landsberg metric) [15]. Also, they show that every m-th root metric with almost vanishing **H**-curvature has vanishing **H**-curvature [16].

Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let

$$F = \sqrt{\sqrt[m]{A^2} + B},$$

where A is given by $A := a_{i_1} \ldots_{i_m}(x) y^{i_1} \ldots y^{i_m}$ with a_{ijkl} symmetric in all its indices and $B := b_{ij}(x) y^i y^j$ is a 2-form on M. Then F is called a generalized m-th root metric. In [17], Tayebi-Peyghan-Shahbazi Nia studied the generalized m-th root Finsler metrics and characterize locally dually flat generalized m-th root Finsler metrics. They found a necessary and sufficient condition under which a generalized m-th root metric is projectively related to an m-th root metric. Also, they proved that if a generalized m-th root is conformal to an m-th root Finsler metric, then both of them reduce to Riemannian metrics. In [14], Tayebi studied the class of generalized 4-th root metrics. These metrics generalize 4-th root metrics which are used in Biology as ecological metrics. He found the necessary and sufficient condition under which a generalized 4-th root metric is of isotropic scalar curvature. Then, he obtained the necessary and sufficient condition under which the conformal change of a generalized 4-th root metric is of isotropic scalar curvature. Also, he characterized the Bryant metrics of isotropic scalar curvature.

In this paper, we study generalized cubic metric with vanishing Landsberg curvature and extend the Matsumoto Theorem. More precisely, we prove the following.

Theorem 1.1. *Every generalized cubic metric with vanishing Landsberg curvature is a Berwald metric.*

Moreover, we consider the generalized 4-th root with vanishing Landsberg curvature and prove the following.

Theorem 1.2. *Every generalized 4-th root with vanishing Landsberg curvature is Berwald metric.*

2. Preliminary

A Finsler metric on an n-dimensional C^∞ manifold M is a function $F : TM \to [0,\infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 = TM \setminus \{0\}$,
(ii) F is positively 1-homogeneous on the fibers of the tangent bundle TM,
(iii) for each $y \in T_xM$, the quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is positive definite, where

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y+su+tv)\Big]|_{s,t=0}, \quad u,v \in T_xM.$$

Then (M,F) is called an n-dimensional Finsler manifold.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]|_{t=0}, \quad u,v,w \in T_xM.$$

The quantity $\mathbf{C}$ is called the Cartan torsion of F. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. Taking the trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ defined by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_i), \quad u \in T_xM,$$

where $g^{ij} := (g_{ij})^{-1}$ and $\{\partial_i\}$ is a basis for T_xM at $x \in M$.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in the standard coordinate system (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}\Big], \quad y \in T_xM. \tag{2.1}$$

Here, $\mathbf{G}$ is called the associated spray to (M,F) and G^i are called the spray coefficients.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called Berwald curvature. F is called a Berwald metric if $\mathbf{B} = 0$.

For $y \in T_xM_0$, define $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u, v, w), y\big).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where $L_{ijk} := -\frac{1}{2}y_l B^l{}_{ijk}$. $\mathbf{L}$ is called the Landsberg curvature. F is called a Landsberg metric if $\mathbf{L} = 0$.

3. Proof of Theorems

Let $F = \sqrt[4]{A}$ and $\bar{F} = \sqrt{\sqrt{A} + B}$ are 4-th root and generalized 4-th root Finsler metrics, respectively, where $A := a_{ijkl}y^i y^j y^k y^l$ and $B := b_{ij}y^i y^j$. By definition, $\bar{F}$ is a Landsberg metric and a Berwald metric if and only if $C_{ijk|0} = 0$ and $C_{ijk|l} = 0$, respectively, where we use the Cartan connection of F. With respect to the Cartan connection, we have $A_{i|j} = 0$ and $A_{ij|k} = 0$. Let us define

$$A_i = \frac{\partial A}{\partial y^i}, \qquad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}$$
$$B_i = \frac{\partial B}{\partial y^i}, \qquad B_{ij} = \frac{\partial^2 B}{\partial y^i \partial y^j}$$

From $F^4 = a_{ijkl}y^i y^j y^k y^l$, we define Finslerian symmetric tensors of order r $(1 \le r \le 3)$ with the components

$$b_{i_1 \ldots i_r} := \frac{A_{i_1 \ldots i_r}}{F^{4-r}}$$

Among this tensor, we have the fundamental tensor g_{ij} and Cartan tensor C_{ijk} as follows

$$g_{ij} = \frac{1}{16}\Big\{4b_{ij} - 2b_i b_j\Big\}, \tag{3.1}$$

$$C_{ijk} = \frac{1}{4F}\Big\{b_{ijk} + \frac{3}{4}b_i b_j b_k - \frac{1}{2}\big[b_i b_{jk} + b_j b_{ik} + b_k b_{ij}\big]\Big\}. \tag{3.2}$$

Equation (3.2) leads to the following.

Lemma 3.1. *A generalized 4-th root metric $\bar{F}$ is a Landsberg (resp. Berwald) metric if and only if $b_{ijk|0} = 0$ (resp. $b_{ijk|l} = 0$).*

Now, we are going to prove the main results.

Proof of Theorem 1.1: Suppose that $\bar{F} = \sqrt{\sqrt[3]{A^2} + B}$ is a generalized cubic metric with vanishing Landsberg curvature. Then by Lemma 3.1, we have

$$b_{ijk|0} = b_{ijk|l}y^l = 0. \tag{3.3}$$

By differentiating of (3.3) with respect to y^m, we get

$$b_{ijk|l} + b_{ijk|m,l}y^m = 0 \tag{3.4}$$

It is easy to see that for the Landsberg metric, the hh-curvature of Berwald and Cartan connection coincide. Thus the following Ricci identity of Berwald connection holds

$$b_{ijk|m,l} - b_{ijk,l|m} = -b_{tjk}G^t_{iml} - b_{tik}G^t_{jml} - b_{tij}G^t_{kml}. \tag{3.5}$$

Here it is remarked that b_{ijk} is a cubic metric, and it's a function of x alone. Thus we have $b_{ijk,l|m} = 0$. Contracting (3.5) with y^m and using $G^t_{ilm}y^m = 0$ implies that

$$b_{ijk|m,l}y^m = 0. \tag{3.6}$$

Substituting (3.6) into (3.4) we have $b_{ijk|l} = 0$. Thus space is reduced to a Berwald space. □

In [20], it is proved that every non-Riemannian m-th root Finsler manifold of isotropic Landsberg reduces to a Landsberg metric. Then by Theorem 1.1, we get the following:

To prove Theorem 1.2, let us put

$$b'_{i_1\ldots i_r} := \frac{A_{i_1\ldots i_r}}{F^{4-r-1}}.$$

Then we can rewrite C_{ijk} as the follows

$$C_{ijk} = \frac{1}{4F^2}\Big\{b'_{ijk} + \frac{3}{4}F^{-1}b_ib_jb_k - \frac{1}{2}F^{-1}\big[b_ib_{jk} + b_jb_{ik} + b_kb_{ij}\big]\Big\}. \tag{3.7}$$

By Lemma (3.1), $\bar{F}$ is a Landsberg metric and Berwald metric if and only if $b'_{ijk|0} = 0$ and $b'_{ijk|l} = 0$, respectively. By (3.7), we can prove Theorem 1.1 for generalized 4-th root metrics.

Proof of Theorem 1.2: By differentiating of $b'_{ijk|l}y^l = 0$ with respect to y^m, we have

$$b'_{ijk|l} + b'_{ijk|m,l}y^m = 0. \tag{3.8}$$

By contracting the Ricci identity (3.5) with y^m, we get

$$b'_{ijk|m,l}y^m = b'_{ijk,l|m}y^m. \tag{3.9}$$

By substituting (3.9) into (3.8), we have

$$b'_{ijk|l} + b'_{ijk,l|m}y^m = 0 \tag{3.10}$$

Differentiate (3.10) with respect to y^s yields

$$b'_{ijk|l,s} + b'_{ijk,l|s} + b'_{ijk,l|m,s}y^m = 0 \tag{3.11}$$

For fourth root metric $b'_{ijk,l|m,s}y^m = 0$. By substituting (3.11) into Ricci identity we have

$$2b'_{ijk,l|m} = b'_{tjk}G^t_{iml} + b'_{tik}G^t_{jml} + b'_{tij}G^t_{kml} \tag{3.12}$$

By contracting y^m into (3.12) we have

$$b'_{ijk,l|m}y^m = 0 \tag{3.13}$$

By (3.8), (3.9) and (3.13) we have $b'_{ijk|l} = 0$. Then the generalized 4-th root metric reduces to a Berwald metric. □

References

1. P. L. Antonelli, R. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Acad. Publ. Netherlands, 1993.
2. G.S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht, 1984.
3. V. Balan, *Spectra of symmetric tensors and m-root Finsler models*, Linear Algebra and its Applications, **436**(1) (2012), 152-162.
4. V. Balan, N. Brinzei and S. Lebedev, *Geodesics, paths and Jacobi fields for Berwald-Moór quartic metrics*, Hypercomplex Numbers in Geometry and Physics, accepted.
5. V. Balan and N. Brinzei, *Einstein equations for (h, v)-Berwald-Moór relativistic models*, Balkan. J. Geom. Appl. **11**(2) (2006), 20-26.
6. V. Balan and N. Brinzei, *Berwald-Moór-type (h, v)-metric physical models*, Hypercomplex Numbers in Geometry and Physics. **2**(4) (2005), 114-122.
7. S.V. Lebedev, *The generalized Finslerian metric tensors*, to appear.
8. B. Li and Z. Shen, *On projectively flat fourth root metrics*, Canad. Math. Bull. **55**(2012), 138-145.
9. D.G. Pavlov (Ed.), *Space-Time Structure, Algebra and Geometry*, Collected papers, TETRU, 2006.
10. D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, **1**(2004), 31-39.
11. Z. Shen, *Riemann-Finsler geometry with applications to information geometry*, Chin. Ann. Math. **27**(2006), 73-94.
12. H. Shimada, *On Finsler spaces with metric $L = \sqrt[m]{a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}}$*, Tensor, N.S., **33**(1979), 365-372.
13. L. Tamàssy, *Finsler spaces with polynomial metric*, Hypercomplex Numbers in Geometry and Physics, **3**(6) (2006), 85-92.
14. A. Tayebi, *On generalized 4-th root metrics of isotropic scalar curvature*, Mathematica Slovaca. **68**(2018), 907-928.
15. A. Tayebi and B. Najafi, *On m-th toot Finsler metrics*, J. Geom. Phys. **61**(2011), 1479-1484.
16. A. Tayebi and B. Najafi, *On m-th root metrics with special curvature properties*, C. R. Acad. Sci. Paris, Ser. I, **349**(2011), 691-693.
17. A. Tayebi, E. Peyghan and M. Shahbazi Nia, *On generalized m-th root Finsler metrics*, Linear Algebra. Appl. **437**(2012) 675-683.
18. Y. Yu and Y. You, *On Einstein m-th root metrics*, Diff. Geom. Appl. **28**(2010) 290-294.
19. M. Matsumoto, *Theory of Finsler spaces with m-th root metric II*, Math. Debrecen, 1996.
20. A. Tayebi and B. Najafi, *On m-th root metrics with special curvature properties*, Journal of Geometry and Physics, **61** (2011) 1479-1484.

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